Richard Huber
richard.huber@uni-graz.at

Basic information: Sobolev Spaces

Let $\Omega \subset \mathbb{R}^d$ be open for some $d \in \mathbb{N}_0$. We denote by $\mathcal{C}_c^\infty(\Omega) = \{\phi \in \mathcal{C}^\infty(\Omega) \mid \operatorname{supp}(\phi) \subset\subset \Omega\}$ the space of test functions, where $\subset\subset$ stands for compactly contained. For a function $f \in L^1_{\mathrm{loc}}(\Omega)$, a function $\partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$ is called the weak derivative of f with respect to $\alpha \in \mathbb{N}_0^d$ if

$$\int_\Omega f(x) \frac{\partial \phi}{\partial x^\alpha}(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega \phi(x) \partial^\alpha f(x)\,\mathrm{d}x \qquad \text{for all } \phi \in \mathcal{C}_c^\infty(\Omega), \tag{1}$$

where $|\alpha| = \sum_{i=1}^d \alpha_i$ and for $x = (x_1, \dots x_d)$ the notation $\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial x_1^{\alpha 1}} \cdots \frac{\partial}{\partial x_d^{\alpha d}}$ denotes α_i times derivation with respect to x_i. In particular, the ith unit vector is denoted by e_i and $\partial^{e_i} u = \frac{\partial u}{\partial x_i}$. Note that weak derivatives are unique (in the sense of $L^1_{\mathrm{loc}}(\Omega)$) and $\partial^{e_i} u = 0$ holds for each $i \in \{1, \dots, d\}$ if and only if u is constant on every connected component of Ω.
For $p \in [1, \infty]$ and $m \in \mathbb{N}_0$ the Sobolev space $W^{m,p}(\Omega)$ is defined as the space of functions

$$W^{m,p}(\Omega) = \left\{ f \in L^p(\Omega) \;\middle|\; \partial^\alpha f \in L^p(\Omega)\ \forall \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le m \right\}, \tag{2}$$

in particular, said $\partial^\alpha f$ are required to exist. We endow these spaces with the norms

$$\|f\|_{W^{m,p}} = \left(\sum_{|\alpha| \le m} \|\partial^\alpha f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \quad \text{for } p \in [1, \infty[, \qquad \text{and} \qquad \|f\|_{W^{m,\infty}} = \max_{|\alpha| \le m} \|\partial^\alpha f\|_{L^\infty(\Omega)}.$$

Additionally, we define the space $W_0^{m,p}(\Omega)$ as the closure of $\mathcal{C}_c^\infty(\Omega) \cap W^{m,p}(\Omega)$ with respect to $\|\cdot\|_{W^{m,p}}$.

Example 1.1) [Partition of unity]
Let $N \in \mathbb{N}$ and for $i \in \{1, \dots, N\}$ let $U_i \subset \mathbb{R}^d$ open and $\Omega \subset \bigcup_{i=1}^N U_i$ compact. Show that there exist functions $\xi_i \in \mathcal{C}^\infty(\mathbb{R}^d)$ for $i \in \{1, \dots, N\}$ such that

$$\begin{cases} 0 \le \xi_i(x) \le 1 & \forall x \in \Omega \quad \forall i \in \{1, \dots, N\}, \\ \operatorname{supp}(\xi_i) \subset U_i, & \quad \forall i \in \{1, \dots, N\}, \\ \sum_{i=1}^N \xi_i(x) = 1 & \forall x \in \Omega. \end{cases} \tag{3}$$

Remark. *A set of functions with property* (3) *is called a partition of unity and is useful as it allows to first consider functions locally in small domains before extending to the entirety of* Ω.

Example 1.2) [Fundamental theorem of calculus and Hölder continuity]
Let $\Omega =]0, 1[\subset \mathbb{R}$, $p \in [1, \infty]$ and $u \in W^{1,p}(\Omega)$.

a) Show that

$$u(y) - u(x) = \int_x^y u'(s)\,\mathrm{d}s \qquad \text{for almost every } (x, y) \subset \Omega \times \Omega, \tag{4}$$

where $u' = \partial^{e_1} u$ is the weak derivative of u and conclude that

$$|u(y) - u(x)| \le \|u'\|_{L^p(\Omega)} |x - y|^{1 - \frac{1}{p}} \qquad \text{for almost every } (x, y) \subset \Omega \times \Omega, \tag{5}$$

where we understand $\frac{1}{\infty} = 0$.

b) Conclude that there exists a continuous function $\widetilde{u}$ with $\widetilde{u} = u$ almost everywhere on Ω.

Remark. *This means that in 1d, the fundamental theorem of integration holds for Sobolev spaces, and in particular, this implies that Sobolev functions are Hölder continuous for $p > 1$. For $p = \infty$, we even get Lipschitz continuity almost everywhere, and thus, almost everywhere differentiability (Rademacher theorem). Note that this proof only works in 1d and cannot be generalized to higher dimensions, in particular, for higher dimensions, it is not true that Sobolev functions are necessarily continuous, see the embedding theorems in the lecture.*

Example 1.3) [Classical identities for Sobolev spaces]
Let $\Omega \subset \mathbb{R}^d$ open.

a) Show for $f \in W^{m,p}(\Omega)$, $g \in W_0^{m,p^*}(\Omega)$ (with $p \in [1,\infty]$, $\frac{1}{p} + \frac{1}{p^*} = 1$) and $|\alpha| \leq m$ that

$$\int_\Omega g \partial^\alpha f \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha g \,\mathrm{d}x. \tag{6}$$

b) Show that there is a constant $c > 0$, such that for $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$,

$$\|\nabla u\|_{L^2(\Omega)}^2 \leq c \|\Delta u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \tag{7}$$

holds, where $\nabla u = (\partial^{e_1} u, \ldots, \partial^{e_d} u)$ denotes the gradient and $\Delta u = \sum_{i=1}^d \partial^{2e_i} u$ the Laplace operator, both in a weak sense.

Remark. *Many identities concerning integrals of derivatives for smooth functions can be extended to Sobolev functions, in particular,* (1) *also holds for suitable Sobolev functions.*

Example 1.4) [$W^{m,p}$ as continuous functionals]
Let $\Omega \subset \mathbb{R}^d$ open, $u \in L^1_{\text{loc}}(\Omega)$ and $p \in]1,\infty[$, $p^* \in]1,\infty[$ such that $\frac{1}{p} + \frac{1}{p^*} = 1$.

a) Show that the following are equivalent:

- $u \in W^{m,p}(\Omega)$,
- There is a constant $c > 0$ such that $\int_\Omega u \partial^\alpha \varphi \,\mathrm{d}x \leq c \|\varphi\|_{L^{p^*}}$ for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and $|\alpha| \leq m$.

b) Does the analogue hold for $W^{1,1}(\Omega)$?

Example 1.5) [Density of $C^\infty(\overline{\Omega})$]
We consider the open set $\Omega = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; |x| \in]0,1[,\ y \in]0,1[\right\}$ and let $p \in [1,\infty]$. Show that $\mathcal{C}^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is not dense in $W^{1,p}(\Omega)$.

Remark. *As will be shown in the lecture, $\mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$ is always dense, but the same cannot be guaranteed for $\mathcal{C}^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ unless Ω is sufficiently nice (extension domain).*