MAT.258 / SS 2017
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Introduction to Functional Analysis

Problem Sheet 1

Due date: March 17, 2017

Remark: In the lecture we will show that a subset M of a metric space is compact if and only if every sequence in M admits a convergent subsequence whose limit is in M. You can use this result.

Problem 1.1. Let (X, d) be a metric space. Show that, if d is induced by a norm, then for any $\epsilon > 0$,

$$\overline{B_\epsilon(x)} = \overline{\{y \in X \mid d(x,y) < \epsilon\}} = \{y \in X \mid d(x,y) \leq \epsilon\}.$$

Show that this assertion does not hold true in a general metric space.

Problem 1.2. Let $(X, \|\cdot\|)$ be a normed space and $A \subset X$ closed.

- Show that the mapping $d(\cdot, A) : X \to \mathbb{R}$, $d(x, A) := \inf_{a \in A} \|x - a\|$ is Lipschitz continuous on X and that $A = \{x \in X \mid d(x, A) = 0\}$
- Show the Riesz Lemma: If $A \neq X$ is a closed subspace, for any $\delta > 0$ there exists $x \in X$ such that $\|x\| = 1$ and $d(x, A) \geq 1 - \delta$.

Problem 1.3. Show that, if $(X, \|\cdot\|)$ is a finite dimensional normed space, a subset $M \subset X$ is compact if and only if it is complete and bounded.
(Hint: You might use the Bolzano-Weierstraß theorem from analysis.)

Problem 1.4. Let $(X, \|\cdot\|)$ be a complete normed space and define $B_1 = \{x \in X \mid \|x\| \leq 1\}$. Use the Riesz Lemma of Problem 1.1 to show that, if X is infinite dimensional, B_1 is closed and bounded but not compact. Conclude that B_1 is compact if and only if X is finite dimensional and, consequently, all bounded sequences in X admit convergent subsequences if and only if X is finite dimensional.

Problem 1.5 (Bonus question). Provide an explicit example of a bounded sequence in a complete normed space which does not admit a convergent subsequence.
(Hint: You might consider the space of all bounded real valued functions defined on $[0, 1]$ with the supremum norm.)