Institute for Mathematics
and Scientific Computing

Introduction to Functional Analysis

Problem Sheet 7

Due date: June 23, 2017

In order to learn most from the exercises below, I strongly recommend to first seriously try to complete them on your own, and only afterwards discuss and compare with colleagues.

Problem 7.1. For $f, g \in L^2(\Omega, \mathbb{K}^M; \mu)$, define

$$(f, g)_{L^2} := \int_\Omega (f(x), g(x))\, \mathrm{d}\mu(x),$$

where we set $(a, b) = \sum_{i=1}^M a_i \bar{b}_i$ for $a, b \in \mathbb{K}^M$. Show that $(\cdot, \cdot)_{L^2}$ is an inner product and that $\|f\|_{L^2}^2 = (f, f)_{L^2}$ for all $f \in L^2(\Omega, \mathbb{K}^M, \mu)$.

Problem 7.2. Let X be an inner product space and $A \subset X$ a subspace. Show

- For all $x, y \in X$, $x \perp y \Rightarrow \|x\|^2 + \|y\|^2 = \|x + y\|^2$.
- $A^\perp$ is a closed subspace of X.
- $\overline{A} \subset (A^\perp)^\perp$.
- $A^\perp = (\overline{\mathcal{L}(A)})^\perp$ and $A^\perp = \overline{A}^\perp$

Remember that here, $A^\perp$ denotes the orthogonal complement of A as subspace of X.

Problem 7.3. Proof the following theorem of the lecture: If X, Y are Banach spaces, $T \in \mathcal{L}(X, Y)$ then t.f.a.e.

i) T^* is surjective.

ii) There exists $C > 0$ such that $\|x\| \leq C\|Tx\|$ for all $x \in X$.

iii) $\ker(T) = \{0\}$ and $\mathrm{rg}(T)$ is closed.

Problem 7.4. (Counterexample-Operator) For $p \in [1, \infty)$ define $T : \ell^p \to \ell^p$ as

$$T((x_n)_n) = (\frac{1}{n} x_n)_n.$$

Show that is well defined and $T \in \mathcal{L}(\ell^p, \ell^p)$. Determine T^*, $\ker(T)$ $\ker(T^*)$, $\mathrm{rg}(T)$, $\mathrm{rg}(T^*)$, $\overline{\mathrm{rg}(T)}$ and $\overline{\mathrm{rg}(T^*)}$. Use T and T^* to argue that, for general bounded linear operators A between Banach spaces we have:

- $\overline{\mathrm{rg}(A^*)} \subset \ker(A)^\perp$ but equality does not hold.
- A surjective implies A^* injective, but the converse does not hold.
- A^* surjective implies A injective, but the converse does not hold.

Problem 7.5. Let X be a Banach space and $(x_n)_n \subset X$ and $x \in X$. Show that, if $x_n \rightharpoonup x$, then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

Problem 7.6. (Uniformly convex space) A Banach space $(X, \|\cdot\|)$ is called uniformly convex if for any two sequences $(x_n)_n$, $(y_n)_n$ in X it holds that

$$\left[\|x_n\| \to 1,\ \|y_n\| \to 1 \text{ and } \|\frac{x_n + y_n}{2}\| \to 1\right] \Rightarrow \|x_n - y_n\| \to 0$$

i) Interpret the notion of uniform convexity geometrically.

ii) Show that Hilbert spaces, in particular ℓ^2, are uniformly convex but ℓ^1 is not uniformly convex.

iii) Show that in a uniformly convex space it holds that, if $(x_n)_n \subset X$, $x \in X$ and $\|x_n\| \to \|x\|$ as well as $x_n \rightharpoonup x$ then $\|x_n - x\| \to 0$.