University of Graz
Institute for Mathematics and Scientific Computing

Dr. Martin Holler
martin.holler@uni-graz.at

Inverse Problems - Exercise Sheet 7

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Exercise 1 - Defining semi-norms via operators

Let X, Z be Banach spaces with X being reflexive and Z being the dual of a separable space. Further, let $A : \mathrm{dom}(A) \subset X \to Z$ be a densely defined, linear operator (not necessarily bounded!) with finite dimensional kernel. Further, assume that A is weak-to-weak* closed, which means that for each sequence $(x_n)_n$, $x \in X$, $z \in Z$ such that $x_n \rightharpoonup x$, $Ax_n \overset{*}{\rightharpoonup} z$ it follows that $x \in \mathrm{dom}(A)$ and $Ax = z$. Define $R : X \to [0, \infty]$ as

$$R(u) = \begin{cases} \|Au\|_Z & \text{if } u \in \mathrm{dom}(A), \\ \infty & \text{else.} \end{cases}$$

Show that R is convex and lower semi-continuous. Further, with $P : X \to \ker(A)$ a linear and continuous projection, show that $\mathrm{rg}(A)$ is closed if and only if there exists $C > 0$ such that

$$\|u - Pu\|_X \leq CR(u)$$

for all $u \in X$. In particular, if $\mathrm{rg}(A)$ is closed, R fulfills the assumptions ($\mathrm{Ass_R}$) of the lecture.
Hint: You can use the open mapping theorem for bounded operators, but if you want to use any extension of it you should provide a proof.

Exercise 2 - Huber-Total-Variation Regularization

Consider $X = \mathbb{R}^{n\times n}$ and $T : X \to X$ the convolution operator of Exercise 1.4. Define a discrete gradient operator $\nabla : X \to \mathbb{R}^{n\times n\times 2}$ via

$$(\nabla u)_{i,j,1} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{if } i < n, \\ 0 & \text{else,} \end{cases} \qquad (\nabla u)_{i,j,2} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{if } j < n, \\ 0 & \text{else.} \end{cases}$$

Further, for $\epsilon > 0$, define,

$$h_\epsilon(t) = \begin{cases} \frac{t^2}{2\epsilon} & \text{if } |t| \leq \epsilon \\ |t| - \frac{\epsilon}{2} & \text{else,} \end{cases}$$

and $H_\epsilon(p) = \sum_{i,j=1}^{n,n} h_\epsilon\left(\sqrt{p_{i,j,1}^2 + p_{i,j,2}^2}\right)$. For given $y \in X$, consider the minimiziation problem

$$\min_{u\in X} \frac{1}{2}\|Tu - y\|_2^2 + \alpha H_\epsilon(\nabla u), \tag{1}$$

where $\|y\|_2^2 = \sum_{i,j=1}^{n,n} y_{i,j}^2$.

- Noting that the energy in (1) is C^1, compute its gradient.
- Using the test script *huber_tv.m*, implement the gradient descent method to solve (1) numerically.
- Using the values of the testscript as starting point, try different values of ϵ and α. What do you observe?