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Inverse Problems - Exercise Sheet 2

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Exercise 1 - Backwards heat equation

Consider the one-dimensional heat equation: Given $u_0 \in \mathcal{C}([0,\pi])$ (compatible with the boundary conditions), find $u \in \{v \in C([0,\pi] \times [0,1]) \,:\, \partial_x v,\, \partial^2_{xx} v,\, \partial_t v \in C((0,\pi) \times (0,T])\}$ such that

$$\begin{aligned} \partial_t u(x,t) &= \partial^2_{xx} u(x,t),\ x \in (0,\pi), t \in (0,T] \\ u(0,t) &= u(\pi,t) = 0,\ t \in (0,T] \\ u(x,0) &= u_0(x),\ x \in [0,\pi] \end{aligned}$$

By separation of variables we get that the unique solution to the above equation can be written as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \varphi_n(x)$$

with $\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$ and $c_n = \int_0^\pi u_0(\tau)\varphi_n(\tau) \mathrm{d}\tau$. We are interested in obtaining the initial temperature u_0 from the final temperature $f(x) = u(x,1)$.

- **a)** Show that this problem can be written as solving $Ku_0 = f$ with $K : L^2([0,\pi]) \to L^2([0,\pi])$ a compact, self-adjoint operator.
- **b)** Determine the eigenvalues and eigenvectors of K
- **c)** Show that the problem is severely-ill-posed: Determine when $f \in \mathcal{D}(K^\dagger)$ and show that already an error of about 10^{-8} in (f, φ_5) (the fifth Fourier coefficient of the data) can lead to an error larger than 10^3 in the solution obtained via $K^\dagger$.

For the next exercises, we remember that for $d \in \mathbb{N}$, space of Schwartz functions is defined as

$$\mathcal{S}(\mathbb{R}^d) = \left\{ u \in C^\infty(\mathbb{R}^d) \,|\, \forall \alpha, \beta \in \mathbb{N}^d : C_{\alpha,\beta}(u) := \sup_{x \in \mathbb{R}^d} |x^\alpha \frac{\partial^\beta}{\partial x^\beta} u(x)| < \infty \right\},$$

where α, β are multi-indices for the exponent and derivatives. Also, remember that the continuous Fourier transform can be defined for $\phi \in \mathcal{S}(\mathbb{R}^d)$ as

$$\mathcal{F}(\phi)(\xi) := \hat{\phi}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \phi(x) \mathrm{d}x,$$

and is bijective from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ and its inverse is given via

$$\mathcal{F}^{-1}(\psi)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi \cdot x} \psi(\xi) \mathrm{d}\xi.$$

Further, remember that for $\phi \in L^1(\mathbb{R}^d)$ both $\mathcal{F}(\phi)$ and $\mathcal{F}^{-1}(\phi)$ are well defined and contained in $C(\mathbb{R}^d)$.

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Exercise 2 - Fourier Slice Theorem

For $f \in \mathcal{S}(\mathbb{R}^2)$ and $\mathbf{w}(\theta) = (\cos(\theta), \sin(\theta))^t$, $\mathbf{w}^\perp(\theta) = (-\sin(\theta), \cos(\theta))^t$, show that the Radon transform

$$\mathcal{R}f(\theta, s) = \int_{-\infty}^{\infty} f\left(s\ \mathbf{w}(\theta) + t\ \mathbf{w}^\perp(\theta)\right) dt, \quad (\theta, s) \in [0, 2\pi] \times \mathbb{R},$$

is well defined and that $s \mapsto \mathcal{R}_\theta(s) := \mathcal{R}f(\theta, s) \in \mathcal{S}(\mathbb{R})$ for any $\theta \in [0, 2\pi]$. Also, prove that for all $(\theta, s) \in [0, 2\pi] \times \mathbb{R}$,

$$\hat{R_\theta}(s) = \hat{f}(s\mathbf{w}(\theta)).$$

Use this to prove that the Radon transform is injective on $\mathcal{S}(\mathbb{R}^2)$.

Exercise 3 - Inversion Formula

Show that, for $\alpha < 2$, the linear operator $I^\alpha : \mathcal{S}(\mathbb{R}^2) \to C(\mathbb{R}^2)$, given via $I^\alpha f = \mathcal{F}^{-1}(\xi \mapsto |\xi|^{-\alpha}\hat{f}(\xi))$ is well-defined and prove that for any $f \in \mathcal{S}(\mathbb{R}^2)$ and any $\alpha \in [0, 2)$,

$$I^\alpha f = \frac{1}{4\pi} R^\# I^{\alpha-1} R f,$$

where $I^{\alpha-1}$ applied to Rf acts only on the second variable and

$$\mathcal{R}^\# g(x) = \int_0^{2\pi} g\left(\theta, x \cdot \mathbf{w}(\theta)\right) d\theta$$

with $\mathbf{w}(\theta) = (\cos(\theta), \sin(\theta))^t$. In particular, for $\alpha = 0$ note that this implies

$$f = \frac{1}{4\pi} R^\# I^{-1} R f.$$

Exercise 4 - Programming Exercise

Using the Matlab functions *radon*, *iradon* and *phantom*, write a Matlab script that helps you to visualize and explain the meaning of the Radon transform, its adjoint and its inverse. In particular, your script should help you to visualize and explain the following:

- The effect of measuring only a subset of the angles $\theta \in [0, 2\pi]$ for the inverse Radon transform.
- The effect of adding additional noise on the data.
- The effect of using the default filter in *iradon* versus the effect of using no filter.

If you find it useful, you can of course also use in addition, e.g., the characteristic function of a square instead of *phantom* for your visualizations.