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Basic information: Distributions

Let $\Omega \subset \mathbb{R}^d$ be open and non-empty. For $K \subset\subset \Omega$ we denote with $\mathcal{D}_K(\Omega) := \{\varphi \in \mathcal{C}^\infty(\Omega) \mid \operatorname{supp}(\varphi) \subset K\}$ endowed with the locally convex topology τ_K generated by the seminorms $p_K^\alpha(\varphi) := \sup_{x\in K} |\partial^\alpha \varphi(x)|$ for multiindices $\alpha \in \mathbb{N}_0^d$. We denote with $\mathcal{D}(\Omega) = \mathcal{C}_c^\infty(\Omega)$ the space of tests functions endowed with the topology τ, created by all norms p, such that the restriction $p_{|\mathcal{D}_K}$ is continuous on $(\mathcal{D}_K, \tau_K)$ for each $K \subset\subset \Omega$. In particular, a sequence $(\varphi_n)_n$ converges to zero in τ if and only if $\bigcup_{n\in N} \operatorname{supp}(\varphi_n) \subset K$ for some compact $K \subset \Omega$ and $\lim_{n\to\infty} p_K^\alpha(\varphi_n) = 0$ for all $\alpha \in \mathbb{N}_0^d$.
The space of distributions is $\mathcal{D}'(\Omega) := \mathcal{D}(\Omega)'$ (the dual space with respect to τ). Note that linear $T : \mathcal{D}(\Omega) \to \mathbb{K}$ is a distribution if and only if $T_{|\mathcal{D}_K} : \mathcal{D}_K(\Omega) \to \mathbb{K}$ is τ_K continuous for any $K \subset\subset \Omega$ (which is equivalent to the existence of $c = c(K) > 0$ and finite $F = F(K) \subset \mathbb{N}_0^d$ such that $|T\varphi| \le c \max_{\alpha\in F} p_K^\alpha(\varphi)$ for all $\varphi \in D_K(\Omega)$). Due to the construction of τ, continuity is also equivalent to sequential continuity. A distribution T is called regular if there is a function $g \in L^1_{\mathrm{loc}}(\Omega)$ with $T\varphi = \int_\Omega \varphi g \,\mathrm{d}x$ for all $\varphi \in \mathcal{D}(\Omega)$.
A distribution $u \in \mathcal{D}'(\Omega)$ has compact support, if there is $K \subset\subset \Omega$ such that $u(\varphi) = 0$ for all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(\varphi) \cap K = \emptyset$ (the support $\operatorname{supp}(u)$ is the smallest closed set satisfying this property). Distributions with compact support have the special property

$$|u(\varphi)| \le c \max_{|\alpha|\le N} p_K^\alpha(\varphi) \qquad \text{for all } \varphi \in \mathcal{D}(\Omega) \tag{1}$$

for some constant $c > 0$ some $K \subset\subset \Omega$ and some $N \in \mathbb{N}$.

Example 6.1) [Distributions for different domains]
We consider for $\Omega :=]0,1[\subset \mathbb{R}$ the operation $T\colon \mathcal{C}_c^\infty(\Omega) \to \mathbb{C}$ according to $T\varphi = \sum_{k=2}^\infty \varphi^{(k)}(\frac{1}{k})$.

- **a)** Show that $T \in \mathcal{D}'(\Omega)$ (and in particular is well defined).
- **b)** Let $\widetilde{\Omega} \supset [0,1[$ be open. Show that there is no $\widetilde{T} \in \mathcal{D}'(\widetilde{\Omega})$ such that $\widetilde{T}_{|\mathcal{C}_c^\infty(\Omega)} = T$ (i.e., T cannot be continuously extended onto $\mathcal{D}(\widetilde{\Omega})$). Why does this not contradict the Hahn-Banach theorem? **Hint:** Be aware, $\widetilde{T}$ does not necessarily satisfy the sum representation for all $\mathcal{D}(\widetilde{\Omega})$. Consider functions $g_{n,k}(x) = \psi_{\frac{1}{n}}(x - \frac{1}{k}) = n\psi\big(n(x - \frac{1}{k})\big)$ a (rescaled and shifted) mollifier function to show that $\widetilde{T}$ cannot be continuous.

Remark. *This exemplifies that the choice of domain can have a significant impact on the resulting topologies of $\mathcal{D}$ and $\mathcal{D}'$. Hence one has to be careful when extending results from one domain to others.*

Example 6.2) [Cauchy principal value]
We consider the functional $T\colon \mathcal{C}_c^\infty(\mathbb{R}) \to \mathbb{C}$ according to $T\varphi = \lim_{\varepsilon\to 0} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{\varphi(x)}{x} \,\mathrm{d}x$ (the Cauchy principal value of $\frac{\varphi(x)}{x}$). Show that T is well-defined and $T \in \mathcal{D}'(\mathbb{R})$. Moreover, show that T is not regular.

Hint. *Use Taylor expansion of φ close to zero to control the term $\frac{1}{x}$.*

Remark. *Note that (as the nonregularity implies) the function* $f(x) = \frac{\varphi(x)}{x}$ *is not integrable in the sense of Lebesgue measure theory as both the positive and negative part of* f *possess mass infinity, nonetheless, due to a kind of symmetry the terms cancel out allowing the limit to exist. This principle can be applied to a larger class of functions with singularities.*

Example 6.3) [$\mathcal{E}'$ distributions]
For $\Omega \subset \mathbb{R}^d$ open, we consider the (locally convex) space $\mathcal{C}^\infty(\Omega)$ generated by seminorms p_K^α (as in the 'basic information') denoted by $\mathcal{E}(\Omega)$.

a) Show that given $\nu \in \mathcal{E}(\Omega)$, there is a sequence φ_n in $\mathcal{C}_c^\infty(\Omega)$ such that for any $K \subset\subset \Omega$ $\varphi_n = \nu$ on K for n sufficiently large and in particular $\varphi_n \overset{\mathcal{E}}{\to} \nu$. (Note that this also implies density of $\mathcal{D}$ in $\mathcal{E}$ in a topological sense).

b) For $M := \big\{u \in \mathcal{D}'(\Omega) \; : \; \mathrm{supp}(u) \subset\subset \Omega\big\}$, show that each $u \in M$ is continuous w.r.t. the $\mathcal{E}(\Omega)$ topology (and hence can be uniquely extended to a function in $\mathcal{E}'(\Omega)$), and conversely, each $v \in \mathcal{E}'(\Omega)$ restricted to $\mathcal{C}_c^\infty(\Omega)$ possesses compact support.

Remark. *Sometimes one wishes constructions similar to distributions which however need to be applicable also to functions without compact support. As can be seen, the construction is analogous and such* $\mathcal{E}'$ *distributions have a relation with compactly supported distributions. Roughly speaking, for a compactly supported distribution* u *it does not matter what the test function does outside the support of* u *and in particular whether it is vanishes.*

Example 6.4) [Tensor products and convolution]

a) Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be open. Given $u \in \mathcal{D}'(\Omega_1)$ and $v \in \mathcal{D}'(\Omega_2)$, show that

$$[u \otimes v](\varphi) := \langle v(y), \langle u(x), \varphi(x,y)\rangle_x\rangle_y \qquad \text{for all } \varphi \in \mathcal{D}(\Omega_1 \times \Omega_2) \tag{2}$$

(i.e., for fixed y one applies u to $\varphi(\cdot, y)$ and v to the resulting function w.r.t. y) is well defined and $u \otimes v \in \mathcal{D}'(\Omega_1 \times \Omega_2)$.

b) Given $u \in \mathcal{E}'(\mathbb{R}^d)$ and $v \in \mathcal{D}'(\mathbb{R}^d)$, we define the convolution

$$[u * v](\varphi) := [u \otimes v](\rho(\cdot_x)\varphi(\cdot_x + \cdot_y)) \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{3}$$

where $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ with $\rho = 1$ on a neighborhood of the support of u (which is compact according to 6.3 b). Show that $u * v$ defines a distribution on $\mathbb{R}^d$ which does not depend on the choice of ρ. Further, for u and v additionally being regular (with functions f and g), show that $u * v$ is regular and the corresponding L^1_{loc} function coincides with the (classical) convolution $f * g$.

Remark. *This construction of the convolution coincides with the considerations of the lecture but allows for a wider and more structured analysis.*