Richard Huber
richard.huber@uni-graz.at

3. Exercise Sheet

Basic information: Spectral measures and functional calculus

A selfadjoint bounded $S \in L(H,H) := L(H)$ (H Hilbert space) possesses a (compact) spectrum $\sigma(S) \subset \{\lambda \in \mathbb{R} \ : \ |\lambda| \leq \|S\|\}$; if S is positive semi-definite (such is T^*T), then $\sigma(S) \subset [0, \|S\|]$. Based on this spectrum, there is a (unique) spectral measure $E := E^S$, with the following properties: $E^S \colon \mathcal{B}(\sigma(S)) \to L(H)$ s.t. $E^S(A)$ is an orthogonal projection for $A \in \mathcal{B}(\sigma(S))$ (Borel measurable), with $E^S(\emptyset) = 0$, $E^S(\sigma(S)) = id_H$ and $E^S(\bigcup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} E^S(A_i)$ for measurable, pairwise disjoint A_i. Then

$$S = \int_{\sigma(S)} \lambda \,\mathrm{d}E^S(\lambda), \tag{1}$$

where the integral is understood as an integral of a real-valued function with respect to a vector valued (sigma additive) measure (see lecture on Advanced Analysis). In particular, for compact operators, E^S is a discrete measure and $E^S(\{\lambda\})$ is the projection onto the eigenspace with respect to the eigenvalue λ (and zero otherwise).

Moreover, for a selfadjoint $S \in L(H)$ and for continuous (more generally bounded Borel measurable) functions $f \colon \sigma(S) \to \mathbb{R}$ one can define the selfadjoint operator $f(S) \in L(H)$ in the following way called functional calculus: For polynomials $f(t) = \sum_{i=0}^N a_i t^i$, the corresponding operator $f(S) := \sum_{i=0}^N a_i S^i$, which satisfies

$$(f+g)(S) = f(S) + g(S), \qquad (f\cdot g)(S) = f(S)g(S), \qquad \|f(S)\| \leq \|f\|_\infty, \tag{2}$$

and can be extended to continuous operators (by density). In particular, if $f_n \to f$ pointwise such that $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$, then $f_n(S)x \to f(S)x$ for all $x \in H$ (convergence in the strong operator topology).

In particular, these two concepts are connected via $\chi_A(S) = E^S(A)$, and

$$f(S) = \int_{\sigma(S)} f(\lambda) \,\mathrm{d}E^S(\lambda) \qquad \text{or equivalently} \qquad f(S)x = \int_{\sigma(S)} f(\lambda) \,\mathrm{d}\big(E^S(\lambda)x\big). \tag{3}$$

In the special case of S compact, $f(S)x = \sum_{n\in\mathbb{N}} f(\lambda_n)\langle u_n, x\rangle u_n + f(0)\mathcal{P}_{\ker(S)}x$, where $(\lambda_n, u_n)_{n\in\mathbb{N}}$ is an eigenvalue decomposition of S with $\lambda_n \neq 0$ and $\mathcal{P}_{\ker(S)}$ is the orthonormal projection onto $\ker(S)$.

The integral formula (3) with given x (on the right) can be extended to unbounded measurable functions f, however, $f(S)$ is not necessarily continuous and $\mathrm{Dom}(f(S)) = \{x \in X \mid \|f(S)x\| < \infty\}$, where $\|f(S)x\|^2 = \int_{\sigma(S)} |f(\lambda)|^2 \,\mathrm{d}\|E^S(\lambda)x\|^2$ if finite. The properties (2) remain valid.

Example 3.1) [Singular value decomposition of Hilbert-Schmidt operators]
Prove the statement of example 2.4b).

Example 3.2) [Functional Calculus]
Let $T \in L(X,Y)$ between Hilbert spaces X and Y. The following statements hold: For continuous $f \colon \sigma(S) \to \mathbb{R}$ it holds that $f(T^*T)T^* = T^* f(TT^*)$. Also, $\mathrm{Rg}(T^*) = \mathrm{Rg}(|T|)$ (with $|T| := (T^*T)^{\frac{1}{2}}$).

a) Show these statements for compact T via the sum representation of the functional calculus and the singular value decomposition.

b) Bonus: Show the statement for bounded T (not necessarily compact).

Hint. *There is a unitary mapping* $U\colon \overline{\mathrm{Rg}(|T|)} \to \overline{\mathrm{Rg}(T)}$ *such that* $T = U|T|$ *(see polar decomposition in Functional Analysis).*

Example 3.3) [Pseudo-inverse via spectral measure]
Let $T \in L(X,Y)$ between Hilbert spaces. In the lecture, it was shown that $T^\dagger = (T^*T)^\dagger T^*$.

a) Show that $(T^*T)^\dagger = \int_{\sigma(T^*T)\setminus\{0\}} \frac{1}{\lambda}\,\mathrm{d}E^{T^*T}(\lambda)$. What is $\mathrm{Dom}((T^*T)^\dagger)$ and in which way does this characterize $\mathrm{Dom}(T^\dagger)$?

b) For $\mu \geq 0$, the set $X_\mu = \mathrm{Rg}((T^*T)^\mu)$ is called the source set to μ. Find a characterization of X_μ via the spectral measure E^{T^*T}.

Hint. $E^{T^*T}(\{0\})$ *corresponds to the orthogonal projection onto* $\ker(T^*T) = \ker(T)$.

Remark. *Point a) gives us a concrete definition of* $T^\dagger$ *via the spectral measure and* T^*y*. This will be useful in finding suitable approximations (regularization) by replacing* $\frac{1}{\lambda}$ *by bounded functions, see later in the lecture. Since* T *compact is understood to possess some kind of smoothing properties, elements of* X_μ *are somehow more 'regular'; the higher* μ *the more so. This regularity will be exploited in the lecture to find convergence rates of regularization methods.*

Example 3.4) [Showalter regularization]
Let $T \in L(X,Y)$ between Hilbert spaces and $y \in Y$ fixed. We consider the corresponding (functional) differential equation

$$\begin{cases} \frac{\partial x}{\partial t}(t) + T^*Tx(t) = T^*y & t \geq 0, \\ x(0) = 0, \end{cases} \tag{4}$$

where $x \in \mathcal{C}^1([0,\infty), X)$ (continuously Fréchet differentiable with $\frac{\partial x}{\partial t}(t) := \lim_{|h|\to 0} \frac{x(t+h)-x(t)}{h}$).

a) Assuming that T is compact, compute a solution of (4) by decoupling into suitable differential equations for complex valued functions. (Here it is not so important to be perfectly rigorous). Find functions $(f_t)_{t\geq 0}$ with $f_t\colon \sigma(T^*T) \to \mathbb{R}$ such that $x(t) = f_t(T^*T)T^*y$.

b) Show that there exists a unique solution to (4). **Hint:** Show rigorously that $x(t) = f_t(T^*T)T^*y$ is a solution. Uniqueness can be shown by classical energy estimates, i.e., consider $\frac{\partial \|x(t)\|^2}{\partial t}$ if $y = 0$.

c) Show that $\|x(t)\| \to \infty$ for $t \to \infty$ if $y \notin \mathrm{Dom}(T^\dagger)$, while if $y \in \mathrm{Dom}(T^\dagger)$, then $x(t) \to T^\dagger y$. **Hint:** Consider the derivative of $\|x(t) - T^\dagger y\|^2$ when $y \in \mathrm{Dom}(T^\dagger)$.

Remark. *As stated in Example 3.3 one could replace* $\frac{1}{\lambda}$ *with a bounded function. Here, this function* f_t *is given implicitly via (4), with the motivation that* $x(t)$ *is always moving in a direction reducing the residue, i.e., a gradient flow (gradient descent). Concerning the solvability, the theorem of Picard-Lindelöf would also holds in Hilbert spaces.*