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Basic information: Lipschitz boundary and trace

A bounded domain (i.e., non-empty, open, connected and bounded) $\Omega \subset \mathbb{R}^d$ is said to possess Lipschitz boundary (or be a bounded Lipschitz domain), if there is $N \in \mathbb{N}$, open sets $(U_i)_{i\in\{1,\dots,N\}}$ such that $\partial\Omega \subset \bigcup_{i=1}^N U_i$, associated orthonormal systems $(e_i^1,\dots,e_i^d) = E_i \in \mathbb{R}^{d\times d}$ and Lipschitz continuous functions $g_i : \mathbb{R}^{d-1} \to \mathbb{R}$ such that for $x \in U_i$ with $(x_i^1,\dots x_i^d) := E_i^* x$ (the parameter representation):

$$\begin{cases} x \in \partial\Omega & \text{if and only if} \quad x_i^d = g_i(\overline{x}_i),\\ x \in \Omega & \text{if and only if} \quad x_i^d < g_i(\overline{x}_i),\\ x \notin \Omega & \text{if and only if} \quad x_i^d > g_i(\overline{x}_i), \end{cases} \tag{1}$$

where $\overline{x}_i = (x_i^1,\dots,x_i^{d-1})$. Without loss of generality, it can always be assumed that the U_i are hypercubes, i.e., $U_i = \{x \in \mathbb{R}^d \mid x_i^j \in]\alpha_i^j,\beta_i^j[\text{ for } j \in \{1,\dots,d\}\}$ for some constants $\alpha_i^j, \beta_i^j \in \mathbb{R}$ and restrict g_i to $U_i' = \{z \in \mathbb{R}^{d-1} \mid z^j \in]\alpha_i^j,\beta_i^j[\text{ for } j \in \{1,\dots,d-1\}\}$. Recall, that (by the Rademacher theorem) a Lipschitz continuous function is almost everywhere (totally) differentiable.

Setting $\tilde{g}_i(x) = x_i^d - g(\overline{x}_i)$ and $\tilde{\nu}_i := E_i \nabla \tilde{g}_i = E_i \left(-\frac{\partial g_i}{\partial x^1},\dots,-\frac{\partial g_i}{\partial x^{d-1}},1\right)^T$, the outer normal is (almost everywhere) given via $\nu = \tilde{\nu}_i/\|\tilde{\nu}_i\|$. In particular, ν does depend only on x, and not on U_i, g_i and E_i (altough this construction does).

Given $(x^1,\dots,x^{d-1}) \in U_i'$ we set $\underline{x}_i = \sum x^j e_i^j + g_i(x^1,\dots,x^{d-1}) e_i^d = E_i(x^1,\dots,x^{d-1},g_i)^T \in \partial\Omega \cap U_i$ (the associated boundary point). For a function $f\colon U_i \cap \partial\Omega \to \mathbb{R}$ such that $(x^1,\dots,x^{d-1}) \mapsto f(\underline{x}_i)$ is measurable on U_i', one can define the integral

$$\int_{\partial\Omega\cap U_i} f \,\mathrm{d}s := \int_{U_i'} f(\underline{x}_i)\|\tilde{\nu}_i(\underline{x}_i)\| \,\mathrm{d}(x^1,\dots,x^{d-1}). \tag{2}$$

Note that for a smooth boundary, this formula coincides with the classical formula for surface integrals. For measurable functions $f\colon \partial\Omega \to \mathbb{R}$ (such that each $f_{|\partial\Omega_i}$ is measurable as above), the integral $\int_{\partial\Omega} f \,\mathrm{d}s := \sum_{i=1}^N \int_{\partial\Omega\cap U_i} \xi_i f \,\mathrm{d}s$, where $(\xi_i)_{i=1}^N$ is a suitable partition of unity of $\partial\Omega$ with respect to U_i. So we can now define spaces $L^p(\partial\Omega)$ in the classical way with respect to aforementioned integral formulation (with measure $S(A) = \int_{\partial\Omega} \chi_A \,\mathrm{d}s$). Note that alternatively, one could construct these spaces via Hausdorff measures, whose construction is however quite technical.

On a Lipschitz domain Ω, the subspace $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ for $p < \infty$, and there is a (unique) linear continuous operator (called the trace operator) $\gamma\colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that $\gamma u = u_{|\partial\Omega}$ almost everywhere when $u \in \mathcal{C}^\infty(\overline{\Omega})$.

Example 3.1) [No trace for L^p]

Let Ω be a domain with Lipschitz boundary, and $p \in [1,\infty[$. Show that there is no linear continuous operator $T\colon L^p(\Omega) \to L^p(\partial\Omega)$ such that $Tu = u_{|\partial\Omega}$ for $\mathcal{C}^\infty(\overline{\Omega})$.

Hint. *Show that the boundary measure S is finite and not trivial (not the zero measure), concluding that $L^p(\partial\Omega)$ is not trivial ($L^p(\partial\Omega) \neq \{0\}$).*

Remark. *This result illustrates that it is not possible to define a trace operator from $L^p(\Omega)$, meaning there is no reasonable way to define boundary values for all L^p functions.*

Example 3.2) [Integration by parts]
Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $p, p^* \in]1, \infty[$ with $\frac{1}{p} + \frac{1}{p^*} = 1$.

a) Show for $f \in W^{1,p}(\Omega)$, $g \in W^{1,p^*}(\Omega)$ and $i \in \{1, \ldots, d\}$ that

$$\int_\Omega f \partial^{e_i} g \,\mathrm{d}x = \int_{\partial\Omega} \gamma(g) \nu_i \gamma(f) \,\mathrm{d}s - \int_\Omega g \partial^{e_i} f \,\mathrm{d}x, \tag{3}$$

where ν_i is the i th component of ν.

b) Conclude for $f \in W^{2,p}(\Omega)$ and $g \in W^{1,p^*}(\Omega)$, that

$$\int_\Omega g\Delta f + \nabla g \cdot \nabla f \,\mathrm{d}x = \int_{\partial\Omega} g\gamma(\nabla f) \cdot \nu \,\mathrm{d}s. \tag{4}$$

Here $\gamma(\nabla f)$ is understood to be applied componentwise.

Hint. *You may assume that* (3) *holds if* $f, g \in \mathcal{C}^\infty(\overline{\Omega})$.

Remark. *These results are can be understood as generalizations of integration by parts,* (4) *is known as the Green identity. They find much application in the study of second order linear PDEs.*

Example 3.3) [Decomposition of Lipschitz domains]
Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\{\Omega_i\}_{i=1}^n$ a tuple of bounded Lipschitz domains such that $\overline{\Omega} = \bigcup_{i=1}^n \overline{\Omega}_i$, with $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$.

a) Show that for S almost every $x \in \bigcup_{i=1}^n \partial\Omega_i$, if $x \in \partial\Omega_{i_1} \cap \partial\Omega_{i_2}$ for $i_1 \neq i_2$, then $\nu_{\Omega_{i_1}}(x) = -\nu_{\Omega_{i_2}}(x)$ (the respective outer normal vectors). Conclude that S almost everywhere $x \notin \bigcup_{i \in \{1,\ldots n\} \setminus \{i_1, i_2\}} \partial\Omega_i$, i.e., almost every point is in at most two boundaries.

b) Conclude that a function $f \in \mathcal{C}(\overline{\Omega})$ with $f_{|\Omega_i} \in \mathcal{C}^1(\overline{\Omega_i})$, is weakly differentiable and compute the derivative.

Hint. *A sketch is certainly helpful, but also prove the result rigorously.*

Remark. *Such results are for example of relevance for numerical schemes regarding PDEs which consider smooth ansatz functions on suitable subsets (triangulation) and need to understand the corresponding Sobolev properties.*

Example 3.4) [Functions with zero derivative are constant]
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain (open and connected) with Lipschitz boundary. It is well known, that ‘the function is constant’ and ‘$\nabla f = 0$ on Ω’ are equivalent when f is smooth. Show that this still holds when $f \in L^1(\Omega)$ and ∇ is the weak gradient.