Institute for Mathematics
and Scientific Computing

Introduction to Functional Analysis

Problem Sheet 5

Due date: Mai 26, 2017

In order to learn most from the exercises below, I strongly recommend to first seriously try to complete them on your own, and only afterwards discuss and compare with colleagues.

Problem 5.1. (Completion of proofs from the lecture) Let X be a normed space and $G, L \subset X$ be closed subspaces and $V \subset X^*$ be a subspace. Show

- If X is reflexive, then $i_X(V_\perp) = V^\perp$, where $i_X : X \to X^{**}$ is the canonical embedding.
- If X is reflexive, $(V_\perp)^\perp = \overline{V}$
- $G^\perp \cap L^\perp = (G+L)^\perp$

Problem 5.2. (Completion of proofs from the lecture) As mentioned in the lecture, the results we have obtained so far allow to directly conclude that, in general, neither $L^1(\Omega)$ nor ℓ^1 can be reflexive. Provide this argumentation and conclude also that, in general, neither $L^\infty(\Omega)$ nor ℓ^∞ can be reflexive.

Problem 5.3. (The finite dimensional setting) Let X be a normed space. Show the following:

- X is finite dimensional $\Leftrightarrow$ X^* is finite dimensional. In this case, $\dim(X) = \dim(X^*)$.
- If X is finite dimensional, it is reflexive.
- If X is finite dimensional the notions of norm convergence, weak convergence and weak* convergence coincide.

Hint: For $\Rightarrow$ of the first point, consider the mapping we have used to show that finite dimensional subspaces admit a complement. For the other direction, use what you already have.

Problem 5.4. Let X be a normed space, $(x_n)_n$ a bounded sequence in X and $x \in X$. Show that $(x_n)_n$ weakly converges to x if and only if there exists a $D \subset X^*$ with $X^* = \overline{\mathcal{L}(D)}$ such that $x^*(x_n) \to x^*(x)$ for all $x^* \in D$.

Problem 5.5. Let X be a normed space and $M, E \subset X$ be two subspaces such that E is finite dimensional and M is closed. Show

- E is closed
- If $M + E = X$ (i.e. M is of finite codimension), then M admits a complement in X.
- $M + E$ is closed.

Can you use one of the previous exercises to give an examples of two closed subspaces of a Banach space whose sum is not closed?
Hint: For the second point, consider a complement of $M \cap E$ in E. For the third point, first assume that $M \cap E = \{0\}$, take a convergent sequence $x_n = u_n + v_n \in M + E$ and show boundedness of v_n.