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Basic information: Weak topologies and distributions

Let X be a vector space and Y a subspace of $\mathcal{L}(X, \mathbb{K})$ (the space of linear functions from X onto $\mathbb{K}$). Then we denote with $\sigma(X, Y)$ the coarsest topology on X for which each $y \in Y$ is continuous, i.e., the topology is induced by $P = \{p_y\}_{y \in Y}$ with $p_y(x) := |\langle y, x\rangle| = |y(x)|$. This topology indeed exists, is locally convex, and is Hausdorff if and only if Y is seperating. Note that in normed spaces the special cases $\sigma(X, X^*)$ and $\sigma(X^*, X)$ coincide with the classical definitions of weak and weak* (weak-star) topologies, respectively. Conversely, given a locally convex topology τ on X, the dual space is $X' = X'_\tau := \{\xi \in \mathcal{L}(X, \mathbb{K}) \mid \xi \text{ continuous w.r.t. } \tau\}$. Indeed, $X'_{\sigma(X,Y)} = Y$, i.e., the dual space of the topology induced by Y is Y itself. For clarity, for a normed space X the classical dual space (i.e., linear functions which are continuous with respect to the norm) is referred to as X^*.
The theorem of Hahn-Banach remains true for locally convex spaces, i.e., for U subspace of X and $l \in U'_{\tau \cap U}$ (this is a slight abuse of notation for the relative topology, $\tau \cap U := \{U \cap O \mid O \in \tau\}$), there is $\tilde{l} \in X'_\tau$ with $\tilde{l}_{|U} = l$. This yields so-called separation theorems, one of which states: If X locally convex, $V \subset X$ closed, convex and non-empty, $x \notin V$, then there is $\xi \in X'$ and $\varepsilon > 0$ such that $\mathrm{Re}(\xi(x)) + \varepsilon \leq \mathrm{Re}(\xi(v))$ for all $v \in V$.

Example 5.1) [**Metrification of locally convex spaces**]
Let (X, τ) be a locally convex Hausdorff space.

a) Show that there is a countable family of seminorms $P = \{p_1, p_2, \dots\}$ creating τ if and only if τ is metrizable (i.e., there is a metric inducing the same topology).

b) Given Y subspace of $\mathcal{L}(X, \mathbb{K})$, conclude that the weak topology $\sigma(X, Y)$ is metrizable if and only if the space of Y possesses a countable generating system (i.e., $B \subset Y$ such that every $y \in Y$ can be represented by a (finite) linear combination of elements in B).

Hint. *For a), show that* $d(x, y) := \sum_{n \in \mathbb{N}} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}$ *is a metric. You may assume this metric to induce a vector space topology on* X*. Two topologies (of topological vector spaces) with respective neighborhood basis of the origin* $\mathcal{A}$ *and* $\mathcal{B}$ *are equivalent if and only if for each* $A \in \mathcal{A}$ *there is* $B \in \mathcal{B}$ *with* $B \subset A$ *and conversely. A neighborhood basis of the origin in a metric space is given by all open balls with rational radius and center zero, i.e.,* $\mathcal{U} = \big\{\{x \in X \mid d(0, x) < r\} \mid 0 < r \in \mathbb{Q}\big\}$*. For b), recall from the lecture that there is* $c > 0$ *such that* $|\langle y, x\rangle| \leq c \max_{i=1,\dots,N} |\langle e_i, x\rangle|$ *for all* $x \in X$ *if and only if* $y \in \mathrm{span}\{e_i\}_{i=1}^N$*. Also, continuity of* $f\colon X \to \mathbb{K}$ *in* x *in a metric space* X *can be rephrased as: for any* $\varepsilon \in \mathbb{Q}$ *there is* $\delta \in \mathbb{Q}$ *such that* $d(x, y) \leq \delta$ *implies* $|f(y) - f(x)| \leq \varepsilon$*.*

Remark. *Note that a topology being represented by a metric (or more generally possessing countable neighborhood bases) has the advantage that sequential definitions of topologic properties are equivalent, i.e., continuity, compactness, closedness can be defined by sequences. Note that any vector space possesses a (vector space) basis (generating system with unique representations) by the axiom of choice and the statement of b) could also be rephrased as 'countable vector space basis' instead of 'generating system'.*

Example 5.2) [Lemma of Farkas]
Let X be a real locally convex space and $\xi, \xi_1, \dots \xi_n \in X'$, such that if $x \in X$ with $\xi_i(x) \geq 0$ for all $i = 1, \dots, n$, then $\xi(x) \geq 0$. Show the existence of $a_i \geq 0$ such that $\xi = \sum_{i=1}^n a_i \xi_i$.

Hint. *Use the separation theorem on $\{\sum_{i=1}^n a_i \xi_i \mid a_i \geq 0 \text{ for } i = 1, \dots, n\}$. Also, recall the results from Exercise 4.2 that finite-dimensional subspaces of locally convex Hausdorff spaces are closed, the same statement holds for norm-closed subsets (in the sense of any norm on the finite-dimensional subspace) of finite-dimensional subspaces (with identical proof). Be careful which topology you use and whether it is Hausdorff.*

Example 5.3) [Sequential vs topological closedness]
We consider $X = l^2(\mathbb{N})$ with the (standard) weak topology $\sigma(l^2(\mathbb{N}), l^2(\mathbb{N}))$ (where $\langle f, g \rangle = \sum_{i \in \mathbb{N}} f_i g_i$). We consider the set $V = \{x_n\}_{n \in \mathbb{N}} \subset X$ according to $x_n = \sqrt{n} e_n$ for $e_n(k) = 1$ if $k = n$ and zero otherwise. Show that there is no sequence in V which converges to zero in the weak topology, but still zero is in the (weak) closure of V.

Hint. *By the theorem of Banach-Steinhaus, weakly convergent sequences (in the classical weak topology) are norm-bounded. Recall, topologically speaking x is in the closure of V if and only if there is no neighborhood of x which is disjoint from V.*

Remark. *This shows, as was eluded to before, that sequential closedness and topological closedness are not equivalent in locally convex spaces. Hence checking whether limits of sequences in a set is not sufficient for confirming closedness, while the reverse statement is true (in any topology). When considering nets (generalization of sequences as functions on $\mathcal{U}$ instead of $\mathbb{N}$ with a suitable notion of convergence) instead of sequences, the statement would be true. In particular, in view of 5.1), $l^2(\mathbb{N})$ does not possess a countable vector space basis.*

Example 5.4) [Operator topologies]
Let X and Y be normed spaces, and $L(X, Y)$ denotes the space of (norm-norm) continuous linear functions between the two. It is well known, that $L(X, Y)$ is again a normed space when equipped with the operator norm $\|T\| = \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X}$. Alternatively, one may consider the strong operator topology induced by the seminorms $P = \{p_x\}_{x \in X}$ according to $p_x(T) = \|Tx\|_Y$ and the weak operator topology induced by $P = \{p_{x,y^*}\}_{x \in X, y^* \in Y^*}$ according to $p_{x,y^*}(T) = |\langle y^*, Tx \rangle|$. We consider on $X = L^2(\mathbb{R})$ the translation operator $T_t \colon X \to X$ according to $[T_t f](\cdot) := f(t + \cdot)$ in a Lebesgue almost everywhere sense and a real sequence $t_n \to \infty$.

- **a)** Show that there is no $T \in L(X, X)$ such that T_{t_n} converges to T in the strong operator topology, and conclude that neither does a limit in the operator norm topology exist.
- **b)** Show that in the weak operator topology T_{t_n} converges towards zero.

Hint. *For $L^2(\mathbb{R})$ functions there exist compact sets, such that in an L^2 sense outside these sets the function possesses very little mass.*

Remark. *Note that convergence in the operator norm can be understood as locally uniform convergence, while in the strong operator topology convergence means pointwise (norm) convergence and in the weak operator topology pointwise weak convergence.*