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Inverse Problems - Exercise Sheet 4

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Exercise 1 - Source condition for convolution

With $X = L^2([-\pi, \pi]^2)$, define $T : X \to X$ to be the convolution operator of Exercise 1.2. with convolution kernel $k \in X$.

- For $x^\dagger \in X$, $\mu > 0$, express the condition

$$x^\dagger \in X_\mu = \{x \in X \,:\, x = (T^*T)^\mu w \text{ for } w \in X\}$$

  in terms of the Fourier coefficient of $x^\dagger$

- In case there exist $c, C > 0$ and $p \geq 2$ such that $c|l|^p \leq |\hat{k}|_l^{-1} \leq C|l|^p$ for all $l \in \mathbb{Z}^2$, provide a $\mu > 0$ such that, for $x^\dagger \in X$ with compact support in $(-\pi, \pi)^2$,

$$x^\dagger \in X_\mu \quad \Leftrightarrow \quad x^\dagger \in H^1((-\pi, \pi)^2).$$

Exercise 2 - Convergence rates for convolution

Consider the convolution operator T of Exercise 1.2 on $H = L^2([-\pi, \pi]^2)$. Write down Tikhonov and truncated-SVD regularization for this operator in the form $R_\alpha y = g_\alpha(T^*T)T^*y$. For $T^\dagger y^\dagger = x^\dagger = (T^*T)^\mu w$ with $\|w\|_H \leq 1$, a suitable parameter choice and $\|y^\delta - y^\dagger\|_H \leq \delta$, provide an explicit proof (without relying on the general results of the lecture) of the optimal convergence rate for both Tikhonov and truncated-SVD regularization.

Exercise 3 - Testing convergence rates

Define the point-wise operator $T : \mathbb{R}^n \to \mathbb{R}^n$ as $Tx = h{*}x$ where $h{*}x$ denotes the point-wise multiplication and $h_i = 1/(linspace(1, 10, n)^\theta)$ with $\theta \geq 1$ and the exponent and division are taken point-wise. Use this operator to test the convergence rate of regularization by Tikhonov and truncated singular value decomposition.

More precisely, given $x^\dagger = T^\dagger y^\dagger \in X_{\mu,1}$ for $\mu \in \{1/2, 1, 1.5\}$ and a parameter choice strategy $\alpha(\delta) \sim (\delta)^{2/(2\mu+1)}$ as in the lecture, provide a convergence plot for $\|R_{\alpha(\delta)} y^\delta - y^\dagger\|_{\ell^2} =: e(\delta)$ as $\delta \to 0$.

Remarks:

- Plot $(\log(\delta), \log(e(\delta)))$ to visualize the rate of convergence.
- Use linear regression on $(\log(\delta), \log(e(\delta)))$ to estimate the rate of convergence.
- You should be able to observe what the theory predicts.