University of Graz
Institute for Mathematics and Scientific Computing

Dr. Martin Holler
martin.holler@uni-graz.at

Inverse Problems - Exercise Sheet 6

Publication date: December 19, 2019

Due date: January 7, 2020

Exercise 1 - A range characterization

Let X, Y be Hilbert spaces, $T \in \mathcal{L}(X,Y)$. Show that $\mathrm{rg}(T^*) = \mathrm{rg}((T^*T)^{1/2})$.

Hint: You can use that for a spectral family $(E_\lambda)_\lambda$ and $C > 0$, $\lim_{\rho \to 0} \int_\rho^C f(\lambda) dE_\lambda x$ exists if and only if $\lim_{\rho \to 0} \int_\rho^C f(\lambda)^2 d\|E_\lambda x\|^2 < \infty$.

Exercise 2 - Variational source condition

Let X, Y be Hilbert spaces and $F : \mathcal{D}(F) \subset X \to Y$ with $\mathcal{D}(F) \neq \emptyset$ be continuous and weakly sequentially closed. For $x^* \in X$, $y^\delta \in Y$ we let x_α^δ be a solution of

$$\min_{x \in \mathcal{D}(F)} \|F(x) - y^\delta\|_Y^2 + \alpha \|x - x^*\|_X^2. \tag{1}$$

Assume that $y^\dagger \in Y$, $x^\dagger \in X$ are such that $\|y^\delta - y^\dagger\|_Y \leq \delta$ and $x^\dagger$ is a x^*-minimum norm solution of $F(x) = y^\dagger$. Assume further that $x^\dagger$ satisfies a variational source condition of the form

$$(x^\dagger - x^*, x^\dagger - x)_X \leq \frac{\beta_1}{2} \|x - x^\dagger\|_X^2 + \beta_2 \|F(x) - F(x^\dagger)\|_Y$$

with $\beta_1 \in [0,1)$,$\beta_2 \geq 0$ and for all $x \in \mathcal{D}(F)$.

Show that in this case, for the parameter choice strategy $\alpha = \alpha(\delta) \sim \delta$ it holds that

$$\|x_{\alpha(\delta)}^\delta - x^\dagger\|_X = O(\sqrt{\delta}), \quad \|F(x_{\alpha(\delta)}^\delta) - y^\delta\|_Y = O(\delta)$$

as $\delta \to 0$.

Exercise 3 - Interpretation of the variational source condition

With the setting of Exercise 1 above, show that each of the following assertions implies the variational source condition of Exercise 1:

- $\mathcal{D}(F) = X$, F is linear and $x^\dagger - x^* \in X_{\frac{1}{2}}$.
- F is Frechét differentiable, there exists $w \in Y$ with $x^\dagger - x^* = DF(x^\dagger)^* w$ and DF is Lipschitz continuous with constant γ such that $\gamma \|w\|_Y < 1$.

Defining $E_\alpha(x,y) = \|F(x) - y\|_Y + \alpha \|x - x^*\|_X^2$, show that the variational source condition is equivalent to the existence of constants $\alpha > 0$, $\nu \in (0,1]$ such that

$$E_\alpha(x^\dagger, y^\dagger) + \alpha\nu \|x - x^\dagger\|^2 \leq E_\alpha(x, y^\dagger)$$

for all x. Interpret this condition and its consequences.