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2. Exercise Sheet

Basic information: Singular systems
For Hilbert spaces X, Y we call a tuple $(\sigma_n, u_n, v_n)_{n\in\{1,\dots,N\}}$ or sequence $(\sigma_n, u_n, v_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+ \times X \times Y$ a singular system, if σ_n is monotone decreasing and converging to zero (if sequence), $(u_n)_n$ and $(v_n)_n$ orthonormal systems in X and Y respectively. It is well known (see lectures on Functional Analysis), that $T \in L(X,Y)$ is compact if and only if there exists a singular system (with index set $M \in \{\{1,\dots,N\} \mid N \in \mathbb{N}\} \cup \mathbb{N}$) such that

$$Tx = \sum_{n\in M} \sigma_n \langle u_n, x\rangle v_n \qquad \text{and } T^*y = \sum_{n\in M} \sigma_n \langle v_n, y\rangle u_n. \tag{1}$$

In particular, T^*T is compact and selfadjoint, thus possessing a spectral decomposition with eigenvalues and orthonormal eigenvectors $(\lambda_n, u_n)_{n\in\mathbb{N}}$ with $\lambda_n \to 0$, and the singular system is given via $\sigma_n = \sqrt{\lambda_n}$, $v_n = \frac{Tu_n}{\sigma_n}$ (and the same u_n) for n s.t. $\lambda_n > 0$.
Analogously, for a matrix $A\colon \mathbb{R}^m \to \mathbb{R}^k$ (which is trivially compact), there are matrices $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{k\times k}$ both unitary, and $\Sigma \in \mathbb{R}^{k\times m}$ diagonal such that $A = V\Sigma U^*$. More precisely, the columns of U and V are the vectors u_n and v_n of a singular system of A respectively (filling with an orthonormal basis orthonormal to u_n or v_n respectively if necessary to generate the necessary dimensions) and $\Sigma_{nn} = \sigma_n$ (singular values) for $1 \le n \le r$ (r the rank of A) and $\Sigma_{ij} = 0$ otherwise.

Example 2.1) [Inversion via SVD]
We consider a matrix $A\colon \mathbb{R}^m \to \mathbb{R}^k$, and the corresponding inverse problem $Ax = y$. Find a matrix representation of $A^\dagger$ as well as a sum representation in the form of (1) of $A^\dagger$, and determine the spectral norm $\|A^\dagger\|$ (operator norm for euklidean norms on $\mathbb{R}^k$ and $\mathbb{R}^m$).

Remark. *As this example shows, the pseudo-inverse (in a finite dimensional setting) can be determined by the singular value decomposition. Also note the connection between the singular values and the norm of the pseudo inverse implying that the smaller singular values the bigger the norm. As will be shown in the lecture, these results can be carried over to compact operators T, hence knowledge of the singular system of T is useful in finding and understanding $T^\dagger$.*

Example 2.2) [Quotient spaces]
Let X, Y be Banach spaces and $T \in L(X,Y)$. We define $\widehat{X} := X/\ker(T)$ the quotient space of X subject to the kernel of T, i.e., we consider classes according to the equivalence relation $x \approx y$ if and only if $x - y \in \ker(T)$. It is easy to check, that with $[x] + [y] = [x+y]$ and $\alpha[x] = [\alpha x]$ for $x, y \in X$, $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), $\widehat{X}$ is a vector space. We endow $\widehat{X}$ with $\|[x]\|_{\widehat{X}} = \inf_{m\in\ker(T)} \|x - m\|$ for $x \in X$ and $[x]$ the associated equivalence class.

a) Show that $\|[\cdot]\|_{\widehat{X}}$ is welldefined and constitutes a norm on $\widehat{X}$ such that it is complete (i.e., a Banach space). Further, show that there is a linear bijective $\widehat{T}\colon \widehat{X} \to \mathrm{Rg}(T)$ with $\widehat{T}[x] = Tx$ for $x \in X$ and in particular $\|\widehat{T}\| = \|T\|$.

b) Show that there is a minimizer to the problem $\min_{z\in x_0+\ker(T)} \|z\|$ for given $x_0 \in X$ in case X is reflexive and separable (implying weak sequential compactness of bounded sets). Find an example (in finite dimensions) where given any x_0 the corresponding minimizer z is unique but the mapping $x_0 \mapsto z$ is not linear.

Remark. *Note that the construction of the Moore Penrose inverse on the previous sheet was based on using* $\ker(T)^\perp$ *which only works in Hilbert spaces. The quotient space is (one) natural way to generalize* $\ker(T)^\perp$*, but it is not a native subspace of* X*, though linear (not necessarily continuous) embeddings can be found. A natural way of finding such an embedding might appear to be projection via minimzation as in b) (in particular this would be continuous), however, this does not yield a (linear) subspace. Note that in Hilbert spaces this projection by minimization coincides with the orthonormal projection onto* $\ker(T)^\perp$ *and is in particular linear (thus* $\widehat{X}$ *can be associated with* $\ker(T)^\perp$*).*

Example 2.3) [Singular decomposition of integration operator]
We consider again the operator $T \in L^2([0,1]) \to L^2([0,1])$ with $[Tx](t) = \int_0^t x(s)\,\mathrm{d}s$ (see Example 1.2b). In fact, $\mathrm{Rg}(T) = \{f \in H^1([0,1]) \mid f(0) = 0 \text{ (via trace)}\}$. Show that T is compact (**hint** 2.4a), and compute the corresponding singular system.

Example 2.4) [Singular value decomposition of Hilbert-Schmidt operators]
Let Ω, Ω' be open in $\mathbb{R}^d$ and $k \in L^2(\Omega \times \Omega')$. Let $T\colon L^2(\Omega) \to L^2(\Omega')$ be the Hilbert-Schmidt operator defined according to

$$[Tx](t) = \int_\Omega k(s,t)x(s)\,\mathrm{d}s \qquad \text{for almost every } t \in \Omega'. \tag{2}$$

a) Show that the operator T is compact.

b) Conclude that the singular values of T satisfy $\sigma_n = \mathcal{O}(n^{-\frac{1}{2}})$.

Hint. *For complete orthonormal systems* U *and* V *of* $L^2(\Omega)$ *and* $L^2(\Omega')$ *respectively, the set*

$$J = \big\{u \otimes v \;\big|\; u \in U,\ v \in V\big\} \qquad \textit{with } [f \otimes g](s,t) = f(s)g(t) \textit{ a.e.} \tag{3}$$

is a complete orthonormal system in $L^2(\Omega \times \Omega')$*. Also note that* L^2 *spaces are separable, hence any orthonormal system is at most countable.*

Remark. *A large class of compact operators can be described as Hilbert-Schmidt operators, which allows for general considerations. It is said that* T *possesses degree of ill-posedness p if* $\sigma_n \approx n^{-p}$ *and as the name suggests this represents how bad an ill-posedness is. In particular Hilbert Schmidt operators always possess the degree of ill-posedness* $\frac{1}{2}$ *or worse.*