University of Graz
Institute for Mathematics and Scientific Computing

Dr. Martin Holler
martin.holler@uni-graz.at

Inverse Problems - Exercise Sheet 1

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Exercise 1 - Closed range

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a bounded, linear operator. With the equivalence relation $x_1 \sim x_2 :\Leftrightarrow x_1 - x_2 \in \ker(T)$ on X, define $\hat{X} = X/_\sim$ to be the quotient space w.r.t. $\sim$,

$$\begin{aligned}\|\cdot\|_{\hat{X}} :& \hat{X} \to \mathbb{R} \\ &[x]_\sim \mapsto \inf_{\hat{x}\in[x]_\sim} \|\hat{x}\|_X\end{aligned} \qquad \text{and} \qquad \begin{aligned}\hat{T} :& \hat{X} \to \mathrm{rg}(T) \\ &[x]_\sim \mapsto Tx,\end{aligned}$$

where $\mathrm{rg}(T)$ is the range of $\hat{T}$.

- Show that $(\hat{X}, \|\cdot\|_{\hat{X}})$ is a Banach space and $\hat{T}$ is well defined, bounded and linear.
- Noting that $\hat{T}$ is bijective, use the open mapping theorem to argue that if $\mathrm{rg}(T)$ is closed, the inverse of $\hat{T}$ is continuous.
- The other way around, prove that $\mathrm{rg}(T)$ is closed if $\hat{T}^{-1}$ is continuous.

Hint: You can use that a normed space is complete if and only if every absolutely convergent series is convergent.

For the next exercise, we remember that any function $u \in L^2([-\pi,\pi]^2)$ admits a representation in terms of its Fourier series

$$u = \sum_{l=(l_1,l_2)\in\mathbb{Z}^2} (u, e_l)_{L^2([-\pi,\pi]^2)} e_l, \quad \text{where } (u,v)_{L^2([-\pi,\pi]^2)} := \int_{[-\pi,\pi]^2} u(x)\overline{v}(x)\mathrm{d}x,$$

$e_l(x_1,x_2) := e^{i(l_1x_1+l_2x_2)}$ and $(e_l)_{l\in\mathbb{Z}^2}$ is an orthonormal Basis of $L^2([-\pi,\pi]^2)$ such that $\hat{u} := ((u,e_l)_{L^2([-\pi,\pi]^2)})_l \in \ell^2(\mathbb{Z}^2)$. Also we remember that any operator is compact if it is the limit of finite-range operators in operator norm. Further, remember that a space is finite dimensional if and only if its closed unit ball is compact.

Exercise 2 - Ill-posedness of inverting the convolution

For $k \in L^2([-\pi,\pi]^2)$, define the convolution operator

$$\begin{aligned}T :& L^2([-\pi,\pi]^2) \to L^2([-\pi,\pi]^2) \\ &u \mapsto k * u := \left(x \mapsto \int_{[-\pi,\pi]^2} k(x-y)u(y)\mathrm{d}y\right)\end{aligned}$$

where we use periodic boundary extension.

- Show that
$$\widehat{Tu}_l = \hat{k}_l\hat{u}_l \text{ for all } l \in \mathbb{Z}^2$$
and provide the inverse of T in case $\hat{k}_l \neq 0$ for all $l \in \mathbb{Z}^2$.
- Proof that T is compact and deduce that, in case $\hat{k}_l \neq 0$ for infinitely many $l \in \mathbb{Z}^2$, T cannot have closed range.

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Exercise 3 - Radon transform

For $f : B \to \mathbb{R}$ continuous, where $B := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, we define the Radon transform as

$$\mathcal{R}f(\theta, s) = \int_{-\infty}^{\infty} f\left(s\ \mathbf{w}(\theta) + t\ \mathbf{w}^{\perp}(\theta)\right) dt, \quad (\theta, s) \in [0, 2\pi] \times \mathbb{R},$$

with f extended by 0 outside B and $\mathbf{w}(\theta) = (\cos(\theta), \sin(\theta))^t$, $\mathbf{w}^{\perp}(\theta) = (\text{-}\sin(\theta), \cos(\theta))^t$.

a) Show that the Radon transform can be extended to a linear continuous operator from $L^p(B)$ to $\Omega = L^p([0, 2\pi] \times [-1, 1])$, where $p \in [1, \infty)$.

b) Prove that, for $p \in (1, \infty)$, the adjoint of the Radon transform $\mathcal{R}^*$, also called **backprojection operator**, has the form

$$\mathcal{R}^* g(x) = \int_0^{2\pi} g\left(\theta, x \cdot \mathbf{w}(\theta)\right) d\theta. \tag{1}$$

Hint: You can use Jensen's inequality for measures. Also, remember that $\mathcal{R}^*$ is the adjoint if and only if $\mathcal{R}^* g \in L^{p*}(B)$ and $\int_\Omega (\mathcal{R}f) g = \int_B f(\mathcal{R}^* g), \quad \forall f \in L^p(B), g \in L^{p*}(\Omega)$ with $p* = p/(p-1)$.

Exercise 4 - Programming exercise

Use then script "convolution_test.m" to implement and test a convolution. The main tasks as described in the script are as follows:

- Implement a convolution both using an actual convolution (such as with "conv2") and by using multiplication in the Fourier domain, e.g., with "fft2". Ensure that both implementations produce the same result. Note: The correct boundary extension matters!
- Implement a deconvolution operator using division in the Fourier domain. Observe the instability of this operator as implemented in the script.
- Bonus: Can you come up with a better direct inversion of the convolution operator?