DIOPHANTINE EQUATIONS OF PELLIAN TYPE

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Abstract. We investigate the solutions of diophantine equations of the form $dx^2 - d^*y^2 = \pm t$ for $t \in \{1, 2, 4\}$ and their connections with ideal theory, continued fractions and Jacobi symbols.

1. Introduction and History

The aim of this article is a thorough study of diophantine equations of the form

$$dx^2 - d^*y^2 = \pm 1,$$

where $d, d^* \in \mathbb{N}$ and $dd^*$ is not a square.

For $d = 1$, this is Pell’s equation, while the general equation (1) is sometimes called antipellian. Multiplication of (1) with $d$ implies (with $X = dx$, $Y = dy$ and $D = dd^*$) the norm equation

$$X^2 - DY^2 = \pm d,$$

where $d | D$ and $(X, Y) = 1$.

Conversely, if $d$ is squarefree, then (2) implies (1). The solubility of (2) can be rephrased in the language of binary quadratic forms. Explicitly, this was done by G. Pall in [15], where the following result was stated and essentially attributed to C. F. Gauss (see the English Edition [2]). A special case was later rediscovered by H.F. Trotter [17].

**Theorem A.** Let $\Delta > 0$ be a discriminant of binary quadratic forms. Then precisely two divisors of $\Delta$ can be properly represented by the principal class of discriminant $\Delta$.

The special case of (1) where $D = dd^*$ is squarefree was frequently investigated in the literature, using different methods. In this case, the result reads as follows.

**Theorem B.** Let $D \in \mathbb{N}$ be a squarefree positive integer, and

$$D^* = \begin{cases} 2D & \text{if } D \equiv 3 \mod 4, \\ D & \text{if } D \not\equiv 3 \mod 4. \end{cases}$$

Then there is exactly one $1 < m \mid D^*$ such that the diophantine equation

$$x^2 - Dy^2 = m \quad x^2 - Dy^2 = m \quad x^2 - Dy^2 = m \quad x^2 - Dy^2 = m$$

has a solution $(x, y) \in \mathbb{Z}^2$.

An elementary proof of Theorem B, only using the theory of Pell’s equation, was given in [8], a proof within the theory of continued fractions is in [3], and a proof using the theory of quadratic number fields can be found in [6].

Partial results in the general case (also addressing the connection with ideal theory, continued fractions and Jacobi symbols) were obtained only recently by various authors, see [12], [10], [14], [1], [18] [7].

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There is a significant overlap with R.A. Mollin’s paper [13]. There he investigates antipellian equations within the theory of continued fractions, however ignoring the structural point of view taken in the main theorems 4.3 and 4.4 of the present paper. Nevertheless, some of his explicit results there are more general than the applications given in our section 5 below.

The basic tools for the present investigations are the theory of ambiguous ideals in quadratic number fields as developed in [4] and their connection with continued fractions. This interrelation is principally known and republished several times (I refer to R. Mollin’s book [11] and to the article [9]). Unfortunately, the terminology on these subjects is far from being standardized. Thus I decided to give an overview of the necessary basic result, at least to fix the notation. This will be done in the sections 2 and 3.

Section 4 contains the main results concerning equation (1) and their connection with ideal theory, continued fractions and Jacobi symbols. By the way, it turns out that it is natural to consider the more general equations $dx^2 - dt^2 = \pm t$, where $t \in \{1, 2\}$ if $\Delta \equiv 12 \mod 16$, and $t \in \{1, 4\}$ if $\Delta \equiv 1 \mod 4$. Finally, section 5 contains several applications for small discriminants.

2. Quadratic Orders

A non-square integer $\Delta \in \mathbb{Z}$ is called a discriminant if $\Delta \equiv 0$ or $1 \mod 4$, and we set

$$\sigma_{\Delta} = \begin{cases} 0 & \text{if } \Delta \equiv 0 \mod 4, \\ 1 & \text{if } \Delta \equiv 1 \mod 4, \end{cases}$$

and

$$\mathcal{O}_{\Delta} = \mathbb{Z}[\sqrt{\Delta}] = \left\{ \frac{a + b\sqrt{\Delta}}{2} \mid a, b \in \mathbb{Z}, \ a \equiv b\Delta \mod 2 \right\}.$$ We call $\omega_{\Delta}$ the basis number and $\mathcal{O}_{\Delta}$ the order of discriminant $\Delta$. A quadratic discriminant $\Delta$ is called a fundamental discriminant if it admits no factorization $\Delta = \Delta_1 m^2$ such that $\Delta_1$ is a discriminant and $m \in \mathbb{N}_{\geq 2}$. Every discriminant $\Delta$ has a unique factorization $\Delta = \Delta_0 f^2$, where $\Delta_0$ is a fundamental discriminant and $f \in \mathbb{N}$. In this factorization, $\Delta_0 = \Delta_K$ is the field discriminant of the quadratic number field $K = \mathbb{Q}(\sqrt{\Delta})$, $\mathcal{O}_{\Delta_0} = \mathcal{O}_K$ is its maximal order, and $f = (\mathcal{O}_K : \mathcal{O}_{\Delta})$.

We denote by $(\xi \mapsto \xi')$ the non-trivial automorphism of $K$, and for a subset $X \subset K$, we set $X' = \{\xi' \mid \xi \in X\}$. For $\xi \in K$, we call $\xi'$ its conjugate and $\mathcal{N}(\xi) = \xi\xi'$ in $\mathbb{Q}$ its norm.

If $\Delta$ is a quadratic discriminant, then the unit group $\mathcal{O}_{\Delta}^\times$ of $\mathcal{O}_{\Delta}$ is given by

$$\mathcal{O}_{\Delta}^\times = \{ e \in \mathcal{O}_{\Delta} \mid |\mathcal{N}(e)| = 1 \} = \left\{ \frac{a + b\sqrt{\Delta}}{2} \mid a, b \in \mathbb{Z}, \ |a^2 - b^2\Delta| = 4 \right\},$$

and $\mathcal{O}_{\Delta} = \langle -1, \varepsilon_{\Delta} \rangle$, where $\varepsilon_{\Delta} = \min(\mathcal{O}_{\Delta} \cap \mathbb{R}_{>1})$ is the fundamental unit of discriminant $\Delta$ (see [5, §16.4]).

An algebraic number $\xi \in \mathbb{C}$ of degree 2 is called a quadratic irrational. For an integer $D \in \mathbb{Z}$, we normalize its square root by $\sqrt{D} \geq 0$ if $D \geq 0$, and $3\sqrt{D} \geq 0$ if $D < 0$. Then every quadratic irrational $\xi \in \mathbb{C}$ has a unique representation

$$\xi = \frac{b + \sqrt{b^2 - 4ac}}{2a}, \quad \text{where } a, b, c \in \mathbb{Z} \quad \text{and} \quad (a, b, c) = 1.$$ In this representation, the triple $(a, b, c) \in \mathbb{Z}^3$ is called the type and $\Delta = b^2 - 4ac$ is called the discriminant of $\xi$. If $\Delta \in \mathbb{Z}$ is any discriminant, then $\Delta = 4D + \sigma_{\Delta}$, where $D \in \mathbb{Z}$, and the basis number $\omega_{\Delta}$ is a quadratic irrational of type $(1, \sigma_{\Delta}, -D)$ and discriminant $\Delta$. 

Two irrational numbers \( \xi, \xi_1 \in \mathbb{C} \setminus \mathbb{Q} \) are called \textit{equivalent} if
\[
\xi_1 = \frac{\alpha \xi + \beta}{\gamma \xi + \delta} \quad \text{for some} \quad \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \text{GL}_2(\mathbb{Z}).
\]

It is easily checked that equivalent quadratic irrationals have the same discriminant.

Let \( K \) be a quadratic number field. For \( n \in \mathbb{N} \) and \( \alpha_1, \ldots, \alpha_n \in K \), we denote by \([\alpha_1, \ldots, \alpha_n] = \mathbb{Z} \alpha_1 + \cdots + Z \alpha_n \subset K \) the \( \mathbb{Z} \)-module generated by \( \alpha_1, \ldots, \alpha_n \). A free \( \mathbb{Z} \)-submodule \( a \subset K \) of rank 2 is called a \textit{lattice} in \( K \), and \( \mathcal{R}(a) = \{ \lambda \in K \mid \lambda a \subset a \} \) is called its \textit{ring of multipliers}. If \( (\omega_1, \omega_2) \) is a basis of \( a \), then \( a = [\omega_1, \omega_2] \). In particular, for every discriminant \( \Delta \) we have
\[
\mathcal{O}_\Delta = [1, \omega_\Delta] = \left\{ \frac{a + b\sqrt{\Delta}}{2} \mid a, b \in \mathbb{Z}, \ a \equiv b \Delta \mod 2 \right\}.
\]

In a different terminology, the following Propositions 2.1, 2.2 and 2.3 can be found in [4, Propositions 1 and 3].

**Proposition 2.1** (Structure of lattices). \textit{Let \( K \) be a quadratic number field and \( a \subset K \) a lattice. Then \( a = [m, \xi] \), where \( m = \text{min}(a \cap \mathbb{Q}_{>0}) \) and \( \xi \in K \). If \( \xi \) is a quadratic irrational of type \( (a, b, c) \) and discriminant \( \Delta \), then \( \mathcal{R}(a) = \mathcal{O}_\Delta \), and \( a^\prime = m^2 a^{-1} \mathcal{O}_\Delta \). In particular, \( a \) is an invertible fractional ideal of \( \mathcal{O}_\Delta \).}

**Proof.** Observe first that \( a \cap \mathbb{Q} \neq \{0\} \). Indeed, \( a' \) and \( \mathcal{R}(a') \) are lattices as well, and it \( 0 \neq a \in a \), then there is some \( q \in \mathbb{N} \) such that \( qa \in \mathcal{R}(a') \), which implies that \( 0 \neq q\mathcal{N}(a) = qa' \in a \cap \mathbb{Q} \). Now \( a \cap \mathbb{Q} \) is a finitely generated non-zero subgroup of \( \mathbb{Q} \), and therefore \( a \cap \mathbb{Q} = ma \), where \( m = \text{min}(a \cap \mathbb{Q}_{>0}) \). Let \( (\omega_1, \omega_2) \) be a basis of \( a \) and \( m = c_1 \omega_1 + c_2 \omega_2 \), where \( c_1, c_2 \in \mathbb{Z} \). Then \((c_1, c_2) = 1 \) by the minimal choice of \( m \), and there exist \( u_1, u_2 \in \mathbb{Z} \) such that \( c_1 u_2 - c_2 u_1 = 1 \). If \( \xi_1 = u_1 \omega_1 + u_2 \omega_2 \), then
\[
\left( \begin{array}{c} m \\ \xi_1 \end{array} \right) = \left( \begin{array}{cc} c_1 & c_2 \\ u_1 & u_2 \end{array} \right) \left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right) \quad \text{and} \quad (a : [m, \xi]) = |c_1 u_2 - c_2 u_1| = 1.
\]

Hence \( a = [m, \xi] = [m, 1, \xi] \), where \( \xi = m^{-1} \xi_1 \).

Assume now that \( \xi \) if of type \((a, b, c)\) and discriminant \( \Delta = b^2 - 4ac \). We shall prove that \( \mathcal{O}_\Delta a \subset a \) and \( m^{-2} a a' = \mathcal{O}_\Delta \). Then it follows that
\[
\mathcal{O}_\Delta \subset \mathcal{R}(a) = \mathcal{R}(a) \mathcal{O}_\Delta = m^{-2} a a' \mathcal{R}(a) \subset m^{-2} a a' = \mathcal{O}_\Delta,
\]
and therefore equality holds. Since
\[
\omega_\Delta = \frac{\sigma_\Delta - b}{2} + \frac{b + \sqrt{\Delta}}{2a} \in [1, \xi] \quad \text{and} \quad \omega_\Delta \xi = -c + \frac{\sigma_\Delta + b + \sqrt{\Delta}}{2a} \in [1, \xi],
\]
we obtain \( \mathcal{O}_\Delta a = m [1, \omega_\Delta] [1, \xi] = m [1, \xi, \omega_\Delta, \omega_\Delta \xi] \subset a \). On the other hand, as \( b \equiv \sigma_\Delta \mod 2 \),
\[
m^{-2} a a' = [a, a \xi] [1, \xi'] = [a, b, c, \xi', \xi] = \left[ 1, b + \frac{\sqrt{\Delta}}{2} \right] = [1, \omega_\Delta] = \mathcal{O}_\Delta. \quad \Box
\]

**Proposition 2.2** (Equivalences of lattices). \textit{Let \( K \) be a quadratic number field and \( \xi, \xi_1 \in K \setminus \mathbb{Q} \).

1. Let \( \theta \in K^\times \) be such that \([1, \xi] = \theta [1, \xi_1] \). Then there exists some matrix
\[
\left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \text{GL}_2(\mathbb{Z}) \quad \text{such that} \quad \xi_1 = \frac{\alpha \xi + \beta}{\gamma \xi + \delta} \quad \text{and} \quad \theta = \gamma \xi + \delta
\]
\]
2. Suppose that
\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta \\
\end{pmatrix} \in \text{GL}_2(\mathbb{Z}) \quad \text{and} \quad \xi_1 = \frac{\alpha \xi + \beta}{\gamma \xi + \delta}.
\]
Then \([1, \xi_1] = \frac{1}{\gamma \xi + \delta} [1, \xi] = \frac{1}{\gamma \xi + \delta} [1, \xi].\]

Proof. 1. If \([1, \xi] = [0, \theta \xi_1], then
\[
\begin{pmatrix}
\theta_1 \\
\theta \\
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta \\
\end{pmatrix} \begin{pmatrix}
\xi \\
1 \\
\end{pmatrix}
\]
for some \(\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta \\
\end{pmatrix} \in \text{GL}_2(\mathbb{Z}),\)
and consequently
\[
\theta = \gamma \xi + \delta \quad \text{and} \quad \xi_1 = \frac{\theta_1}{\theta} = \frac{\alpha \xi + \beta}{\gamma \xi + \delta}.
\]

2. By assumption, we have
\[
[1, \xi_1] = \frac{1}{\gamma \xi + \delta} [\gamma \xi + \delta, \alpha \xi + \beta] = \frac{1}{\gamma \xi + \delta} [1, \xi].\]
\]

Next we investigate ideals. Let \(\Delta\) be a discriminant and \(K = \mathbb{Q}(\sqrt{\Delta})\). Every non-zero fractional ideal \(a\) of \(\mathcal{O}_\Delta\) is a lattice in \(K\), and by Proposition 2.1 it is invertible if and only if \(R(a) = \mathcal{O}_\Delta\). An ideal \(a \subset \mathcal{O}_\Delta\) is called \(\mathcal{O}_\Delta\)-primitive if \(e^{-1}a \not\subset \mathcal{O}_\Delta\) for all \(e \in \mathbb{N}_{>2}\), and it is called \(\mathcal{O}_\Delta\)-regular if it is \(\mathcal{O}_\Delta\)-primitive and \(R(a) = \mathcal{O}_\Delta\). Consequently, every \(\mathcal{O}_\Delta\)-regular ideal is invertible, and the product of two \(\mathcal{O}_\Delta\)-regular ideals is again \(\mathcal{O}_\Delta\)-regular. A lattice \(\epsilon \subset K\) is an invertible fractional ideal of \(\mathcal{O}_\Delta\) if and only if \(\epsilon = m^{-1}a\) for some \(\mathcal{O}_\Delta\)-regular ideal \(a \subset \mathcal{O}_\Delta\).

Two \(\mathcal{O}_\Delta\)-regular ideals \(a, a_1\) are called equivalent if \(a_1 = \lambda a\) for some \(\lambda \in K^\times\). For an \(\mathcal{O}_\Delta\) regular ideal \(a \subset \mathcal{O}_\Delta\), we denote by \([a]\) its equivalence class and by \(\mathcal{N}_\Delta(a) = (\mathcal{O}_\Delta : a) \in \mathbb{N}\) its absolute norm. The set \(\mathcal{C}_\Delta\) of all ideal classes \([a]\) built by \(\mathcal{O}_\Delta\)-regular ideals \(a \subset \mathcal{O}_\Delta\) is a finite abelian group under the composition \([a][a_1] = [aa_1]\). Its unit element is the principal class \([\mathcal{O}_\Delta]\) which consists of all primitive principal ideals of \(\mathcal{O}_\Delta\). Up to isomorphisms, \(\mathcal{C}_\Delta = \text{Pic}(\mathcal{O}_\Delta)\) is the factor group of invertible fractional ideals modulo fractional principal ideals of \(\mathcal{O}_\Delta\).

Next we describe the fundamental connection between quadratic irrationals and ideals. For a quadratic irrational \(\xi \in \mathbb{C}\) of type \((a, b, c)\) and discriminant \(\Delta\), we define the lattice
\[
I(\xi) = \left[ a, \frac{b + \sqrt{\Delta}}{2} \right] = [a][1, \xi] \subset \mathcal{O}_\Delta.
\]
Clearly, \(I(\xi) = I(-\xi), \quad I(\xi') = I(\xi), \quad \text{and} \quad \mathcal{O}_\Delta = I(\omega_\Delta)\). If \(\xi, \xi_1\) are quadratic irrationals, then it is easily checked that \(I(\xi) = I(\xi_1)\) if and only if \(\xi_1 = \varepsilon \xi + n\) for some \(\varepsilon \in \{\pm 1\}\) and \(n \in \mathbb{Z}\).

**Proposition 2.3** (Structure of regular ideals). Let \(\Delta\) be a discriminant.

1. A subset \(a \subset \mathbb{Q}(\sqrt{\Delta})\) is an \(\mathcal{O}_\Delta\)-regular ideal if and only if \(a = I(\xi)\) for some quadratic irrational \(\xi\) of discriminant \(\Delta\). Moreover, if \(\xi\) is of type \((a, b, c)\), then \(\mathcal{N}_\Delta(a) = [a]\).

2. Let \(\xi, \xi_1\) be quadratic irrationals of discriminant \(\Delta\). Then \(\xi\) and \(\xi_1\) are equivalent if and only if \([I(\xi)] = [I(\xi_1)] \in \mathcal{C}_\Delta\).

**Proof.** 1. By definition, \(I(\xi) \subset \mathcal{O}_\Delta\) is a lattice, \(e^{-1}I(\xi) \not\subset \mathcal{O}_\Delta\) for all \(e \in \mathbb{N}_{>2}\), and \(R(I(\xi)) = \mathcal{O}_\Delta\) by Proposition 2.1. Hence \(I(\xi)\) is an \(\mathcal{O}_\Delta\)-regular ideal.

Let now \(a \subset \mathcal{O}_\Delta\) be an \(\mathcal{O}_\Delta\)-regular ideal. By Proposition 2.1, \(a = m[1, \xi]\), where \(m = \min(a \cap \mathbb{Q}_{>0})\) and \(\xi\) is a quadratic irrational, say of type \((a, b, c)\) and discriminant \(\Delta' = b^2 - 4ac\). Since \(\mathcal{O}_{\Delta'} = R(a) = \mathcal{O}_\Delta\), it follows that \(\Delta = \Delta'\), and
as \( a \cap \mathbb{Q}_{>0} \subset \mathbb{N} \), we obtain \( m \in \mathbb{N} \). Now \( m\xi \in a \subset \mathcal{O}_\Delta \) implies \( a \mid m \), say \( m = ae \) for some \( e \in \mathbb{Z} \). Hence
\[
a = m \left[ 1, \frac{b + \sqrt{\Delta}}{2a} \right] = e \left[ a, \frac{b + \sqrt{\Delta}}{2} \right] = |e| \left[ |a|, \frac{b + \sqrt{\Delta}}{2} \right],
\]
and \( |e|^{-1} a \subset \mathcal{O}_\Delta \) implies \( |e| = 1 \) and \( a = I(\xi) \). Since
\[
\left( \frac{|a|}{b + \sqrt{\Delta}} \right) = \left( \frac{|a|}{b - \sqrt{\Delta}} \right) \left( \frac{1}{\omega_{\Delta}} \right),
\]
it follows that \( \mathcal{N}_\Delta(a) = |a| \).

2. By Proposition 2.2. \qed

From now on we consider positive discriminants and real quadratic irrationals.

**Definition 2.4.**
1. Let \( \xi \in \mathbb{R} \) be a quadratic irrational. Then the quadratic irrational
\[
\xi^+ = \frac{1}{\xi - \lfloor \xi \rfloor},
\]
is called the *successor* of \( \xi \). \( \xi \) is called
   - reduced if \( \xi > 1 \) and \( -1 < \xi' < 0 \);
   - ambiguous if \( \xi + \xi' \in \mathbb{Z} \).
2. Let \( \Delta > 0 \) be a discriminant. An \( \mathcal{O}_\Delta \)-regular ideal \( a \subset \mathcal{O}_\Delta \) is called
   - reduced if \( a = I(\xi) \) for some reduced quadratic irrational \( \xi \);
   - ambiguous if \( a' = a \).

**Proposition 2.5.** Let \( \xi \in \mathbb{R} \) be a quadratic irrational of type \((a, b, c)\) and discriminant \( \Delta \).
1. \( \xi \) is reduced if and only if \( 0 < \sqrt{\Delta} - b < 2a < \sqrt{\Delta} + b \). In particular, if \( \xi \) is reduced, then \( 0 < a < \sqrt{\Delta} \), \( 0 < b < \sqrt{\Delta} \), \( 0 < -c < \sqrt{\Delta} \), and \( \xi^+ \) is also reduced.
2. \( \xi \) is ambiguous if and only if \( a \mid b \), and \( I(\xi) \) is ambiguous if and only if \( \xi \) is ambiguous.
3. If \( \xi^+ = -\xi'^{-1} \), then \( \xi \) is ambiguous, and if \( \xi \) is reduced and ambiguous, then \( \xi^+ = -\xi'^{-1} \).
4. If \( \xi \) and \( \xi_1 \in \mathbb{R} \) are reduced quadratic irrationals and \( I(\xi) = I(\xi_1) \), then \( \xi = \xi_1 \).

*Proof.* All assertions are easily checked (and in fact well known). \qed

It is easily checked that \( \xi \) is ambiguous if and only if \( I(\xi') = I(\xi) \), and in this case the \( \mathcal{O}_\Delta \)-regular ideal \( a = I(\xi) \) is also called *ambiguous*.

If \( \xi \) is reduced, then \( \xi \) is ambiguous if and only if \( \xi^+ = -\xi'^{-1} \). Indeed, if \( \xi^+ = -\xi'^{-1} \), then \( \xi' = \lfloor \xi \rfloor - \xi \), and therefore \( \xi + \xi' \in \mathbb{Z} \). Conversely, if \( \xi \) is reduced and ambiguous, then \( \xi - 1 < \xi + \xi' < \xi \), hence \( \lfloor \xi \rfloor = \xi + \xi' \) and \( \xi^+ = (\xi - \lfloor \xi \rfloor)^{-1} = -\xi'^{-1} \).

If \( \Delta > 0 \) is a discriminant, then an \( \mathcal{O}_\Delta \)-regular ideal \( a \subset \mathcal{O}_\Delta \) is called *reduced* if \( a = I(\xi) \) for some reduced quadratic irrational \( \xi \in \mathbb{R} \). If \( \xi \in \mathbb{R} \) is any quadratic irrational, then \( I(\xi) \) is reduced if and only if \( \xi + \lfloor -\xi' \rfloor > 1 \) (see [4, Lemma 2]). In particular, the unit ideal \( \mathcal{O}_\Delta = I(\omega_\Delta) \) is reduced.
3. Continued fractions and reduction

Our main reference for the classical theory of continued fractions is Perron’s book [16]. It is well known that every \( \xi \in \mathbb{R} \setminus \mathbb{Q} \) has a unique (simple) continued fraction

\[
\xi = [u_0, u_1, \ldots] = \lim_{n \to \infty} [u_0, u_1, \ldots, u_n],
\]

where \( u_0 \in \mathbb{Z}, \ u_i \in \mathbb{N} \) for all \( i \geq 1 \), and

\[
[u_0, u_1, \ldots, u_n] = u_0 + \cfrac{1}{u_1 + \cfrac{1}{u_2 + \cfrac{1}{\ddots + \cfrac{1}{u_n + \frac{1}{\eta_n}}}}}
= \frac{p_n}{q_n},
\]

such that \( p_n \in \mathbb{Z}, \ q_n \in \mathbb{N} \) and \( (p_n, q_n) = 1 \). The sequences \( (p_n)_{n \geq 2} \) of partial numerators of \( \xi \) and \( (q_n)_{n \geq 2} \) of partial denominators of \( \xi \) satisfy the recursion

\[
p_{-2} = 0, \ p_{-1} = 1, \text{ and } p_i = u_ip_{i-1} + p_{i-2} \text{ for all } i \geq 0, \]
\[
q_{-2} = 1, \ q_{-1} = 0, \text{ and } q_i = u_iq_{i-1} + q_{i-2} \text{ for all } i \geq 0.
\]

The numbers \( \xi_n = [u_n, u_{n+1}, \ldots] \) are called the complete quotients of \( \xi \). They are equivalent to \( \xi \) and satisfy the recursion formulas \( \xi_0 = \xi \) and \( \xi_{n+1} = \xi_n^+ \) for all \( n \geq 0 \).

A sequence \( (x_n)_{n \geq 0} \) is called ultimately periodic with period length \( l \geq 1 \) and pre-period length \( k \geq 0 \) if \( x_{n+l} = x_n \) for all \( n \geq k \), and \( k \) and \( l \) are minimal with this property. In this case, we write

\[
(x_n)_{n \geq 0} = (x_0, x_1, \ldots) = (x_0, x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{k+l-1}).
\]

If \( k = 0 \), then the sequence is called purely periodic.

**Proposition 3.1** (Periodicity Theorem). Let \( \xi \in \mathbb{R} \setminus \mathbb{Q} \), \( \xi = [u_0, u_1, \ldots] \) its continued fraction and \( (\xi_n)_{n \geq 0} \) its sequence of complete quotients.

1. For \( k \geq 0 \) and \( l \geq 1 \) the following assertions are equivalent:
   (a) The sequence \( (u_n)_{n \geq 0} \) is ultimately periodic with pre-period length \( k \) and period length \( l \).
   (b) The sequence \( (\xi_n)_{n \geq 0} \) is ultimately periodic with pre-period length \( k \) and period length \( l \).
   (c) The numbers \( \xi = \xi_0, \xi_1, \ldots, \xi_{k+l-1} \) are distinct, and \( \xi_{k+l} = \xi_k \).

2. The sequence \( (u_n)_{n \geq 0} \) is ultimately periodic if and only if \( \xi \) is a quadratic irrational, and it is purely periodic if and only if \( \xi \) is a reduced quadratic irrational.

3. Let \( \xi \) be a quadratic irrational, and suppose that \( (\xi_n)_{n \geq 0} \) has pre-period length \( k \) and period length \( l \). Then \( \{\xi_k, \xi_{k+1}, \ldots, \xi_{k+l-1}\} \) is the set of all reduced quadratic irrationals which are equivalent to \( \xi \).

We call \( l = l(\xi) \) the period length and \( (\xi_k, \xi_{k+1}, \ldots, \xi_{k+l-1}) \) the period of \( \xi \).

**Proof.** [16, §17 and Ch. III] \( \square \)

**Corollary 3.2.** Let \( \Delta > 0 \) be a discriminant, \( \xi \in \mathbb{R} \) a quadratic irrational of discriminant \( \Delta \), \( l = l(\xi) \) and \( \eta_1, \ldots, \eta_l \) the period of \( \xi \). Then \( I(\eta_1), \ldots, I(\eta_l) \) are distinct, and \( \{I(\eta_1), \ldots, I(\eta_l)\} \) is the set of all reduced ideals in the ideal class \( [I(\xi)] \in \mathcal{C}_\Delta \).
Proof. A subset \( a \subset K \) is an \( \mathcal{O}_\Delta \)-regular ideal lying in the ideal class \([I(\xi)]\) if and only if \( a = I(\eta) \) for some reduced quadratic irrational \( \eta \) equivalent to \( \xi \). Hence the assertion follows by the Propositions 3.1 and 2.5.

\[ \square \]

**Theorem 3.3.** Let \( \Delta = 4D + \sigma_\Delta > 0 \) be a discriminant, \( \omega_\Delta = [u_0, u_1, \ldots] \) the continued fraction of its basis number and \( l = \ell(\omega_\Delta) \). Then \( u_n = u_{n+l} \) for all \( n \geq 1 \), \( u_i = 2u_{i-1} - \sigma_\Delta \), \( u_{i+1} = u_i \) for all \( i \in [1, l-1] \), and therefore

\[
\omega_\Delta = \frac{\sigma + \sqrt{\Delta}}{2} = [u_0, u_1, u_2, \ldots, u_2, u_1, 2u_0 - \sigma_\Delta].
\]

Let \( (p_n)_{n \geq 2} \) be the sequence of partial numerators, \( (q_n)_{n \geq 2} \) the sequence of partial denominators and \( (\xi_n)_{n \geq 0} \) the sequence of complete quotients of \( \omega_\Delta \). For \( n \geq 0 \), \( \xi_n \) is of type \((a_n, b_n, c_n)\), where \( a_n \geq 1 \) and \( b_n = 2B_n - \sigma_\Delta \) for some \( B_n \in \mathbb{Z} \).

\( (\xi_1, \ldots, \xi_l) \) is the period of \( \omega_\Delta \), and \((I(\xi_1), \ldots, I(\xi_l))\) is the set of all reduced principal ideals of \( \mathcal{O}_\Delta \). In particular,

\[
(\xi_l, \ldots, \xi_l) = [2u_0 - \sigma_\Delta, u_1, u_2, \ldots, u_2, u_1] = \omega_\Delta + u_0 - \sigma, \quad \text{and} \quad I(\xi_l) = I(\omega_\Delta) = \mathcal{O}_\Delta.
\]

If \( \varepsilon_\Delta \) denotes the fundamental unit of discriminant \( \Delta \), then \( N(\varepsilon_\Delta) = (-1)^l \), and

\[
e_\Delta^n = (p_{l-1} - q_{l-1} \omega'_\Delta)^n = (p_n - q_n \omega'_\Delta)^n = p_{nl-1} - q_{nl-1} \omega'_\Delta \quad \text{for all} \quad m \in \mathbb{N}_0.
\]

If \( \Delta \) has a prime divisor \( q \equiv 3 \mod 4 \), then \( l \) is even and \( N(\varepsilon_\Delta) = 1 \).

1. For all \( n \geq 0 \), the following relations hold:

   (a) \( B_n + B_{n+1} = a_n u_n + \sigma_\Delta \).
   (b) \( p_{n-1} = B_n q_{n-1} + a_n q_{n-2} \).
   (c) \( Dq_{n-1} = (B_n - \sigma_\Delta) p_{n-1} + a_n p_{n-2} \).
   (d) \( 4(-1)^n a_n = (2p_{n-1} - \sigma_\Delta q_{n-1})^2 - \Delta q_n^2 = 4N(p_{n-1} - q_{n-1} \omega_\Delta) \).
   (e) \( (-1)^n a_n = p_{n-1}^2 - \sigma_\Delta p_{n-1} q_{n-1} - Dq_n^2 \).

2. If \( i \geq -1 \) and \( n \geq 0 \), then \( p_i + nt_1 - q_i + nt_2 \omega'_\Delta = (p_i - q_i \omega'_\Delta)(p_{i-1} - q_{i-1} \omega'_\Delta)^n \).

3. If \( l \) is odd, then \( \xi_l \) is the only ambiguous number in the period of \( \omega_\Delta \), and \( \mathcal{O}_\Delta \) is the only reduced ambiguous principal ideal of \( \mathcal{O}_\Delta \).

4. Let \( l = 2k \) be even. Then \( \xi_{2k} \) and \( \xi_l \) are the only ambiguous numbers in the period of \( \omega_\Delta \), \( (p_k - q_k \omega'_\Delta)^2 = a_k \varepsilon_\Delta \), \( 2B_k = a_k + \sigma_\Delta \),

\[
a_k | (2p_k - q_k - \Delta) \quad \text{if} \quad \sigma_\Delta = 1, \quad \text{and} \quad a_k | (2p_k - D) \quad \text{if} \quad \sigma_\Delta = 0.
\]

In particular, \( \mathcal{O}_\Delta \) and \( I(\xi_{2k}) \) are the only reduced ambiguous principal ideals of \( \mathcal{O}_\Delta \).

Proof. We prove 3. and 4. The other assertions can be either found in [16, §§ 20, 27 and 30] or easily derived from the investigations there. The assertion concerning reduced principal ideals follows by Corollary 3.2.

If \( i \in [1, l] \), then

\[
\xi_i = [u_0, u_1, \ldots, u_i, u_1, \ldots, u_i] = [u_{i+1}, \ldots, u_i, u_1, \ldots, u_i] = \xi_{i+1}
\]

(see [16, §23]), and by Proposition 2.5.3 it follows that \( \xi_i \) is ambiguous if and only if \( \xi_{i+1} = -\varepsilon_i^{-1} = \xi_{i+1} \). In particular, \( \xi_i \) is ambiguous. If \( i \in [1, l] \), then \( \xi_i \) is ambiguous and only if \( i = l \) or \( i = l - 1 \), which is, and if and only if \( l = 2i \). This proves 3. and the first assertion of 4.

Assume now that \( l = 2k \). Then \( \xi_{2k+1} = -\varepsilon_i^{-1} \), and therefore

\[
-1 = \xi_{k+1} \xi_l = \frac{b_{k+1} + \sqrt{\Delta}}{a_{k+1}} \frac{b_k - \sqrt{\Delta}}{a_k} = \frac{b_kb_{k+1} - \Delta + (b_k - b_{k+1})\sqrt{\Delta}}{4a_ka_{k+1}}.
\]
Theorem 4.1. Let $\Delta \in \mathbb{N}$ be a discriminant.

1. Suppose that $\Delta = 4D$,
   - $c \in \{1, 2\}$ if $8 \nmid D$, and $c = 1$ if $8 \nmid D$;
   - $t \in \{1, 2\}$ if $D \equiv 3 \mod 4$, and $t = 1$ if $D \not\equiv 3 \mod 4$;
   - $D = c^2dd^*$, where $d, d^* \in \mathbb{N}$ and $(d, d^*) = 1$,
   and set
     \[
     j = \begin{cases} 
     [d, \sqrt{D}] & \text{if } ct = 1, \\
     [2d, d + \sqrt{D}] & \text{if } t = 2, \\
     [4d, 2d + \sqrt{D}] & \text{if } c = 2.
     \end{cases}
     \]
   (a) $j$ is an $O_\Delta$-regular ambiguous ideal of $O_\Delta$ satisfying $R_\Delta(j) = c^2dt$, and every $O_\Delta$-regular ambiguous ideal of $O_\Delta$ is of this form.
   (b) $j$ is reduced if and only if $d < d^*$, and $j$ is a principal ideal of $O_\Delta$ if and only if there exist $x, y \in \mathbb{Z}$ such that $|dx^2 - d^*y^2| = t$ and $(c, xy) = 1$.

4. Main Results

Proof. Let $\Delta \in \mathbb{N}$ be a discriminant.

1. Suppose that $\Delta = 4D$,
   - $c \in \{1, 2\}$ if $8 \nmid D$, and $c = 1$ if $8 \nmid D$;
   - $t \in \{1, 2\}$ if $D \equiv 3 \mod 4$, and $t = 1$ if $D \not\equiv 3 \mod 4$;
   - $D = c^2dd^*$, where $d, d^* \in \mathbb{N}$ and $(d, d^*) = 1$,
   and set
     \[
     j = \begin{cases} 
     [d, \sqrt{D}] & \text{if } ct = 1, \\
     [2d, d + \sqrt{D}] & \text{if } t = 2, \\
     [4d, 2d + \sqrt{D}] & \text{if } c = 2.
     \end{cases}
     \]
   (a) $j$ is an $O_\Delta$-regular ambiguous ideal of $O_\Delta$ satisfying $R_\Delta(j) = c^2dt$, and every $O_\Delta$-regular ambiguous ideal of $O_\Delta$ is of this form.
   (b) $j$ is reduced if and only if $d < d^*$, and $j$ is a principal ideal of $O_\Delta$ if and only if there exist $x, y \in \mathbb{Z}$ such that $|dx^2 - d^*y^2| = t$ and $(c, xy) = 1$.


(b) Let $x, y \in \mathbb{Z}$ be such that $|dx^2 - d^*y^2| = t$ and $(c, xy) = 1$. Then $j = (cdx + y\sqrt{D})\mathcal{O}_\Delta$.

2. Suppose that $\Delta \equiv 1 \pmod{4}$ and $\Delta = dd^*$, where $d, d^* \in \mathbb{N}$ and $(d, d^*) = 1$, and set $j = \left[d, \frac{d + \sqrt{\Delta}}{2}\right]$.

(a) $j$ is an $\mathcal{O}_\Delta$-regular ambiguous ideal of $\mathcal{O}_\Delta$ satisfying $\mathfrak{n}_\Delta(j) = d$, and every $\mathcal{O}_\Delta$-regular ambiguous ideal of $\mathcal{O}_\Delta$ is of this form.

(j) is reduced if and only if $d < d^*$, and $j$ is a principal ideal of $\mathcal{O}_\Delta$ if and only if there exist $x, y \in \mathbb{Z}$ such that $|dx^2 - d^*y^2| = 4$.

(b) Let $x, y \in \mathbb{Z}$ such that $|dx^2 - d^*y^2| = 4$. Then $j = \frac{dx + y\sqrt{\Delta}}{2}\mathcal{O}_\Delta$.

Proof. 1.(a) By [4, Proposition 1] it follows that $j \subset \mathcal{O}_\Delta$ is an $\mathcal{O}_\Delta$-regular ambiguous ideal, every $\mathcal{O}_\Delta$-regular ambiguous ideal is of this form, and $j$ is reduced if and only if $d < d^*$. By Proposition 2.3.1, $\mathfrak{n}_\Delta(j) = c^2dt$.

Let now $j$ be principal, say $j = (u + y\sqrt{D})\mathcal{O}_\Delta$, where $u, y \in \mathbb{Z}$ and $(u, y) = 1$.

If $ct = 1$, then $D = dd^*$, and $u + y\sqrt{D} \in [d, \sqrt{D}]$ implies $u = dx$ for some $x \in \mathbb{Z}$. Since $d = \mathfrak{n}_\Delta(j) = |\mathcal{N}(dx + y\sqrt{D})| = |d^2x^2 - dd^*y^2|$, it follows that $|dx^2 - d^*y^2| = 1$.

If $t = 2$, then $u + y\sqrt{D} \in [2d, d + \sqrt{D}]$ implies $u + y\sqrt{D} = 2dv + (d + \sqrt{D})w$ for some $v, w \in \mathbb{Z}$, and if $x = 2v + w$, then $u = dx$ and $y = w$. Since $D = dd^*$, it follows that $2d = \mathfrak{n}_\Delta(j) = |\mathcal{N}(dx + y\sqrt{D})| = |d^2x^2 - dd^*y^2|$, which implies $|dx^2 - d^*y^2| = 2$. If $c = 2$, then $u + y\sqrt{D} \in [4d, 2d + \sqrt{D}]$ implies that there exist $v, w \in \mathbb{Z}$ such that $u + y\sqrt{D} = 4dv + (2d + \sqrt{D})w$. If $x = 2v + w$, then $u = 2dx$, $y = w$, and $x \equiv xy$. Since $D = 4dd^*$, it follows that $4d = \mathfrak{n}_\Delta(j) = |\mathcal{N}(2dx + y\sqrt{D})| = |4d^2x^2 - 4dd^*y^2|$, which implies $|dx^2 - d^*y^2| = 1$.

The converse follows by (b).

(b) If $ct = 1$, then obviously $dx + y\sqrt{D} \in j$, hence $(dx + y\sqrt{D})\mathcal{O}_\Delta \subset j$, and equality holds, since $\mathfrak{n}_\Delta((dx + y\sqrt{D})\mathcal{O}_\Delta) = |\mathcal{N}(dx + y\sqrt{D})| = |d^2x^2 - dd^*y^2| = d = \mathfrak{n}_\Delta(j)$.

If $t = 2$, then $D = dd^* \equiv 3 \pmod{4}$, hence $2 \nmid xy$, and $x - y = 2u$ for some $u \in \mathbb{Z}$. Now we obtain $dx + y\sqrt{D} = 2du + (d + \sqrt{D})y \in j$, hence $(dx + y\sqrt{D})\mathcal{O}_\Delta \subset j$, and equality holds, since $\mathfrak{n}_\Delta((dx + y\sqrt{D})\mathcal{O}_\Delta) = |\mathcal{N}(dx + y\sqrt{D})| = |d^2x^2 - dd^*y^2| = 2d = \mathfrak{n}_\Delta(j)$.

If $c = 2$ and $2 \nmid xy$, then $D = 4dd^*$ and $x - y = 2u$ for some $u \in \mathbb{Z}$, which implies $2dx + y\sqrt{D} = 4du + (2d + \sqrt{D})y \in j$. Hence we obtain $(dx + y\sqrt{D})\mathcal{O}_\Delta \subset j$, and equality holds, since $\mathfrak{n}_\Delta((2dx + y\sqrt{D})\mathcal{O}_\Delta) = |\mathcal{N}(2dx + y\sqrt{D})| = |4d^2x^2 - 4dd^*y^2| = 4d = \mathfrak{n}_\Delta(j)$.

2.(a) By [4, Proposition 1] it follows that $j \subset \mathcal{O}_\Delta$ is an $\mathcal{O}_\Delta$-regular ambiguous ideal, every $\mathcal{O}_\Delta$-regular ambiguous ideal is of this form, and $j$ is reduced if and only if $d < d^*$. By Proposition 2.3.1, $\mathfrak{n}_\Delta(j) = d$.

Let now $j$ be principal, say $j = \frac{u + y\sqrt{\Delta}}{2}\mathcal{O}_\Delta$, where $u, y \in \mathbb{Z}$ and $u \equiv y \pmod{2}$. Then $\frac{u + y\sqrt{\Delta}}{2} \in j$ implies $\frac{u + y\sqrt{\Delta}}{2} = dv + \frac{d + \sqrt{\Delta}}{2}w$ for some $v, w \in \mathbb{Z}$. Hence it follows
that \( u = dx \), where \( x = 2v + w \), \( w = y \), \( j = \frac{dx + y\sqrt{x}}{2} \), \( d = \mathcal{N}(j) = \frac{|dx^2 - dd^*y^2|}{4} \), and therefore \( |dx^2 - dd^*y^2| = 4 \).

(b) If \( |dx^2 - d^*y^2| = 4 \), then \( x \equiv y \text{ mod } 2 \), \( \frac{dx + y\sqrt{x}}{2} = a \frac{x-y}{2} + \frac{dx + y\sqrt{x}}{2} y \in j \), hence \( \frac{dx + y\sqrt{x}}{2} \mathcal{O}_\Delta \subset j \), and equality holds, since
\[
\mathcal{N}\left(\frac{dx + y\sqrt{x}}{2}\right) = \left|\mathcal{N}\left(\frac{dx + y\sqrt{x}}{2}\right)\right| = \frac{|dx^2 - dd^*y^2|}{4} = d = \mathcal{N}(j). \quad \square
\]

The following remark addresses the diophantine equation \( |dx^2 - d^*y^2| = 1 \) if \( c = 2 \) and \( 2 \nmid xy \).

**Remark 4.2.** Let \( D \in \mathbb{N} \) be not a square, \( 8 \nmid D \) and \( D = 4dd^* \), where \( d, d^* \in \mathbb{N} \) and \( (d, d^*) = 1 \). Let \( x, y \in \mathbb{Z} \) be such that \( |dx^2 - d^*y^2| = 1 \).

1. If \( 2 \nmid x \), then \( (2dx + y\sqrt{D}) \mathcal{O}_{4D} = [4d, \sqrt{D}] \).
   Indeed, if \( x = 2x_1 \), where \( x_1 \in \mathbb{Z} \), then \( |4dx^2_1 - d^*y^2| = 1 \) and \( D = (4d)d^* \). Hence the assertion follows by Theorem 4.1.2(a).

2. If \( 2 \nmid y \) and \( y = 2y_1 \), then \( (dx + y_1\sqrt{D}) \mathcal{O}_{4D} = [d, \sqrt{D}] \). Indeed, in this case \( |dx^2 - 4d^*y_1^2| = 1 \) and \( D = (4d^*) \). Hence again the assertion follows by Theorem 4.1.2(a).

**Theorem 4.3.** Let \( D \in \mathbb{N} \) be not a square and \( l = l(\sqrt{D}) \) the period length of \( \sqrt{D} \). Let \( \mathcal{L}(D) \) be the set of all quadruples \( (d, d^*, t, \sigma) \), where

- \( d, d^* \in \mathbb{N} \) and \( (d, d^*) = 1 \);
- \( D = c^2d^*d \), where \( c \in \{1, 2\} \) if \( 8 \nmid D \), and \( c = 1 \) if \( 8 \nmid D \);
- \( t \in \{1, 2\} \) if \( D \equiv 3 \text{ mod } 4 \), and \( t = 1 \) if \( D \equiv 3 \text{ mod } 4 \);
- \( \sigma \in \{\pm 1\} \);
- there exist \( x, y \in \mathbb{Z} \) such that \( dx^2 - d^*y^2 = \sigma t \) and \( (c, xy) = 1 \).

Then \( |\mathcal{L}(D)| = 4 \), and the structure of \( \mathcal{L}(D) \) is as follows.

1. If \( l \) is odd, then \( \mathcal{L}(D) = \{(1, D, 1, \pm 1), (D, 1, 1, \pm 1)\} \).
2. If \( l = 2k \) is even, then
   \[
   \mathcal{L}(D) = \{(1, D, 1, 1), (D, 1, 1, -1), (d, d^*, t, \sigma), (d^*, d, t, -\sigma)\},
   \]
   where \( 1 \leq d < d^* \) and \( cdt \neq 1 \).

3. Let \( l = 2k \) be even and \( (d, d^*, t, \sigma) \in \mathcal{L}(D) \) such that \( 1 \leq d < d^* \) and \( cdt \neq 1 \). Then \( \sigma = (-1)^k \). If \( (p_n)_{n \geq 2} \) denotes the sequence of partial numerators and \( (q_n)_{n \geq 2} \) the sequence of partial denominators of \( \sqrt{D} \), then
   \[
   p_{k-1}^2 - Dq_{k-1}^2 = (-1)^k c^2 dt, \quad c^2 dt \varepsilon_{4D} = (p_{k-1} + q_{k-1}\sqrt{D})^2,
   \]
   \[
   c^2 dt | 2p_{k-1} \quad \text{and} \quad \varepsilon_{4D} = (-1)^k + 2 \frac{2d^*}{c^2 dt} \frac{q_{k-1}^2}{c^2 dt} .
   \]

**Proof.** Note that \( (d, d^*, t, \sigma) \in \mathcal{L}(D) \) holds if and only if \( (d^*, d, t, -\sigma) \in \mathcal{L}(D) \).

1. If \( l \) is odd, then Theorem 3.3 implies that \( \mathcal{N}(\varepsilon_{4D}) = -1 \), and \( \mathcal{O}_{4D} \) is the only reduced ambiguous principal ideal in \( \mathcal{O}_{4D} \). Hence we obtain \( \mathcal{N}(\mathcal{O}_{4D}^*) = \{\pm 1\}, \{(1, D, 1, \pm 1), (D, 1, 1, \pm 1)\} \subset \mathcal{L}(D), \ D \equiv 3 \text{ mod } 4 \) and \( t = 1 \). Assume now that there exists some \( (d, d^*, 1, \sigma) \in \mathcal{L}(D) \) such that \( 1 \leq d < d^* \) and \( cd > 1 \). Then Theorem 4.1.1 implies the existence of some reduced ambiguous principal ideal \( j \subset \mathcal{O}_{4D} \) such that \( \mathcal{N}(j) = c^2d > 1 \), a contradiction.

2. Let \( l = 2k \) be even. Then Theorem 3.3 implies \( \mathcal{N}(\varepsilon_{4D}) = 1 \) and therefore \( \mathcal{N}(\mathcal{O}_{4D}^*) = \{1\} \). We prove first:
A. If \((d, d^*, t, \sigma) \in \mathcal{L}(D)\), then \((d, d^*, t, -\sigma) \notin \mathcal{L}(D)\).

Proof of A. Assume to the contrary that there is some \((d, d^*, t, \sigma) \in \mathcal{L}(D)\) such that \((d, d^*, t, -\sigma) \in \mathcal{L}(D)\), and let \(x, y, x_1, y_1 \in \mathbb{Z}\) be such that \(dx^2 - d^*y^2 = \sigma t\), \(dx_1^2 - d^*y_1^2 = -\sigma t\) and \((c, xy) = (c, x_1y_1) = 1\). By Theorem 4.1.1(b) it follows that \((cdx + y\sqrt{D})\mathcal{O}_{4D} = (cdx_1 + y_1\sqrt{D})\mathcal{O}_{4D}\), and therefore \(cdx_1 + y_1\sqrt{D} = \varepsilon(cdx + y\sqrt{D})\) for some \(\varepsilon \in \mathcal{O}_{4D}^\times\). Taking norms, we obtain

\[-c^2\sigma dt = \mathcal{N}(cdx_1 + y_1\sqrt{D}) = \mathcal{N}(\varepsilon)\mathcal{N}(cdx + y\sqrt{D}) = \mathcal{N}(\varepsilon)c^2\sigma dt,
\]

and therefore \(\mathcal{N}(\varepsilon) = -1\), a contradiction. \(\Box\) [A]

By Theorem 3.3.4, \(\mathcal{O}_{4D}\) contains precisely one reduced ambiguous principal ideal \(j\) distinct from the unit ideal, and by Theorem 4.1.1 this ideal gives rise to an equation \(|dx^2 - d^*y^2| = t\), where \(d, d^* \in \mathbb{N}\) and \(x, y \in \mathbb{Z}\) are such that \(1 \leq d < d^*\), \((d, d^*) = 1\), \(D = c^2d^*\), \(c dt > 1\) and \((c, xy) = 1\). Hence there exists some \(\sigma \in \{\pm 1\}\) such that \((d, d^*, t, \sigma) \in \mathcal{L}(D)\). To prove uniqueness, we must show:

B. If \((d_1, d_1^*, t_1, \sigma_1), (d_2, d_2^*, t_2, \sigma_2) \in \mathcal{L}(D)\), \(1 \leq d_1 < d_1^*\), \(c_1t_1d_1 > 1\), and \(1 \leq d_2 < d_2^*\), \(c_2t_2d_2 > 1\), then \((d_1, d_1^*, t_1, \sigma_1) = (d_2, d_2^*, t_2, \sigma_2)\).

Proof of B. For \(i \in \{1, 2\}\), suppose that \((d_i, d_i^*, t_i, \sigma_i) \in \mathcal{L}(D)\), \(1 \leq d_i < d_i^*\) and \(c_it_id_i > 1\), where \(c_i \in \{1, 2\}\) are such that \(D = c_i^2d_id_i^*\). By Theorem 4.1 there exist \(x_i, y_i \in \mathbb{Z}\) such that \((c_i, x_i, y_i) = 1\), and

\[j_i = (c_1d_i x_i + y_i\sqrt{D})\mathcal{O}_{4D} = \begin{cases} [d_i, \sqrt{D}] & \text{if } c_i t_i = 1, \\ [2d_i, d_i + \sqrt{D}] & \text{if } t_i = 2, \\ [4d_i, 2d_i + \sqrt{D}] & \text{if } c_i = 2 \end{cases}\]

is a reduced ambiguous ideal distinct from the unit ideal in the principal class of \(\mathcal{O}_{4D}\). Hence it follows that \(j_1 = j_2\), and in particular \(\mathfrak{N}_{4D}(j_1) = \mathfrak{N}_{4D}(j_2)\), which implies \(c_1^2t_1d_1 = c_2^2t_2d_2\).

If \(t_1 = 2\), then \(D \equiv 3 \mod 4\), hence \(c_1 = c_2 = 1\). Since \(2d_1 = t_2d_2\) and \(d_2\) is odd, it follows that \(t_2 = 2\), \(d_1 = d_2\), \(d_1^* = d_2^*\), and A implies \(\sigma_1 = \sigma_2\). By symmetry, we may now assume that \(t_1 = t_2 = 1\).

Assume now that \(c_1 \neq c_2\), say \(c_1 = 2\) and \(c_2 = 1\). Then we obtain \(4d_1 = d_2\) and \([4d_1, 2d_1 + \sqrt{D}] = [d_1, \sqrt{D}] = [4d_1, \sqrt{D}]\), a contradiction. Hence it follows that \(c_1 = c_2\), \(d_1 = d_2\), \(d_1^* = d_2^*\), and A implies \(\sigma_1 = \sigma_2\). \(\Box\) [B]

3. Let again \(l = 2k\) be even and \((d, d^*, t, \sigma) \in \mathcal{L}(D)\), where \(1 \leq d < d^*\) and \(c dt > 1\). Let \(x, y \in \mathbb{Z}\) be such that \(dx^2 - d^*y^2 = \sigma t\). Then \(j = (cdx + y\sqrt{D})\mathcal{O}_{4D}\) is a reduced principal ideal of \(\mathcal{O}_{4D}\) such that \(\mathfrak{N}_{4D}(j) = c^2dt\) by Theorem 4.1.1.

Let \((\xi_n)_{n \geq 0}\) be the sequence of complete quotients of \(\sqrt{D} = \omega_{4D}\), and for \(n \geq 0\) let \((a_n, b_n, c_n)\) be the type of \(\xi_n\). By Theorem 3.3, \(I(\xi) = \mathcal{O}_{4D}\) and \(I(\xi)\) are the only reduced ambiguous principal ideals of \(\mathcal{O}_{4D}\). Hence it follows that \(j = I(\xi_k)\), and \(\mathfrak{N}_{4D}(j) = |\mathcal{N}(\xi_k)| = c^2dt = a_k\). By Theorem 3.3 we obtain

\[\mathcal{N}(\xi_k) = p_{k-1}^2 - c^2d^*q_{k-1}^2 = (-1)^k c^2dt, \quad c^2dt \varepsilon_{4D} = (p_{k-1} + q_{k-1}\sqrt{D})^2\]

and

\[\varepsilon_{4D} = \frac{p_{k-1}^2 + q_{k-1}^2}{c^2dt}D + \frac{2p_{k-1}q_{k-1}\sqrt{D}}{c^2dt} = (-1)^k \left( \frac{2d^*}{l} q_{k-1}^2 + \frac{2p_{k-1}q_{k-1}}{c^2dt} \right) \sqrt{D}\]

(note that \(c^2dt | 2p_{k-1}\) by Theorem 3.3). It remains to prove that \(\sigma = (-1)^k\).

CASE 1: \(c = 2\). Then \(8 | D, \ t = 1, \ a_k = 4d^*2p_{k-1},\) and therefore \(p_{k-1} = 2dx_1,\) where \(x_1 \in \mathbb{Z}\). If \(y_1 = q_{k-1},\) then \((p_{k-1}, q_{k-1}) = 1\) implies \(2 | y_1,\) and it follows that \(dx_1^2 - d^*y_1^2 = (-1)^k\). If \(2 | x_1,\) then \((d, d^*, 1, (-1)^k) \in \mathcal{L}(D),\) hence \(\sigma = (-1)^k,\) and we are done.
We assert that the case $2 \mid x_1$ cannot occur. Indeed, if $2 \mid x_1$, then $x_1 = 2x_2$, where $x_2 \in \mathbb{Z}$, and $4dx_2^2 - d^*y_1^2 = (-1)^k$. But this implies that $(4d, d^*, 1, (-1)^k) \in \mathcal{L}(D)$, hence either $(4d, d^*, 1, (-1)^k) = (d^*, d^*, 1, \sigma)$ or $(4d, d^*, 1, (-1)^k) = (d^*, 4d, 1, -\sigma)$, and both relations are impossible.

**CASE 2**: $c = 1$ and $2 \nmid d$ (in particular, this occurs if $D \equiv 3 \pmod{4}$). As $a_k = td|2pk-1$, it follows that $d \mid pk-1$, say $pk-1 = dx_1$, where $x_1 \in \mathbb{Z}$. If $y_1 = q_k-1$, then $dx_1^2 - d^*y_1^2 = (-1)^k$, hence $(d, d^*, t, (-1)^k) \in \mathcal{L}(D)$ and therefore $\sigma = (-1)^k$.

**CASE 3**: $c = 1$ and $d = 2d_0$, where $d_0 \in \mathbb{N}$ and $2 \nmid d_0$. Since $a_k = 2d_0|2pk-1$, we obtain $p_k-1 = d_0x_1$, where $x_1 \in \mathbb{Z}$. If $y_1 = q_k-1$, then $d_0x_1^2 - 2d^*y_1^2 = 2(-1)^k$, which implies that $2 \mid x_1$. If $x_1 = 2x_2$, where $x_2 \in \mathbb{Z}$, then $dx_2^2 - d^*y_2^2 = (-1)^k$, hence $(d, d^*, 1, (-1)^k) \in \mathcal{L}(D)$ and therefore $\sigma = (-1)^k$.

**CASE 4**: $c = 1$ and $d = 4d_0$, where $d_0 \in \mathbb{N}$ and $4 \nmid d_0$. If $D_0 = 4d_0\sigma$, then $\sigma = dx_1^2 - d^*y_1^2 = d_0(2x_1^2 - d^*y_2^2)$ implies that $(d_0, d^*, 1, \sigma) \in \mathcal{L}(D_0)$. Since $a_k = 4^k d_0^\sigma | 2pk-1$, it follows that $2^\sigma d_0 | 2^{2\sigma-1} d_0 | p_k-1$, and we set $p_k-1 = 2d_0x_1$, where $x_1 \in \mathbb{Z}$ and $y_1 = q_k-1$. Then $(p_k-1, q_k-1) = 1$ implies $2 \nmid y_1$. It follows that $d_0x_1^2 - d^*y_1^2 = (-1)^k$, and therefore $(d_0, d^*, 1, (-1)^k) \in \mathcal{L}(D_0)$. If $d_0 > 1$, then $l(\sqrt{D_0})$ is even, and $B$ (applied with $D_0$ instead of $D$) yields $\sigma = (-1)^k$. If $d_0 = 1$, then $\sigma \equiv -d^* \pmod{4}$. Since $2 \nmid d^*y_1^2$, it follows that $2 \mid x_1$, hence $(-1)^k \equiv -d^* \pmod{4}$, and thus again $\sigma = (-1)^k$. □

**Theorem 4.4.** Let $\Delta \in \mathbb{N}$ be not a square, $\Delta \equiv 1 \pmod{4}$, $l = l(\omega_\Delta)$ the period length of $\omega_\Delta$ and $l^* = l(\sqrt{\Delta})$ the period length of $\sqrt{\Delta}$. Let $\mathcal{L}_0(\Delta)$ be the set of all triples $(d, d^*, \sigma)$ such that

\[d, d^* \in \mathbb{N}, \ (d, d^*) = 1, \ \Delta = dd^*, \ \sigma \in \{\pm 1\}, \text{ and there exist } x, y \in \mathbb{Z} \text{ such that} \]

\[dx^2 - d^*y^2 = 4x.\]

Then $|\mathcal{L}_0(\Delta)| = 4$, and the structure of $\mathcal{L}_0(\Delta)$ is as follows.

1. If $l$ is odd, then $\mathcal{L}_0(\Delta) = \{(1, \Delta, \pm 1), (\Delta, 1, \pm 1)\}$.

2. If $l = 2k$ is even, then

$\mathcal{L}_0(\Delta) = \{(1, \Delta, 1), (\Delta, 1, -1), (d, d^*, \sigma), (d^*, d, -\sigma)\}$,

where $(d, d^*, \sigma) \notin \{(1, \Delta, -1), (\Delta, 1, 1)\}$.

3. Let $l = 2k$ be even and $(d, d^*, \sigma) \in \mathcal{L}_0(\Delta)$ such that $1 < d < d^*$. Then $\sigma = (-1)^k$. Let $(p_n)_{n \geq 2}$ be the sequence of partial numerators and $(q_n)_{n \geq 2}$ the sequence of partial denominators of $\omega_\Delta$. Then $d | 2p_k - q_k$, and $2p_k - q_k = ds_k$, then

\[dx_k^2 - d^*q_k^2 = 4(-1)^k, \quad d \varepsilon_\Delta = \left(\frac{ds_k + q_k\sqrt{\Delta}}{2}\right)^2,\]

and

\[\varepsilon_\Delta = (-1)^k + \frac{d^*q_k^2 + q_k^{-1}s_k\sqrt{\Delta}}{2}.\]

Moreover, $\varepsilon_\Delta$ has half-integer coordinates if and only if there exist $x, y \in \mathbb{Z}$ such that $|dx^2 - d^*y^2| = 4$ and $(x, y) = 1$.

4. If $(d, d^*, \sigma) \in \mathcal{L}_0(\Delta)$, then there exist $x_1, y_1 \in \mathbb{Z}$ such that $dx_1^2 - d^*y_1^2 = \sigma$. In particular, if $l$ is even, then $l \equiv l^* \pmod{4}$.

**Proof.** Note that $(d, d^*, \sigma) \in \mathcal{L}_0(\Delta)$ holds if and only if $(d^*, d, -\sigma) \in \mathcal{L}_0(\Delta)$.

1. If $l$ is odd, then Theorem 3.3 implies that $N(\varepsilon_\Delta) = -1$, and $\mathcal{O}_\Delta$ is the only reduced ambiguous principal ideal in $\mathcal{O}_\Delta$. Hence $N(\mathcal{O}_\Delta^\ast) = \{\pm 1\}$, and therefore $\{(1, \Delta, \pm 1), (\Delta, 1, \pm 1)\} \subset \mathcal{L}_0(\Delta)$. Assume that there is some $(d, d^*, \sigma) \in \mathcal{L}_0(\Delta)$.
such that $1 < d < d^*$. Then Theorem 4.1.2 implies the existence of some reduced ambiguous principal ideal $j \in \mathcal{O}_{4D}$ such that $\mathcal{R}_{4D}(j) = d > 1$, a contradiction.

2. Let $l = 2k$ be even. Then Theorem 3.3 implies $\mathcal{N}(\mathcal{O}_\Delta^0) = \{1\}$. We prove first:

A. If $(d, d^*, \sigma) \in \mathcal{L}_0(\Delta)$, then $(d, d^*, -\sigma) \notin \mathcal{L}_0(\Delta)$.

Proof of A. Assume to the contrary that there is some $(d, d^*, \sigma) \in \mathcal{L}_0(\Delta)$ such that $(d, d^*, -\sigma) \in \mathcal{L}_0(\Delta)$, and let $x, y, x_1, y_1 \in \mathbb{Z}$ be such that $dx^2 - d^*y^2 = 4\sigma$ and $dx_1^2 - d^*y_1^2 = -4\sigma$. By Theorem 4.1.2 it follows that

$$\left[ d, \frac{d + \sqrt{\Delta}}{2} \right] = \frac{dx + y\sqrt{\Delta}}{2} \mathcal{O}_\Delta = \frac{dx_1 + y_1\sqrt{\Delta}}{2} \mathcal{O}_\Delta$$

and therefore $dx_1 + y_1\sqrt{\Delta} = \varepsilon(dx + y\sqrt{\Delta})$ for some $\varepsilon \in \mathcal{O}_\Delta^\times$. Taking norms, we obtain $-\varepsilon dx = \mathcal{N}(dx_1 + y_1\sqrt{\Delta}) = \mathcal{N}(\varepsilon)\mathcal{N}(dx + y\sqrt{\Delta}) = 4\mathcal{N}(\varepsilon) dx$ and therefore $\mathcal{N}(\varepsilon) = -1$, a contradiction. $\Box$ [A]

By Theorem 3.3.4, $\mathcal{O}_{4D}$ contains precisely one reduced ambiguous principal ideal $j$ distinct from the unit ideal, and by Theorem 4.1.2 this ideal gives rise to an equation $|dx^2 - d^*y^2| = 4$, where $d, d^* \in \mathbb{N}$, $1 < d < d^*$, $(d, d^*) = 1$, $\Delta = dd^*$ and $x, y \in \mathbb{Z}$. Hence there exists some $\sigma \in \{\pm 1\}$ such that $(d, d^*, \sigma) \in \mathcal{L}_0(D)$. To prove uniqueness, we must show:

B. If $(d_1, d_1^*, \sigma_1), (d_2, d_2^*, \sigma_2) \in \mathcal{L}_0(\Delta)$, $1 < d_1 < d_1^*$ and $1 < d_2 < d_2^*$, then $(d_1, d_1^*, \sigma_1) = (d_2, d_2^*, \sigma_2)$.

Proof of B. For $i \in \{1, 2\}$, suppose that $(d_i, d_i^*, \sigma_i) \in \mathcal{L}_0(\Delta)$. By Theorem 4.1.2 there exist $x_i, y_i \in \mathbb{Z}$ such that

$$j_i = \frac{d_i x_i + y_i \sqrt{\Delta}}{2} \mathcal{O}_\Delta = \left[ d_i, \frac{d_i + \sqrt{\Delta}}{2} \right]$$

is a reduced ambiguous principal ideal distinct from the unit ideal of $\mathcal{O}_\Delta$. Therefore it follows that $j_1 = j_2$, in particular $d_1 = d_2$, hence $d_1^* = d_2^*$, and A implies $\sigma_1 = \sigma_2$.

3. Let again $l = 2k$ be even and $(d, d^*, \sigma) \in \mathcal{L}_0(\Delta)$, where $1 < d < d^*$. Let $x, y \in \mathbb{Z}$ be such that $dx^2 - d^*y^2 = 4\sigma$. Then

$$j = \left( \frac{dx + y\sqrt{\Delta}}{2} \right) \mathcal{O}_\Delta = \left[ d, \frac{d + \sqrt{\Delta}}{2} \right]$$

is a reduced principal ideal of $\mathcal{O}_{\Delta}$ such that $\mathcal{R}(j) = d$ by Theorem 4.1.2.

Let $(\xi_n)_{n \geq 0}$ be the sequence of complete quotients of $\omega_\Delta$, and for $n \geq 0$ let $(a_n, b_n, c_n)$ be the type of $\xi_n$. By Theorem 3.3, $I(\xi_i) = \mathcal{O}_\Delta$ and $I(\xi_k)$ are the only reduced ambiguous principal ideals of $\mathcal{O}_{4D}$. Hence it follows that $j = I(\xi_k)$ and $\mathcal{R}_{4D}(j) = |N(\xi_k)| = d = a_k$. Since $a_k | (2p_{k-1} - q_{k-1}, \Delta)$ by Theorem 3.3, there exists some $s_k \in \mathbb{Z}$ such that $2p_{k-1} - q_{k-1} = ds_k$, and then $4(-1)^k d = d_0^2 s_k^2 - d\sigma q_{k-1}^2$, which implies $ds_k^2 - d\sigma q_{k-1}^2 = 4(-1)^k$. Moreover,

$$d\varepsilon_\Delta = \left( \frac{ds_k + q_{k-1}\sqrt{\Delta}}{2} \right)^2 \text{ and } \varepsilon_\Delta = (-1)^k + \frac{d\sigma q_{k-1}^2 + q_{k-1} s_k \sqrt{\Delta}}{2}.$$

In particular, $(d, d^*, (-1)^k) \in \mathcal{L}_0(\Delta)$, and by A it follows that $(d, d^*, \sigma) \in \mathcal{L}_0(\Delta)$ if and only if $\sigma = (-1)^k$.

The above formulas show that $\varepsilon_\Delta$ has half-integral coordinates if and only if $2 \nmid q_{k-1}$, and in this case the diophantine equation $|dx^2 - d^*y^2| = 4$ has a solution
\[(x, y) \in \mathbb{Z}^2 \text{ such that } (x, y) = (s_k, q_{k-1}). \text{ Assume now that there exist } x, y \in \mathbb{Z} \text{ such that } (x, y) = 1 \text{ and } d x^2 - d^* y^2 = \sigma \in \{\pm 1\}. \text{ Then }
\varepsilon = \frac{2\sigma + d^* y^2 + xy\sqrt{\Delta}}{2} \in \mathcal{O}_\Delta
\]
is half-integral, and \(\mathcal{N}(\varepsilon) = 1\), which implies that \(\varepsilon \in \mathcal{O}_\Delta \setminus \mathcal{O}_\Delta^\times\). Since \(\mathcal{O}_\Delta^\times \neq \mathcal{O}_\Delta^\times\) if and only if \(\varepsilon_\Delta\) has half-integral coordinates, it follows that \(\varepsilon_\Delta\) has half-integral coordinates.

4. Suppose that \((d, d^*, \sigma) \in \mathcal{L}_0(\Delta)\), and let \(x, y \in \mathbb{Z}\) be such that \(d x^2 - d^* y^2 = 4\sigma\). If \(x \equiv y \equiv 0 \mod 2\), we set \(x = 2x_1, y = 2y_1\), and we obtain \(d x_1^2 - d^* y_1^2 = \sigma\). Thus assume now that \(x \equiv y \equiv 1 \mod 2\). Then we set
\[x_1 = \frac{(dx^2 - 3\sigma)x}{2} \text{ and } y_1 = \frac{(dx^2 - \sigma)y}{2},\]
and we assert that \(d x_1^2 - d^* y_1^2 = \sigma\). For the proof, we start with the identity
\[64\varepsilon d^3 = (d^2 x^2 - \Delta y^2)^3 = [dx(d^2 x^2 + 3\Delta y^2)]^2 - \Delta(y(3d^2 x^2 + \Delta y^2))^2.\]
Now we find
\[dx(d^2 x^2 + 3\Delta y^2) = dx[4d^2 x^2 - 3(d^2 x^2 - \Delta y^2)] = dx(4d^2 x^2 - 12d\sigma) = 4d^2 x(dx^2 - 3\sigma) = 8d^2 x_1\]
and
\[y(3d^2 x^2 + \Delta y^2) = y[4d^2 x^2 - (d^2 x^2 - \Delta y^2)] = y(4d^2 x^2 - 4\sigma) = 4dy(dx^2 - \sigma) = 8dy_1.\]
Hence it follows that \(64\varepsilon d^3 = 64d^2 x_1^2 - 64d^2 y_1^2\Delta\), and therefore \(\sigma = dx_1^2 - d^* y_1^2\).

Suppose now that \(l\) is even. Then there exists some \((d, d^*, \sigma) \in \mathcal{L}_0(\Delta)\) such that \(1 < d < d^*\), and, as we have just proved, this implies that \((d, d^*, 1, \sigma) \in \mathcal{L}(\Delta)\). By Theorem 4.3 it follows that \(l^*\) is even, and if \(l^* = 2k^*\), then \(\sigma = (-1)^k = (-1)^{k^*}\), which implies \(l \equiv l^* \mod 4\).

\textbf{Remark 4.5.} The Theorems 4.3 and 4.4 are closely connected with the results of R. A. Mollin in [13], in particular with his Theorems 3 and 9. There he derives a close connection between the fundamental solutions of pellian and antipellian equations in terms of continued fractions.

5. Applications

\textbf{Theorem 5.1.} (compare [13, Theorem 5 and Corollaries]) Let \(q \equiv 3 \mod 4\) be a prime and \(\Delta = 4q^r\) for some odd \(r \in \mathbb{N}\).

1. Then \(l(\sqrt{q}) = 2k\) is even, \(l(\sqrt{q}) \equiv l(\sqrt{q}) \mod 4\), and there exists exactly one \(\sigma \in \{\pm 1\}\) such that the diophantine equation
\[x^2 - q^r y^2 = 2\sigma\]
is solvable, namely \(\sigma = (-1)^k = \begin{cases} 1 & \text{if } q \equiv 7 \mod 8, \\ -1 & \text{if } q \equiv 3 \mod 8. \end{cases}\)

2. If \(\varepsilon_\Delta = u + v\sqrt{q}\), where \(u, v \in \mathbb{N}\), then \(2|u\) and \(\mathcal{N}(\varepsilon_\Delta) = 1\). 


Proof. 1. By Theorem 3.3, \( N(\varepsilon_D) = 1 \) and \( l(\sqrt{D}) = 2k \) is even. By Theorem 4.3, applied with \( D = q^r \), there exists a unique \( \sigma \in \{ \pm 1 \} \) such that the diophantine equation \( x^2 - q^r y^2 = 2\sigma \) has a solution \((x, y) \in \mathbb{Z}^2\), namely \( \sigma = (-1)^k \). Hence

\[
1 = \left( \frac{2(-1)^k}{q} \right) = (-1)^k \left( \frac{2}{q} \right), \quad \text{and} \quad \sigma = (-1)^k = \begin{cases} 
1 & \text{if } q \equiv 7 \pmod{8}, \\
-1 & \text{if } q \equiv 3 \pmod{8}.
\end{cases}
\]

Therefore the parity of \( k \) does not depend on \( r \).

2. Let \((p_n)_{n \geq -2}\) the sequence of partial numerators and \((q_n)_{n \geq -2}\) the sequence of partial denominators of \( \sqrt{D} \). Since \( (1, q^r, 2, (-1)^k) \in \mathcal{L}(q^r) \), it follows that \( p_{k-1}^2 - q^r q_{k-1}^2 = 2(-1)^k \), hence \( 2 \nmid q_{k-1} \), and \( \varepsilon_D = (-1)^k + q^r q_{k-1}^2 + p_{k-1} q_{k-1} \sqrt{D} \), which implies \( u = (-1)^k + q^r q_{k-1}^2 \equiv 0 \pmod{2}. \)

\( \Box \)

**Theorem 5.2.** Let \( q \equiv 3 \pmod{4} \) be a prime and \( r \in \mathbb{N} \). Then \( l(\sqrt{2Dq^r}) = 2k \) is even, \( l(\sqrt{2Dq^r}) \equiv l(\sqrt{2D}) \pmod{4} \), and there exists exactly one \( \sigma \in \{ \pm 1 \} \) such that the diophantine equation

\[
2x^2 - q^r y^2 = \sigma \quad \text{is solvable, namely} \quad \sigma = (-1)^k = \begin{cases} 
1 & \text{if } q \equiv 7 \pmod{8}, \\
-1 & \text{if } q \equiv 3 \pmod{8}.
\end{cases}
\]

**Proof.** Note that \( l(\sqrt{2Dq^r}) = 2k \) is even by Theorem 3.3. By Theorem 4.3, applied with \( D = 2q^r \), there exists a unique \( \sigma \in \{ \pm 1 \} \) such that the diophantine equation \( 2x^2 - q^r y^2 = \sigma \) has a solution \((x, y) \in \mathbb{Z}^2\), namely \( \sigma = (-1)^k \). Hence

\[
1 = \left( \frac{2(-1)^k}{q} \right) = (-1)^k \left( \frac{2}{q} \right), \quad \text{and} \quad \sigma = (-1)^k = \begin{cases} 
1 & \text{if } q \equiv 7 \pmod{8}, \\
-1 & \text{if } q \equiv 3 \pmod{8}.
\end{cases}
\]

In particular, the parity of \( k \) does not depend on \( r \). \( \Box \)

**Theorem 5.3.** (compare [13, Theorem 10]) Let \( q \) and \( q^s \) be odd primes and \( \Delta = 4p^r q^s \) for some odd \( r, s \in \mathbb{N} \) such that \( p^r < q^s \).

1. If \( N(\varepsilon_D) = -1 \), then the diophantine equation \( \left| p^r x^2 - q^s y^2 \right| = 1 \) is unsolvable.

2. Suppose that \( N(\varepsilon_D) = 1 \) and \( l(\sqrt{p^r q^s}) = 2k \). Then there exists precisely one \( \sigma \in \{ \pm 1 \} \) such that the diophantine equation \( p^r x^2 - q^s y^2 = \sigma \) is solvable, namely \( \sigma = (-1)^k \). In particular,

\[
\left( \frac{-1)^k p}{q} \right) = \left( \frac{-1)^k + 1 q}{p} \right) = 1.
\]

**Proof.** By Theorem 4.3, applied with \( D = p^r q^s \). \( \Box \)

**Theorem 5.4.** Let \( p \) and \( q \) be primes and \( \Delta = 8p^r q^s \) for some odd \( r, s \in \mathbb{N} \). If \( N(\varepsilon_D) = 1 \), we set \( l(\sqrt{2p^r q^s}) = 2k \).

1. Let \( p \equiv 1 \pmod{8} \) and \( q \equiv 5 \pmod{8} \).

\( a) \) The diophantine equations \( \left| 2x^2 - p^r q^s y^2 \right| = 1 \) and \( \left| 2p^r x^2 - q^s y^2 \right| = 1 \) are unsolvable.

\( b) \) If \( N(\varepsilon_D) = 1 \), then there exists precisely one \( \sigma \in \{ \pm 1 \} \) such that the diophantine equation \( p^r x^2 - 2q^s y^2 = \sigma \) is solvable, namely

\[
\sigma = \begin{cases} 
(-1)^k & \text{if } p^r < 2q^s, \\
(-1)^k + 1 & \text{if } p^r > 2q^s,
\end{cases} \quad \text{and} \quad \left( \frac{2}{q} \right) = 1,
\]

**Proof.** By Theorem 4.3, applied with \( D = p^r q^s \). \( \Box \)
(c) If \( N(\varepsilon_{\Delta}) = -1 \), then the diophantine equation \( |p^r x^2 - 2q^s y^2| = 1 \) is unsolvable.

2. Let \( p \equiv 3 \mod 8 \) and \( q \equiv 5 \mod 8 \) (then \( N(\varepsilon_{\Delta}) = 1 \)).
   (a) The diophantine equation \( |2x^2 - p^r q^s y^2| = 1 \) is unsolvable.
   (b) Exactly one of the two diophantine equations
       \[
       2p^r x^2 - q^s y^2 = -\left(\frac{p}{q}\right) \quad \text{and} \quad p^r x^2 - 2q^s y^2 = \left(\frac{p}{q}\right)
       \]
       is solvable, while the two diophantine equations
       \[
       2p^r x^2 - q^s y^2 = \left(\frac{p}{q}\right) \quad \text{and} \quad p^r x^2 - 2q^s y^2 = -\left(\frac{p}{q}\right)
       \]
       are both unsolvable.
   (c) By the preliminary remark.

3. Let \( p \equiv 3 \mod 8 \) and \( q \equiv 7 \mod 8 \) (then \( N(\varepsilon_{\Delta}) = 1 \)).
   (a) The diophantine equations \( |2x^2 - p^r q^s y^2| = 1 \) and \( |p^r x^2 - 2q^s y^2| = 1 \)
       are both unsolvable.
   (b) There exists precisely one \( \sigma \in \{\pm 1\} \) such that the diophantine equation
       \[
       2p^r x^2 - q^s y^2 = \sigma \]
       is solvable, namely
       \[
       \sigma = \begin{cases} (-1)^k & \text{if } 2p^r < q^s, \\ (-1)^{k+1} & \text{if } 2p^r > q^s, \end{cases} \quad \text{and} \quad (-1)^k \left(\frac{p}{q}\right)(q^s - 2p^r) > 0.
       \]
   (c) (compare [13, Corollary 10]) If \( \varepsilon_{\Delta} = u + v\sqrt{2pq\sigma^2} \), then \( v \) is even, and
       \[
       \left(\frac{p}{q}\right) = (-1)^{v/2}.
       \]

Proof. We apply Theorem 4.3 with \( D = 2p^r q^s \). If \( N(\varepsilon_{\Delta}) = 1 \), then exactly one of
the six diophantine equations

(I) \( 2x^2 - p^r q^s y^2 = \pm 1 \), \quad (II) \( 2p^r x^2 - q^s y^2 = \pm 1 \), \quad (III) \( p^r x^2 - 2q^s y^2 = \pm 1 \)

is solvable. Otherwise, if \( N(\varepsilon_{\Delta}) = -1 \), then \( p \equiv q \equiv 1 \mod 4 \), and all these
diophantine equations are unsolvable.

1. (a) If \( x, y \in \mathbb{Z} \) are such that \( 2x^2 - p^r q^s y^2 = \sigma \in \{\pm 1\} \), then \( 2x^2 \equiv \sigma \mod q \), and therefore

   \[
   1 = \left(\frac{\sigma}{q}\right) = \left(\frac{2}{q}\right),
   \]

   a contradiction.

   If \( x, y \in \mathbb{Z} \) are such that \( 2p^r x^2 - q^s y^2 = \sigma \in \{\pm 1\} \), then the congruences
   \( 2p^r x^2 \equiv \sigma \mod q \) and \( q^s y^2 \equiv \sigma \mod p \) imply that

   \[
   1 = \left(\frac{\sigma}{q}\right) = \left(\frac{2p}{q}\right) = -\left(\frac{p}{q}\right) \quad \text{and} \quad 1 = \left(\frac{\sigma}{p}\right) = \left(\frac{2}{p}\right),
   \]

   which contradicts the quadratic reciprocity law.

   (b) By (a) and Theorem 4.3, there exists exactly one \( \sigma \in \{\pm 1\} \) such that the
diophantine equation \( p^r x^2 - 2q^s y^2 = \sigma \) is solvable, and \( \sigma = (-1)^k \) if and only if
   \( p^r < 2q^s \). In particular, it follows that

   \[
   1 = \left(\frac{\sigma}{q}\right) = \left(\frac{p}{q}\right).
   \]

   (c) By the preliminary remark.

2. (a) As in 1.(a), since \( \pm 2 \) is a quadratic non-residue modulo \( q \).
(b) By the preliminary remark, exactly one of the four diophantine equations
\[ 2p' x^2 - q'^2 y^2 = \pm 1 \] and \[ p' x^2 - 2q' y^2 = \pm 1 \] is solvable. Let \( x, y \in \mathbb{Z} \) and \( \sigma \in \{ \pm 1 \} \). If
\[ 2p' x^2 - q'^2 y^2 = \sigma, \] then \( \sigma \equiv -q'^2 y^2 \mod p \) and therefore
\[ \sigma = \left( \frac{\sigma}{p} \right) = -\left( \frac{2}{p} \right) = -\left( \frac{p}{q} \right). \]
If \( p' x^2 - 2q' y^2 = \sigma, \) then \( \sigma \equiv -2q' y^2 \mod p \) and therefore
\[ \sigma = \left( \frac{\sigma}{p} \right) = \left( \frac{-2q}{p} \right) = \left( \frac{q}{p} \right) = \left( \frac{p}{q} \right). \]

3. (a) If \( x, y \in \mathbb{Z} \) are such that \( 2x^2 - p' y^2 = \sigma \in \{ \pm 1 \}, \) then \( 2x^2 \equiv \sigma \mod p \) and \( 2x^2 \equiv \sigma \mod q, \) which implies
\[ -1 = \left( \frac{2}{p} \right) = \left( \frac{x}{p} \right) = \left( \frac{2}{q} \right) = 1, \text{ a contradiction}. \]
If \( x, y \in \mathbb{Z} \) are such that \( p' x^2 - 2q' y^2 = \sigma \in \{ \pm 1 \}, \) then \( p' x^2 \equiv \sigma \mod q \) and \( -2q' y^2 \equiv \sigma \mod p, \) which implies
\[ \sigma = \left( \frac{\sigma}{p} \right) = \left( \frac{-2q}{p} \right) = \left( \frac{q}{p} \right) = -\left( \frac{p}{q} \right) = -\sigma, \text{ a contradiction}. \]

(b) By (a) and the preliminary remark, there is exactly one \( \sigma \in \{ \pm 1 \} \) for which the diophantine equation \( 2p' x^2 - q'^2 y^2 = \sigma \) is solvable, and by Theorem 4.3 we obtain \( \sigma = (-1)^k \) if and only if \( 2p' < q'. \) If \( x, y \in \mathbb{Z} \) are such that \( 2p' x^2 - q'^2 y^2 = \sigma, \) then \( 2p' x^2 \equiv \sigma \mod q, \) and therefore
\[ \sigma = \left( \frac{\sigma}{p} \right) = \left( \frac{2p}{q} \right) = \left( \frac{p}{q} \right), \text{ which implies } (-1)^k \left( \frac{p}{q} \right) (q^a - 2p^r) > 0. \]

(c) Let \((p_n)_{n \geq -2}\) the sequence of partial numerators and \((q_n)_{n \geq -2}\) the sequence of partial denominators of \( \sqrt{2p' q'^2}. \) For \( g \in \mathbb{Z}, \) we denote by \( v_2(g) \) the 2-adic exponent of \( g. \)

Assume first that \( 2p' < q'. \) Then \( (2p', q'^a, 1, (-1)^k) \in L(2p' q'^a), \) and it follows that
\[ p_{k-1}^2 - 2p' q'^a q_{k-1}^2 = (-1)^k 2p' \] \( 2 | p_{k-1}, \) and \( 2 | q_{k-1}, \) and
\[ \varepsilon_\Delta = (-1)^k + 2q'^a q_{k-1}^2 + \frac{p_{k-1} q_{k-1}}{2p'} \sqrt{2p' q'^a}, \] which implies \( v = \frac{p_{k-1} q_{k-1}}{2p'} \) and \( v_2(v) = v_2(p_{k-1}) \geq 1, \) Since
\[ p_{k-1}^2 = 2p' \frac{[(-1)^k + q'^a q_{k-1}^2]}{2} \equiv 2 [1 - (-1)^k] \mod 8, \] it follows that \( 4 | p_{k-1} \) (and thus \( 4 | v \)) if and only if \( 2 | k, \) and therefore
\[ \left( \frac{p}{q} \right) = (-1)^k = (-1)^{v/2}. \]

Assume now that \( q'^a < 2p'. \) Then \( (q'^a, 2p', 1, (-1)^k) \in L(2p' q'^a), \) and it follows that
\[ p_{k-1}^2 - 2p' q'^a q_{k-1}^2 = (-1)^k q'^a, \] \( 2 \nmid p_{k-1}, \) and
\[ \varepsilon_\Delta = (-1)^k + 4q'^a q_{k-1}^2 + \frac{p_{k-1} q_{k-1}}{q'} \sqrt{2p' q'^a}, \] which implies \( v = \frac{2p_{k-1} q_{k-1}}{q'} \) and \( v_2(v) = v_2(q_{k-1}) + 1 \geq 1, \) Since
\[ p_{k-1}^2 = q'^a \frac{[2p' q_{k-1}^2 + (-1)^k]}{2q_{k-1}^2 - (-1)^k} \equiv 2q_{k-1}^2 - (-1)^k \mod 8, \] it follows that \( 2q_{k-1}^2 \equiv 1 + (-1)^k \mod 8. \) Hence \( 2 | q_{k-1} \) (and thus \( 4 | v \)) if and only if \( 2 \nmid k, \) and therefore
\[ \left( \frac{p}{q} \right) = (-1)^{k-1} = (-1)^{v/2}. \]
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