INSTITUT FÜR MATHEMATIK UND
WISSENSCHAFTLICHES RECHNEN

Optimierung I

Übungsblatt 3

Bearbeitung bis 1. April 2014

Aufgabe 3.1: [Canonical form]

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. Put the following linear problem into a canonical form:

$$\min_{x\in\mathbb{R}^n} c^\top x, \text{ subject to } Ax = b, \; x \geq 0, \; x_1 \leq x_2 \leq ... \leq x_n. \tag{1}$$

Aufgabe 3.2: [Piecewise affine approximation of a convex function]

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. We are interested in the following non-linear optimization problem:

$$\min_{x\in\mathbb{R}^n} f(x), \text{ subject to: } Ax = b, \; x \geq 0. \tag{P}$$

We assume that the feasible set is bounded. Let s be the optimal value of problem (P). We recall that since f is convex, then for all x and x' in $\mathbb{R}^n$,

$$f(x') - f(x) \geq Df(x)(x' - x), \tag{2}$$

where $Df(x)$ is the derivative of f. Let x_1,...,x_N be N points of $\mathbb{R}^n$. We define:

$$f_N(x) = \max_{i=1,...,N} \{f(x_i) + Df(x_i)(x - x_i)\} \tag{3}$$

and we consider the problem

$$\min_{x\in\mathbb{R}^n} f_N(x), \text{ subject to: } Ax = b, \; x \geq 0. \tag{P_N}$$

We consider that f_N is a piecewise affine approximation of f and that problem (P_N) is an approximation of (P).

i) For a function f and points x_1,...,x_N of your choice, draw f and f_N.

ii) Prove that for all $x \in \mathbb{R}^n$, $f(x) \geq f_N(x)$.

iii) Compute $f_N(x_i)$, for all i.

iv) Formulate problem (P_N) as a linear programming problem, justify the reformulation. Put the obtained problem into a canonical form.

v) Let x^* be a solution to problem (P_N), prove that

$$f_N(x^*) \leq s \leq f(x^*). \tag{4}$$

vi) How could we iteratively improve the approximation (P_N) of problem (P) ? Describe an algorithm which enables to solve approximately problem (P) by solving only linear programming problems.

Aufgabe 3.3: [Application exercise]

A manufacturer wishes to product a special alloy (Legierung), made of a proportion b_1 of a metal 1 and a proportion b_2 of a metal 2 (with $b_1 + b_2 = 1$). To this purpose, he can mix n different alloys. The i-th alloy is made of a proportion $a_{1,i}$ and $a_{2,i}$ of metals 1 and 2 respectively, with $a_{1,i} + a_{2,i} = 1$. The prices (per unit of weight) of the n alloys are c_1, c_2,...,c_n.

i) Denoting by x_i the proportion of alloy i used to make the special alloy, formulate the manufacturer's problem as a linear programming problem. Warning: use as few constraints as possible.

ii) What is the maximal number of bases of the problem ?

iii) Give a condition on the data to ensure that the problem has a solution.

iv) We assume that $a_{1,1} \leq a_{1,2} \leq ... \leq a_{1,n}$. We assume that for some k, $a_{1,k} < b_1 < a_{1,k+1}$. What is the number of feasible bases ?

v) We consider a case with 4 alloys:

	Alloy 1	Alloy 2	Alloy 3	Alloy 4	Special alloy
Metal 1	0.2	0.3	0.5	0.7	0.4
Metal 2	0.8	0.7	0.5	0.3	0.6
Cost per unit	5	3	7	6	

Determine the feasible bases, the associated basic solutions and finally the optimal solution.

Aufgabe 3.4: [Optimal transportation]

Quantities a_1,...,a_m, respectively, of a certain product are to be shipped from each of m locations and received in amounts b_1, b_2,...,b_n, respectively at each of n destinations. Associated with the shipping of a unit of product from origin i to destination j is a unit shipping cost c_{ij}. It is desired to determine the amounts $x_{i,j}$ to be shipped between each origin-destination pair $i = 1,...,m$, $j = 1,...,n$ so as to satisfy the shipping requirements and minimize the total cost of transportation. To make the problem consistent, we assume that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

i) Formulate the problem as a linear programming problem. Denoting by

$$y = (x_{1,1}, ..., x_{1,n}, x_{2,1}, ..., x_{2,n}, ..., x_{m,1}, ..., x_{m,n}) \qquad (5)$$

the vector of size nm, write the constraints of the problem in the canonical form $Ay = w$, with $w \in \mathbb{R}^{(n+m)\times 1}$ and $A \in \mathbb{R}^{(n+m)\times nm}$.

ii) Is it possible that the rank of the matrix A is equal to $n + m$?

iii) We consider the following case: $m = 2$, $n = 3$, $a = (5, 8)$, $b = (4, 3, 6)$, with the costs:

	Destination 1	Destination 2	Destination 3
Origin 1	1	1	2
Origin 2	2	1	1

Express the variables $x_{1,2}$, $x_{1,3}$, $x_{2,1}$, and $x_{2,3}$ in function of $x_{1,1}$ and $x_{2,2}$. Reduce the optimal transportation problem into a problem with two optimization variables. Solve it graphically.