Institute for Mathematics
and Scientific Computing

Introduction to Functional Analysis

Problem Sheet 2

Due date: March 31, 2017

Problem 2.1. Let (X, d) be a metric space and $(A_i)_{i=1}^n$ a finite sequence of open sets in X such that each A_i is dense in X. Show that also $\bigcap_{i=1}^n A_i$ is dense in X. Does the same hold for countably many sets A_i (without further assumptions)?

Problem 2.2. Let (X, d) be a separable metric space and $M \subset X$. Show that (M, d) is a separable metric space.

Problem 2.3. Let $(X, \|\cdot\|)$ be a normed space. Show that X is a Banach space if and only if every absolutely convergent series converges.

Problem 2.4. Let X be a vector space and $\|\cdot\|^*$ be a semi-norm on X. Show

i) $N := \{x \in X \mid \|x\|^* = 0\}$ is a subspace of X.

ii) $\|[x]\| := \|x\|^*$ is a norm on X/N

iii) If X is complete, X/N is a Banach space.

Note: Here complete means w.r.t. the semi-norm $\|\cdot\|^*$, i.e., any Cauchy sequence w.r.t. $\|\cdot\|^*$ has a subsequence converging w.r.t. $\|\cdot\|^*$ to some element in X

Problem 2.5. Use the result of Baire to show that there exist uniformly continuous functions which are nowhere differentiable.
Hint 1: Consider $C([0, 1])$, the set of continuous functions mapping from $[0, 1]$ to $\mathbb{R}$. You can use the Weierstraß approximation theorem stating that polynomials are dense in $C([0, 1])$ w.r.t. the $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$.
Hint 2: Consider the sets

$$O_n = \left\{ x \in C([0,1]) \, : \, \sup_{\substack{0<|h|\leq 1/n \\ t+h\in[0,1]}} \left| \frac{x(t+h) - x(t)}{h} \right| > n \, \forall t \in [0,1] \right\}.$$