Institute for Mathematics
and Scientific Computing

Introduction to Functional Analysis

Problem Sheet 4

Due date: Mai 12, 2017

Please remember that the intermediate exam will take place on Mai 12!

Problem 4.1. Let X be a normed space, $f : X \to \mathbb{K}$ be linear, $\alpha \in \mathbb{R}$ and define the hyperplane H as

$$H = \{x \in X \,:\, \mathrm{Re} f(x) = \alpha\}.$$

Show that H is closed if and only if f is continuous.
A second, not directly related question: Why is it not possible to just use a zero-extension in the Hahn-Banach theorem, that is, extend $T : U \subset X \to \mathbb{K}$ by $\tilde{T}x = Tx$ if $x \in U$ and $\tilde{T}x = 0$ else?

Problem 4.2. Let X be a normed space and K and L be two closed, convex subsets of X such that $K \cap L = \emptyset$. Show that if either K or L is compact, then there exists $f \in \mathcal{L}(X, \mathbb{K})$ such that

$$\sup_{x \in K} \mathrm{Re} f(x) < \inf_{x \in L} \mathrm{Re} f(x).$$

Problem 4.3. In this example we show that the assertion of Problem 4.2. above (even with $\leq$) does not hold without the compactness: Set

$$X = \{x = (x_n)_{n\in\mathbb{N}} \in \ell^1 \,:\, x_{2n} = 0 \,\forall n \geq 1\}$$

$$Y = \{y = (y_n)_{n\in\mathbb{N}} \in \ell^1 \,:\, y_{2n} = \frac{1}{2^n} y_{2n-1} \,\forall n \geq 1\}$$

i) Show that X and Y are closed linear subspaces of ℓ^1 and that $\overline{X + Y} = \ell^1$.

ii) Let $c = (c_n)_{n\in\mathbb{N}} \in \ell^1$ be such that $c_{2n} = \frac{1}{2^n}$, $c_{2n-1} = 0$ for all $n \geq 1$. Show that $c \notin X + Y$.

iii) Set $Z = X - \{c\}$. Confirm that both Y and Z are closed, convex sets and that $Y \cap Z = \emptyset$. Show that Y and Z cannot be separated by a closed hyperplane, i.e., there does not exist $f \in \mathcal{L}(\ell^1, \mathbb{R})$ such that $f \neq 0$ and

$$f(x) \leq f(y) \quad \text{for all } x \in Z,\, y \in Y.$$