Basic information: Regularization

A family of functions $(R_\alpha)_{\alpha\in]0,\alpha_0]}$ (for some $\alpha_0 > 0$) is called regularization, if for every $y^\dagger \in \mathrm{Dom}(T^\dagger)$ there is a parameter choice rule $\alpha = \alpha_{y^\dagger}\colon]0,\infty[\times Y \to]0,\alpha_0]$ such that

$$\lim_{\delta\to 0} \sup\{\|R_{\alpha_{y^\dagger}(\delta,y^\delta)}y^\delta - T^\dagger y^\dagger\| \mid y^\delta \in Y,\ \|y^\delta - y^\dagger\| \le \delta\} = 0 \tag{1}$$

$$\lim_{\delta\to 0} \sup\{\alpha_{y^\dagger}(\delta,y^\delta) \mid y^\delta \in Y,\ \|y^\delta - y^\dagger\| \le \delta\} = 0. \tag{2}$$

The pair $((R_\alpha)_\alpha, \{\alpha_{y^\dagger}\}_{y^\dagger\in\mathrm{Dom}(T^\dagger)})$ is called a regularization method. When $(R_\alpha)_\alpha$ is a family of continuous operators such that $R_\alpha y \to T^\dagger y$ as $\alpha \to 0$ for all $y \in \mathrm{Dom}(T^\dagger)$, then $(R_\alpha)_\alpha$ is a regularization. Moreover, a linear regularization $(R_\alpha)_\alpha$ is a regularization method together with an a-priori parameter choice α if and only if $\lim_{\delta\to 0}\alpha(\delta) = 0$ and $\lim_{\delta\to 0}\delta\|R_{\alpha(\delta)}\| = 0$.
We consider the source set $X_{\mu,\rho} = (T^*T)^\mu(\overline{B(0,\rho)})$ for $\mu,\rho > 0$ and call a regularization method order optimal (for $\mu > 0$), if there is a constant c such that $\sup\{\|R_\alpha y^\delta - x\| \mid x \in X_{\mu,\rho},\ y^\delta \in Y,\ \|y^\delta - Tx\| \le \delta\} \le c\delta^{\frac{2\mu}{2\mu+1}}\rho^{\frac{1}{2\mu+1}}$ for all $\delta > 0$.
Spectral filtering consists in constructing R_α via spectral integrals, i.e., $R_\alpha = g_\alpha(T^*T)T^*$ for some functions $g_\alpha\colon]0,\|T\|^2] \to \mathbb{R}$ bounded and measurable. Indeed, if

$$\lim_{\alpha\to 0} g_\alpha(\lambda) = \frac{1}{\lambda} \text{ for every } \lambda \qquad \text{and} \qquad \lambda g_\alpha(\lambda) \le c \tag{3}$$

for some constant c, then R_α is a regularization. Moreover, if

$$\sup_{\lambda\in]0,\|T\|^2]} |g_\alpha(\lambda)| = \mathcal{O}(\alpha^{-1}) \qquad \text{and} \qquad \lambda^\mu|1 - \lambda g_\alpha(\lambda)| < c\alpha^\mu, \tag{4}$$

then R_α together with $\alpha(\delta) \approx \left(\frac{\delta}{\rho}\right)^{\frac{2}{2\mu+1}}$ yields an order optimal regularization method (w.r.t. this μ).

Example 4.1) [Matrix free regularization]
Given $T \in L(X,Y)$ between Hilbert spaces, we consider a family of operators $(R_\alpha)_{\alpha>0}$ according to $R_\alpha = g_\alpha(T^*T)T^* = \frac{p_\alpha}{q_\alpha}(T^*T)T^*$ for families of polynomials $p_\alpha(\lambda) = \sum_{k=0}^{N-1} a_k(\alpha)\lambda^k$ and $q_\alpha(\lambda) = \sum_{k=0}^{N} b_k(\alpha)\lambda^k$ for some $a_k, b_k\colon]0,\alpha_0] \to [0,\infty]$. Further, we assume $b_0(\alpha) > 0$ and (without loss of generality) $\max_{k\in\{0,\dots,N\}} b_k = 1$.

a) Let $\frac{a_{k-1}(\alpha)}{b_k(\alpha)} \to 1$ for $k \in \{1,\dots,N\}$ (understanding $\frac{0}{0} = 1$) and $b_0 \to 0$. Show that $(R_\alpha)_\alpha$ is a regularization. Further, show that $x = R_\alpha y$ is equivalent to solution of $q_\alpha(T^*T)x = p_\alpha(T^*T)T^*y$.

b) Assume for some $l \in \{1,\dots,N\}$ that $\frac{a_{k-1}(\alpha)}{b_k(\alpha)} - 1 = \mathcal{O}(\alpha)$ for all $k \in \{1,\dots,N\}$ and $b_0(\alpha) = \alpha$ as $\alpha \to 0$. Show that the regularization is order optimal at least up to $\mu \le 1$.

c) Show that for R_α to satisfy (3), the $a_k(\alpha)$ must be bounded; conclude, that $\lim_{\alpha\to 0}|a_{k-1}(\alpha) - b_k(\alpha)| = 0$ for $k \in \{1,\dots,N\}$ and $b_0 \to 0$ are necessary.

Remark. *Note that for many operators, the spectral measures are unknown or it is non-viable to compute spectral integrals, even in a discrete setting. Hence, it is not possible to use (naively) spectral filtering to such operators. However, recall that the functional calculus for polynomials* *can be computed without the actual spectral measure or spectral integrals, but only by iterative application of T^*T. Hence, many regularization methods used in practice are of a form that can be computed by evaluations of T and T^*.*

Example 4.2) [Source set of periodic convolution]
Let $\Omega = [0,1[$. We consider the (linear and continuous) periodic convolution operator

$$T\colon L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C}) \qquad [Tf](x) = \int_\Omega f(y)k(x-y)\,\mathrm{d}y \text{ for almost every } x \in \Omega, \tag{5}$$

for a given $k \in L^2(\mathbb{R}/\mathbb{Z},\mathbb{C})$ (i.e., $k \in L^2(\Omega,\mathbb{C})$ periodically extended). It is easy to check, that T is compact and possesses eigenvectors $(e_n)_{n\in\mathbb{Z}}$ with $e_n = e^{2\pi i n x}$, and corresponding eigenvalues $\hat{k}_n = \int_\Omega k(x)\overline{e_n}\,\mathrm{d}x = \langle k, e_n\rangle$. We say $f \in H^l_\#(\mathbb{R}/\mathbb{Z},\mathbb{C})$ if $f \in L^2(\mathbb{R}/\mathbb{Z},\mathbb{C})$ is l times weakly differentiable (on $\mathbb{R}$) with derivatives in $L^2(\mathbb{R}/\mathbb{Z},\mathbb{C})$. Note that the weak differentiation operator $\partial^l : H^l_\#(\mathbb{R}/\mathbb{Z},\mathbb{C}) \subset L^2(\Omega) \to L^2(\mathbb{R}/\mathbb{Z})$ is closed.

- **a)** Show that given $f \in L^2(\Omega,\mathbb{C})$, $f \in H^1_\#(\mathbb{R}/\mathbb{Z},\mathbb{C})$ if and only if $\sum_{n\in\mathbb{Z}} |n|^2 |\langle f, e_n\rangle|^2 < \infty$.
- **b)** We assume $\hat{k}_0 = 1$ and $c|n|^{-1} \le |\hat{k}_n| \le C|n|^{-1}$ for some constants $c, C > 0$ and all $n \in \mathbb{Z}\setminus\{0\}$. (This corresponds to the assumption $T'\colon f \mapsto \partial T f$ is well defined on $L^2(\mathbb{R}/\mathbb{Z})$, with closed range and $\ker(T') = \operatorname{span}\{1\}$). Show $X_l = H^{2l}_\#(\mathbb{R}/\mathbb{Z},\mathbb{C})$ for $l \in \mathbb{N}$.

Remark. *This exemplifies how X_μ contains more "regular" elements for greater μ. Note that this can be generalized to higher dimensions and fractional Sobolev spaces.*

Example 4.3) [Radon transform]
Let $\Omega = B(0,1) \subset \mathbb{R}^2$, $\Omega' = [-1,1]\times[0,\pi[$ and $\nu(\varphi) = (\cos(\varphi),\sin(\varphi))$, $\nu^\perp(\varphi) = (-\sin(\varphi),\cos(\varphi))$. For $f \in \mathcal{C}(\overline{\Omega})$, the Radon transform is defined according to

$$[\widetilde{\mathcal{R}}f](s,\varphi) = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} f(s\nu(\varphi) + t\nu^\perp(\varphi))\,\mathrm{d}t \quad \text{ for } (s,\varphi) \in \Omega'. \tag{6}$$

- **a)** Show that $\widetilde{\mathcal{R}}\colon \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega'})$, and the existence of a $c > 0$ such that $\|\widetilde{\mathcal{R}}f\|_{L^2} \le c\|f\|_{L^2}$ for all $f \in \mathcal{C}(\overline{\Omega})$. Consequently, $\widetilde{\mathcal{R}}$ can be uniquely extended to $\mathcal{R}\colon L^2(\Omega) \to L^2(\Omega')$.
- **b)** Show that $[\mathcal{R}^*g](x) = \int_0^\pi g(x\cdot\nu(\varphi),\varphi)\,\mathrm{d}\varphi$ for $g \in \mathcal{C}(\overline{\Omega'})$. Conclude that $\mathcal{R}^*\mathcal{R}f = k * f$ (restricted to Ω) for $k(x) = \frac{1}{\|x\|}\chi_{B(0,2)}$.
- **c)** Let $\widehat{\Omega}' = \mathbb{R}\times[0,\pi[$ and $\mathcal{F}_1\colon L^2(\Omega') \to L^2(\widehat{\Omega}')$ with $[\mathcal{F}g](\xi,\varphi) = \frac{1}{\sqrt{2\pi}}\int_{-1}^1 g(s,\varphi)e^{-is\xi}\,\mathrm{d}s$ the Fourier transform with respect to the first variable. Further, let $M\colon \operatorname{Dom}(M) \subset L^2(\widehat{\Omega}') \to L^2(\widehat{\Omega}')\}$ be defined according to $[Mg](\xi,\varphi) = |\xi| g(\xi,\varphi)$, with $\operatorname{Dom}(M) = \{g \in L^2(\widehat{\Omega}') \mid Mg \in L^2(\widehat{\Omega}')\}$. One can show (though this gets quite technical)

$$\mathcal{R}^\dagger = \frac{1}{2\pi}\mathcal{R}^*\mathcal{F}_1^* M \mathcal{F}_1. \tag{7}$$

We modify M to (the linear and continuous) $M_\alpha \colon L^2(\widehat{\Omega}') \to L^2(\widehat{\Omega}')$ with $[M_\alpha g](\xi, \varphi) = \chi_{[-\alpha^{-1}, \alpha^{-1}]}(\xi)|\xi| g(\xi, \varphi)$ and consider the corresponding family of functions $(R_\alpha)_{\alpha > 0}$ with $R_\alpha = \frac{1}{2\pi} \mathcal{R}^* \mathcal{F}_1^* M_\alpha \mathcal{F}_1$.

Show, that R_α is a regularization of the ill-posed inverse problem $\mathcal{R} f = g$.

Remark. *The Radon transform plays an important role in the context of computed tomography (e.g. reconstructing the density of a human body from a set of x-ray images from various directions). The formulation of $\mathcal{R}^\dagger$ is known as the Filtered Backprojection, while this regularization is known as the Ram-Lak filter. Since the Radon transform has some similarities to convolution (see $\mathcal{R}^* \mathcal{R}$), it makes sense that the pseudo inverse has some relation to the pseudo-inverse of a deconvolution problem (see Exercise 1.3).*