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Inverse Problems - Exercise Sheet 3

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Exercise 1 - Showalter regularization method

Let X, Y be Hilbert spaces and $T \in \mathcal{L}(X, Y)$. Our goal is to obtain a regularized solution of the inverse problem $Tx = y$. To this aim, given y^δ, we consider the initial problem of finding $u_\delta : [0, \infty) \to X$ Fréchet differentiable such that

$$\begin{aligned} u_\delta'(t) + T^*Tu_\delta(t) &= T^*y^\delta \quad \text{for all } t > 0 \\ u_\delta(0) &= 0. \end{aligned}$$

For $\lambda, t \geq 0$, define $\gamma(t, \lambda) = (1 - e^{-\lambda t})/\lambda$ and $v(t) = \int_0^{\|T\|^2} \gamma(t, \lambda) dE_\lambda T^* y^\delta$, where $(E_\lambda)_\lambda$ are the spectral measures associated to T^*T. Further, we define $R_\alpha y^\delta := v(1/\alpha)$.

- Show that v is Fréchet differentiable on $(0, \infty)$ and a solution to the initial value problem above.
- Show that $(R_\alpha)_\alpha$ is a regularization.
- Provide a parameter choice $\alpha : [0, \alpha_0) \to [0, \infty)$ such that $(R_\alpha, \alpha)_\alpha$ is an order-optimal regularization method.
- Show that $\lim_{t\to\infty} v(t)$ exists if $y^\delta \in \mathcal{D}(T^\dagger)$ and $\lim_{t\to\infty} \|v(t)\| = \infty$ otherwise.

Exercise 2 - General non-convergence of continuous regularization methods

Let $T \in \mathcal{L}(X, Y)$ with X, Y Hilbert spaces. Following the notation from the course, consider a regularization $(R_\alpha)_\alpha$ defined as $R_\alpha(y) = g_\alpha(T^*T)T^*y$, for $g_\alpha : [0, \|T\|^2] \to \mathbb{R}$ that satisfies for a fixed $C > 0$:

$$|\lambda g_\alpha(\lambda)| \leq C \quad \text{and} \quad \lim_{\alpha\to 0} g_\alpha(\lambda) = \frac{1}{\lambda} \quad \forall \lambda \in (0, \|T\|^2].$$

Define the value $G_\alpha := \sup\{|g_\alpha(\lambda)|, \lambda \in [0, \|T\|^2]\}$, and take $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ a continuous function such that $\alpha(0) = 0$. Prove the following for $y \in \mathcal{D}(K^\dagger)$, $y^\delta \in Y$ such that $\|y - y^\delta\| \leq \delta$.

a) If $\lim_{\delta\to 0} \delta^2 G_{\alpha(\delta)} = 0$, then $\left\|R_{\alpha(\delta)} y^\delta - T^\dagger y\right\| \to 0$ as $\delta \to 0$.

b) For a general T which is compact, provide a concrete choice of g_α and a continuous parameter choice rule $\alpha : (0, \alpha_0) \to (0, \infty)$ such that $\alpha(0) = 0$ and $\lim_{\delta\to 0} \delta^{2+\epsilon} G_{\alpha(\delta)} = 0$ for all $\epsilon > 0$, but the regularization method $(R_\alpha, \alpha)_\alpha$ is not convergent.