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Basic information: Locally convex vector spaces

A vector space X with a topology τ is called topological vector space, if $\{x\}$ is closed (i.e., $\{x\}^c \in \tau$) for any $x \in X$ and for $x, y \in X$ and $\lambda \in \mathbb{K}$ the functions $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda x$ are continuous. Given an $x \in X$, we call a set $A \subset X$ a neighborhood of x if there is $O \in \tau$ with $x \in O \subset A$; the set of neighborhoods of x is denoted by $\mathcal{N}_x$.

The topology on a topological vector space is uniquely determined by the neighborhoods $\mathcal{N}_0$ of the origin ($O \in \tau$ if and only if $\lambda O + x_0 \in \tau$ for arbitrary $x_0, \lambda \neq 0$). We call a set $\mathcal{U} \subset \mathcal{N}_0$ a neighborhood basis of the origin, if given arbitrary $O \in \mathcal{N}_0$, there is $U \in \mathcal{U}$ such that $U \subset O$. Equivalently, $O \in \tau$ if and only if for any $x \in O$ there is $U \in \mathcal{U}$ with $x + U \subset O$.

A set A is called absolutely convex (= convex + balanced) if $x, y \in A$ and $\lambda, \mu \in \mathbb{K}$ with $|\lambda| + |\mu| \leq 1$ implies $\lambda x + \mu y \in A$. A set A is called absorbent if given $x \in X$, there is $r > 0$ such that $x \in rA := \{rx \mid x \in A\}$.

A topological vector space (X, τ) is called locally convex if there is an absolutely convex, absorbent neighborhood basis of the origin (in fact, this is equivalent to having a convex neighborhood basis). Equivalently, (X, τ) is locally convex if there is a family of seminorms $P = (p_i)_{i \in I}$, and τ is generated by the neighborhood basis $\mathcal{U} := \{U_{F,\varepsilon} \mid F \subset P$ finite and $\varepsilon > 0\}$, where we set

$$U_{F,\varepsilon} := \{x \in X \mid p(x) \leq \varepsilon \ \forall p \in F\}. \tag{1}$$

These two definitions are indeed equivalent, as given P, the corresponding $\mathcal{U}$ satisfies the requested properties, while given $\mathcal{U}$, one can define a family of seminorms $P = \{p_U\}_{U \in \mathcal{U}}$ according to $p_U := \inf\{r \in [0, \infty) \mid x \in rU\}$.

It can be shown, that a locally convex topology is Hausdorff if and only if the associated family of seminorms P is separating (i.e., for each $x \in X$ there is a $p \in P$ with $p(x) > 0$).

Example 4.1) [Weak topology]

Let X be a Banach space and X^* the corresponding dual space. The weak topology $\sigma(X, X^*)$ is generated by the family of seminorms $P = \{p_\xi\}_{\xi \in X^*}$ with $p_\xi(x) := |\xi(x)|$. Show that

- **a)** for $\xi \in X^*$, p_ξ is indeed a seminorm and that the corresponding $U_{F,\varepsilon}$ are absorbent and absolutely convex,
- **b)** a sequence $(x_n)_n$ converges to x in $\sigma(X, X^*)$ if and only if $\xi(x_n) \to \xi(x)$ for all $\xi \in X^*$.

Hint. *Recall that a sequence x_n converges to x in τ, if for any $N \in \mathcal{N}_x$ there is $n_0 \in \mathbb{N}$ such that $x_n \in N$ for $n > n_0$.*

Remark. *You might have learned about weak convergence without a deeper insight into its topology, but in fact, the weak convergence is associated with a locally convex topology.*

Example 4.2) [Finite-dimensional locally convex spaces]

- **a)** Show that a locally convex Hausdorff topology τ on $\mathbb{R}^d$ coincides with any norm topology (it is well-known that on finite-dimensional spaces, all norm topologies coincide).
- **b)** Conclude that any finite-dimensional subspace M of a locally convex Hausdorff space (X, τ) is closed. (Be aware that in general, closedness is not equivalent to containing limits of convergent sequences, but rather being the complement of an open set).

Hint. *For a), iteratively consider* $p^n = \sum_{i=1}^{n} p_i$ *for suitable* $p_i \in P$ *to construct a norm out of* P*. For b), consider finite-dimensional subspaces* $\mathbb{K}x + M$ *for given* $x \in M^c$*.*

Remark. *Since locally convex topologies are a generalization of normed spaces, some properties can be extended to locally convex spaces. However, there are also some classical properties in normed spaces which do not hold on locally convex spaces (such as topological properties – e.g., closedness, continuity, etc – being characterized by sequential properties).*

Example 4.3) [Continuous operators between locally convex spaces]
Let X and Y be vector spaces and $T\colon X \to Y$ linear.

a) Let τ_X and τ_Y be locally convex topologies on X and Y, respectively, with associated families of seminorms P and Q. Show that the following are equivalent:

- The operator T is continuous with respect to τ_X and τ_Y,
- For any $q \in Q$, there is $c > 0$ and $F \subset P$ finite such that $q(Tx) \leq c \max_{p \in F} p(x)$ for all $x \in X$.

b) Assume X and Y are Banach spaces. Show that T is norm-norm continuous if and only if T is weak-weak continuous.

Hint. *Be aware, that a function is continuous if the preimages of open sets are open (and not characterized by sequences). For b) use the Banach Steinhaus theorem (uniform boundedness principle).*

Remark. *Similar to linear functions between Banach spaces, linear functions between locally convex spaces are of interest. The lecture will go into more detail concerning linear continuous functions onto* $\mathbb{K}$*, the resulting weak topologies, and their properties.*

Example 4.4) [Equivalence of locally convex toplogies]
Let X be a vector space with either of the two topologies τ_1 or τ_2 being locally convex. The topology τ_1 is said to be finer than τ_2 if $\tau_2 \subset \tau_1$ (i.e., any τ_2 open set is also τ_1 open). Equivalently, τ_1 is finer than τ_2 if the identity $id : (X, \tau_1) \to (X, \tau_2)$ is continuous. Trivially $\tau_1 \subset \tau_2$ and $\tau_2 \subset \tau_1$ implies $\tau_1 = \tau_2$. We say families of seminorms or neighborhood bases are equivalent if they generate the same topology. Let $X = \mathcal{C}^\infty(K)$ for $K \subset \mathbb{R}$ compact. Show that the following three families of seminorms are equivalent:

- $P = \{p_0, p_1, \dots\}$ with $p_n(f) := \sup_{x \in K} |f^{(n)}(x)|$, where $f^{(n)}$ denotes the nth derivative,
- $Q = \{q_0, q_1, \dots\}$ with $q_n(f) := \int_K |f^{(n)}(x)| \,\mathrm{d}x$,
- $R = \{r_0, r_1, \dots\}$ with $r_n(f) := \sqrt{\int_K |f^{(n)}(x)|^2 \,\mathrm{d}x}$.

Remark. *Note that for a locally convex topology, neither the neighborhood basis nor the corresponding family of seminorms are unique. Naturally, some properties (such as convergence) can be transferred from a finer topology onto a coarser one.*