MAT.258 / SS 2017
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Introduction to Functional Analysis

Problem Sheet 6

Due date: June 9, 2017

In order to learn most from the exercises below, I strongly recommend to first seriously try to complete them on your own, and only afterwards discuss and compare with colleagues.

Problem 6.1. Let X, Y, Z be normed spaces. Show

i) The mapping $T \mapsto T^*$ is linear and isometric from $\mathcal{L}(X,Y)$ to $\mathcal{L}(Y^*, X^*)$

ii) $(ST)^* = T^*S^*$ for $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$.

iii) For $T \in \mathcal{L}(X,Y)$, $T^{**} \circ i_X = i_y \circ T$.

Problem 6.2. We define $c_0 = \{(x_n)_n \subset \mathbb{R} : \lim_{n\to\infty} x_n = 0\}$ with the norm $\|(x_n)_n\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$. Show that the mapping

$$\begin{aligned} T :& \ell^1 \to (c_0)^* \\ & x = (x_n)_n \mapsto Tx \text{ with } (Tx)((y_n)_n) = \sum_{n\in\mathbb{N}} x_n y_n \end{aligned}$$

is an isometric isomorphism. Does your proof also work with c_0 replaced by ℓ^∞? Bonus question: Show that the mapping in Problem 6.1.(i) is, in general, not surjective.
Hint: For surjectivity of T, consider $(y^*(e_n))_n$ for $y^* \in (c_0)^*$. For the Bonus question, consider the opertor $S \in \mathcal{L}(\ell^1, \ell^1)$, $S(x_n) := (\sum_{n\in\mathbb{N}} x_n, 0, 0, \ldots)$ and its adjoint.

Problem 6.3. (Completion of proofs of the lecture) Let $((E_i, \|\cdot\|_{E_i}))_{i\in\mathbb{N}}$ be a sequence of Banach spaces and let $p \in (1,\infty)$. We define

$$E := \bigoplus_{i=1}^{\infty}{}_p E_i = \left\{ (x_n)_n : x_n \in E_n,\ \|(x_n)_n\|_p := \left(\sum_{i=1}^{\infty} \|x_i\|_{E_i}^p \right)^{1/p} < \infty \right\}.$$

Prove the following:

i) $(E, \|\cdot\|_p)$ is a Banach space

ii) For $q = p/(p-1)$ we get

$$E^* \hat{=} \bigoplus_{i=1}^{\infty}{}_q E_i^*$$

Hint: Try to imitate the proof of $\ell^p \hat{=} \ell^q$ as provided, for example, in Satz II.2.3 in the book "Funktionalanalysis" by D. Werner (which is available online at the library of the KFU).

Problem 6.4. (Completion of proofs of the lecture) Let (Ω, Σ, μ) be a measure space, $p \in (1,\infty)$ and assume that $\Omega = \bigcup_{i=1}^{\infty} \Omega_n$ with pairwise disjoint $\Omega_n \in \Sigma$. Show that

$$\bigoplus_{i=1}^{\infty}{}_p L^p(\Omega_i, \mu) \hat{=} L^p(\Omega, \mu).$$

Using this result and the result of Problem 6.3, complete the proof of the lecture that $L^q(\Omega, \mu) \tilde{=} L^p(\Omega, \mu)^*$ with $q = p/(p-1)$ for the case of a sigma-finite measure space (instead of a finite measure space).

Problem 6.5. (Bonus question) Show the assertions of Problems 6.3 and 6.4 for the case $p = 1$ and $q = \infty$.