INSTITUT FÜR MATHEMATIK UND
WISSENSCHAFTLICHES RECHNEN

Optimierung I

Test 1

May 6, 2014

Problem 1: [Extremal points, 2 points]

Compute the set of extremal points of the following convex subset of $\mathbb{R}^n$:

$$K = \big\{ x \in \mathbb{R}^n \,\big|\, \text{for } i = 1, ..., n,\; |x_i| \leq 1 \big\}.$$

Justify your answer carefully.

Problem 2: [A min-max problem, 4 points]

Consider the following optimization problem:

$$\begin{aligned} \max_{x \in \mathbb{R}^2} & \Big\{ \min \{ 8x_1,\, 7x_2 \} \Big\}, \\ \text{subject to:} \quad & 2x_1 + 3x_2 \leq 20 \\ & x_1 + 5x_2 \geq 4 \\ & x_1 \geq 0. \end{aligned}$$

i) Reformulate the problem, without rigorous justification, into a linear program in canonical form.

ii) Given an optimal solution to the initial problem, give an optimal solution to the reformulated problem. Conversely, given an optimal solution to the reformulated problem, give an optimal solution to the initial one.

Problem 3: [Simplex method, 4 points]

Solve the following two problems with the simplex algorithm. At each iteration, select the non-basic variable with the lowest reduced cost.

$$\begin{aligned} \min_{x \in \mathbb{R}^3} \; & -2x_1 + 3x_2 - 4x_3 \\ \text{subject to:} \quad & 4x_1 + 3x_2 + x_3 \leq 3 \\ & 2x_1 - 4x_2 - x_3 \leq 10 \\ & x_1 + x_2 + x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

and

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \; & -x_1 - 3x_2 \\ \text{subject to:} \quad & -x_1 + 2x_2 \leq 6 \\ & x_1 + 2x_2 \leq 10 \\ & 3x_1 + x_2 \leq 15 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Problem 4: [Optimality conditions, 4 points]

i) Show that after an iteration of the simplex algorithm, if for all non-basic variable x_j, the associated reduced cost r_j is strictly greater than 0, then the current basic feasible solution is the unique optimal solution.

ii) Show that a degenerate basic feasible solution may be optimal without satisfying $r_j \geq 0$ for all non-basic variable x_j.

Problem 5: [An optimal transportation problem, 4 points]

A large textile firm has two manufacturing plants, two sources of raw material, and three market centers. The transportation costs between the sources and the plants and between the plants and the markets are as follows:

	Plant A	Plant B
Source 1	\$1/ton	\$1.5/ton
Source 2	\$2/ton	\$1.5/ton

	Market 1	Market 2	Market 3
Plant A	\$4/ton	\$2/ton	\$1/ton
Plant B	\$3/ton	\$4/ton	\$2/ton.

Ten tons are available from source 1 and 15 tons from source 2. The three market centers require 8 tons, 14 tons, and 3 tons. The plants have unlimited processing capacity.

i) Formulate the problem of finding the shipping patterns from sources to plants to markets that minimizes the total transportation cost.

ii) Reduce the problem to a single standard transportation problem with two sources and three destinations. (Hint: Find minimum cost paths from sources to markets.)

iii) Reduce the dimension of the problem and solve it with the method of your choice.