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Inverse Problems - Exercise Sheet 5

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In following exercises, you might need the Chebyshev polynomials of second kind, which are given as

$$U_n(t) := \frac{\sin((n+1)\arccos(t))}{\sin(\arccos(t))} \text{ for } x \in [-1,1], n \in \mathbb{N}_0$$

They satisfy the following properties:

- The functions U_n are well-defined polynomials of order n satisfying the recursion

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t) \quad n \geq 2,$$

  with $U_0(t) = 1$ and $U_1(t) = 2t$
- The U_n satisfy the following bounds on $[-1,1]$

$$|U_n(t)| \leq U_n(1) = n+1, \quad |U_n'(t)| \leq n^2(n+1)$$

Exercise 1 - Accelerated Landweber

Let $T \in \mathcal{L}(X,Y)$ with X, Y Hilbert spaces such that $\|T\| < 1$. For a given $y^\delta, y^\dagger \in Y$ such that $\|y^\delta - y^\dagger\| \leq \delta$, define the recursive sequence $(x_k)_k$ via

$$\begin{cases} x_0 = 0,\ x_1 = 2T^*y^\delta \\ x_{k+1} = \frac{2k+2}{k+2}x_k - \frac{k}{k+2}x_{k-1} + \frac{4k+4}{k+2}T^*(y^\delta - Tx_k). \end{cases}$$

For $\alpha \in (0,1]$, define $R_\alpha y = x_k$ where $\alpha \in ((k+1)^{-1}, k^{-1}]$. Proof that

- $R_\alpha y \to T^\dagger y$ if $y \in \mathcal{D}(T^\dagger)$ and $\|R_\alpha y\|_X \to \infty$ else, as $\alpha \to 0$.
- $(R_\alpha)_\alpha$ is a regularization
- Any parameter choice $\alpha(\delta) = k(\delta)^{-1}$ with $k(\delta) \to \infty$, $\delta k(\delta) \to 0$ as $\delta \to 0$ yields a convergent regularization method (R_α, α).

Exercise 2 - Accelerated Landweber – Rates

With the assumptions of Exercise 1, assume further that $y^\dagger \in \mathrm{rg}(T)$.

- Show that $\|x_k - x^\dagger\| = O(k^{-2\mu})$ for $\mu \in (0, 1/2]$, $x^\dagger \in X_{\mu,1}$ and $(x_k)_k$ obtained with the accelerated Landweber algorithm with initialization $y^\dagger = Tx^\dagger$.
- Proof that the discrepancy principle provides a well-defined stopping rule $k = k(\delta, y^\delta)$ for the iterates of Exercise 1, and that $k(\delta, y^\delta) = O(\delta^{-1/(2\mu)})$ on X_μ for $\mu \in (0, 1/2]$.

Compare this to corresponding results on the Landweber iteration of the Lecture. Depending on μ for $x^\dagger \in X_{\mu,1}$, when do you recommend to use Landweber and when accelerated Landweber for 1) noise-free data and 2) in combination with the discrepancy principle for noisy data?

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Exercise 3 - Implementation of the Landweber iteration

Implement the Landweber iteration and accelerated Landweber iteration as in Exercise 1 above, using the discrepancy principle as stopping criterion. As example, consider the inverse problem of Exercise 1.4. Plot the functions $k \mapsto \|Tx_k - y^\delta\|$ $k \mapsto \|x_k - x^\dagger\|$ in a log-log plot and display the result of both algorithms as well as of taking the generalized inverse. Also, use linear regression as in Exercise 4.3 to estimate the exponent in the convergence rate. Consider all combinations of the following situations:

- $x^\dagger \in X_{\mu,1}$ with $\mu \in \{0, 0.5, 1, 2\}$.
- $x^\dagger = (T^*T)^\mu w$ with μ as above and w a normalized version of either $rand(n)$ or $phantom(n)$, with n the image dimension, e.g., $n = 128$.
- Low-noise and no-noise, where for the later, e.g., 5000 iterations is used as stopping criterion.

What do you observe? How do your results compare to the rates of Exercise 2? Hint: Don't forget to ensure that $\|T\| < 1$ by re-normalization.