Mixed invertibility and Prüfer-like monoids and domains

F. Halter-Koch

Abstract. We give a systematic theory of Prüfer-like domains using ideal systems on commutative cancellative monoids. Based on criteria for mixed invertibility of ideals, we unify and generalize characterizations of various classes of Prüfer-like monoids and domains and furnish them with new proofs. In particular, we generalize and extend criteria for v-domains recently proved by D.D. Anderson, D.F. Anderson, M. Fontana and M. Zafrullah.

Keywords. Prüfer domain, ideal system, star operation, v-domain, (generalized) GCD domain.

AMS classification. 20M12, 13A15, 13F05, 13G05, 20M14, 20m25.

1 Introduction

Prüfer domains and their various generalizations are topics of outstanding interest in non-noetherian multiplicative ideal theory. For an overview of the more classical results we refer to [6], [9] and [15]. Among the various generalizations involving star operations studied in the literature we mention the following ones.

- Prüfer v-multiplication domains (PvMD's, first studied in [10] and called "pseudo-Prüfer domains" in [5, Ch. VII, § 2, Ex. 19]),
- general $*$-multiplication domains (investigated in [13] and in [12]),
- v-domains (called "regularly integrally closed domains" in [5, Ch. VII, §1, Ex. 30, 31], see [19] for an overview and the history of this concept),
- Generalized GCD-domains (GGCD domains, studied in [1]),
- Pseudo-Dedekind domains (introduced in [17] under the name "Generalized Dedekind domains" and thorough investigated in [3]),
- pre-Krull domains (investigated in [18]).

Several of these concepts have only recently sucessfully been generalized to the case of semistar operations (see [7] and [8]).

By the very definitions, the above-mentioned concepts can be defined in a purely multiplicative manner without referring to the ring addition, and thus they can be studied in the context of commutative cancellative monoids. In a systematic way, the ideal theory of commutative cancellative monoids was first developed by P. Lorenzen [16],

Supported by the Austrian Science Fund FWF (Project 18779-N13)

and a thorough presentation of that theory in the language of ordered abelian groups was given by P. Jaffard [14]. A modern treatment of multiplicative ideal theory in the context of commutative monoids (including all above-mentioned generalizations) was given by the author in the monograph [11], which serves as the main reference for the present paper.

A first attempt to a general theory covering the various generalizations of Prüfer domains was made in [4] but not pursued further on. Only recently, these investigations were revived in [2] together with several completely new ideal-theoretic characterizations of v-domains. In this paper we continue these investigations. We show that the results of [2] and several of their refinements and generalizations remain valid in the context of commutative cancellative monoids, and we provide them with new (and simpler) proofs.

The paper is organized as follows. In Section 2 we fix our notations. Section 3 contains the basic result on mixed invertibility (Theorem 3.1) which is fundamental for the following investigations. In Section 4 we apply the concept of mixed invertibility to characterizations of Dedekind-like and Prüfer-like monoids and domains, and finally in Section 5 we continue the investigations of v-domains (resp. v-Prüfer monoids) started in [2].

2 Notations

For any set X, we denote by $|X| \in \mathbb{N}_0 \cup \{\infty\}$ its cardinality, by $\mathbb{P}(X)$ the set of all subsets and by $\mathbb{P}_{\mathsf{f}}(X)$ the set of all finite subsets of X.

Throughout this paper, let D be a commutative multiplicative monoid with unit element $1 \in D$ and a zero element $0 \in D$ (satisfying $0x = 0$ for all $x \in D$) such that $D^\bullet = D \setminus \{0\}$ is cancellative, and let $K = \mathsf{q}(D) = {D^\bullet}^{-1}D$ be its total quotient monoid (then $K^\bullet$ is a quotient group of $D^\bullet$). The most important example we have in mind is when D is the multiplicative monoid of an integral domain (then K is the multiplicative monoid of its quotient field).

For subsets X, $Y \subset K$, we set $(X\!:\!Y) = \{z \in K \mid zY \subset X\}$, $X^{-1} = (D\!:\!X)$, and the set X is called D-fractional if $X^{-1} \cap D^\bullet \neq \emptyset$. We denote by $\mathcal{F}(D)$ the set of all D-fractional subsets of K.

Throughout, we use the language of ideal systems as developed in my book "Ideal Systems" [11], and all undefined notions are as there. For an ideal system r on D, let $\mathcal{F}_r(D) = \{X_r \mid X \in \mathcal{F}(D)\} = \{A \in \mathcal{F}(D) \mid A_r = A\}$ be the semigroup of all fractional r-ideals, equipped with the r-multiplication, defined by $(A,B) \mapsto (AB)_r$ and satisfying $(AB)_r = (A_rB)_r = (A_rB_r)_r$ for all A, $B \in \mathcal{F}(D)$. We denote by $\mathcal{F}_{r,\mathsf{f}}(D) = \{E_r \mid E \in \mathbb{P}_{\mathsf{f}}(K) \subset \mathcal{F}_r(D)$ the subsemigroup of all r-finite (that is, r-finitely generated) fractional r-ideals of D.

For any subset $\mathcal{X} \subset \mathbb{P}(K)$, we set $\mathcal{X}^\bullet = \mathcal{X} \setminus \{\{0\}\}$. In particular, if $\mathcal{J}$ is any set of ideals, then $\mathcal{J}^\bullet = \mathcal{J} \setminus \{0\}$ (where $0 = \{0\}$ denotes the zero ideal). In this way we use the notions $\mathcal{F}(D)^\bullet$, $\mathcal{F}_r(D)^\bullet$, $\mathcal{F}_{r,\mathsf{f}}(D)^\bullet$ etc.

For an ideal system r on D, the associated finitary ideal system of r will be denoted by r_{f} (it is denoted by r_s in [11]). It is given by

$$X_{r_{\mathsf{f}}} = \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(X)} E_r \quad \text{for every } X \in \mathcal{F}(D)\,,$$

and it satisfies $\mathcal{F}_{r,\mathsf{f}}(D) = \mathcal{F}_{r_{\mathsf{f}},\mathsf{f}}(D)$. The ideal system r is called *finitary* if $r = r_{\mathsf{f}}$.

For any two ideal systems r and q on D we write $r \le q$ if $\mathcal{F}_q(D) \subset \mathcal{F}_r(D)$. Note that $r \le q$ holds if and only if $X_r \subset X_q$ [equivalently, $X_q = (X_r)_q$] for all $X \in \mathcal{F}(D)$.

We denote by $s = s(D)$ the system of semigroup ideals, given by $X_s = DX$ for all $X \in \mathcal{F}(D)$; by $v = v(D)$ the ideal system of multiples ("Vielfachenideale"), given by $X_v = (X^{-1})^{-1}$ for all $X \in \mathcal{F}(D)$, and by $t = t(D) = v_{\mathsf{f}}$ the associated finitary system. The systems s and t are finitary, the system v usually not. For every ideal system r on D we have $s \le r_{\mathsf{f}} \le r \le v$ and $r_{\mathsf{f}} \le t$. We shall frequently use that $\mathcal{F}_v(D) = \{A^{-1} \mid A \in \mathcal{F}(D)\}$ (see [11, Theorem 11.8]).

If D is an integral domain, then the (Dedekind) ideal system $d = d(D)$ of usual ring ideals is given by $X_d = {}_D\langle X\rangle$ for all $X \in \mathcal{F}(D)$ (that is, X_d is the fractional D-ideal generated by X). d is a finitary ideal system, and there is a one-to-one correspondence between ideal systems $r \ge d$ and star operations on D, given as follows:

If $*\colon \mathcal{F}_d(D)^\bullet \to \mathcal{F}_d(D)^\bullet$ is a star operation on D and $r^*\colon \mathcal{F}(D) \to \mathcal{F}(D)$ is defined by $X_{r^*} = {}_D\langle X\rangle^*$ for $X \in \mathcal{F}(D)^\bullet$ and $X_{r^*} = \{0\}$ if $X \subset \{0\}$, then r^* is an ideal system satisfying $r^* \ge d$. Conversely, if $r \ge d$ is an ideal system, and if we define $*_r$ by $J^{*_r} = J_r$ for all $J \in \mathcal{F}_d(D)^\bullet$, then $*_r$ is a star operation, and by the very definition we have $r^{*_r} = r$ and $*_{r^*} = *$.

Throughout this paper, we fix a (basic) ideal system δ on D and assume that all ideal systems r considered in this manuscript satisfy $r \ge \delta$. Of course, we may always assume that $\delta = s(D)$, but if D is an integral domain, it may also be convenient to assume that $\delta = d(D)$ in order to make the connection with star operations more apparent.

In any case, we denote by $\boldsymbol{F}(D) = \mathcal{F}_\delta(D)^\bullet$ the set of all non-zero fractional δ-ideals and by $\boldsymbol{f}(D) = \mathcal{F}_{\delta,\mathsf{f}}(D)$ the set of all δ-finite non-zero fractional δ-ideals of D. Then $\mathcal{F}_r(D)^\bullet = \{A_r \mid A \in \boldsymbol{F}(D)\}$ and $\mathcal{F}_{r,\mathsf{f}}(D)^\bullet = \{F_r \mid F \in \boldsymbol{f}(D)\}$ for every ideal system r on D,

3 Mixed invertibility

Mixed invertibility means, that we investigate the invertibility of ideals of one ideal system with respect to another ideal system. We start by recalling some basic facts concerning the concept invertibility in the theory of ideal systems. For details and proofs concerning the following remarks we refer to [11, Theorem 12.1].

Let r be an ideal system on D. A fractional ideal $A \in \boldsymbol{F}(D)$ is called *r-invertible* if $(AA^{-1})_r = D$ [equivalently, $(AB)_r = D$ for some $B \in \boldsymbol{F}(D)$]. Hence a fractional ideal $A \in \boldsymbol{F}(D)$ is r-invertible if and only if A_r is r-invertible. By definition, a fractional r-ideal is r-invertible if and only if it is an invertible element of the semi-group $\mathcal{F}_r(D)$. If $A,\ B \in \boldsymbol{F}(D)$, then AB is r-invertible if and only if A and B are both r-invertible.

We denote by $\mathcal{F}_r(D)^\times$ the group of all r-invertible fractional r-ideals. If q is an ideal system such that $r \le q$, then $\mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)^\times$ is a subgroup (this holds in particular, if $q = v$). If r is finitary, then $\mathcal{F}_r(D)^\times = \mathcal{F}_{r,\mathsf{f}}(D)^\times$ (that is, if $A \in \mathcal{F}_r(D)$ is r-invertible, then both A and A^{-1} are r-finite). This may fail if r is not finitary; then it may occur that $\mathcal{F}_{r,\mathsf{f}}(D)^\times \subsetneq \mathcal{F}_r(D)^\times \cap \mathcal{F}_{r,\mathsf{f}}(D)$ (it is well known that not every v-domain is a PvMD).

Theorem 3.1 (Mixed Invertibility). *Let r, q and y be ideal systems on D, $q \le y$, and $B \in \boldsymbol{F}(D)$. Then the following assertions are equivalent:*

(a) B_q *is* r*-invertible.*

(b) B^{-1} *is* r*-invertible, and* $B_q = B_v$.

(c) *For every* $A \in \boldsymbol{F}(D)$ *such that* $A_r \subset B_q$ *there exists some* $C \in \mathcal{F}_r(D)$ *satisfying* $A_r = (B_qC)_r$.

(d) $(A\!:\!B^{-1})_r = (AB_q)_r$ *for all* $A \in \boldsymbol{F}(D)$.

(e) $(A\!:\!B_q)_r = (AB^{-1})_r$ *for all* $A \in \boldsymbol{F}(D)$.

(f) $[(A\!:\!B)_q]_r = (A_qB^{-1})_r$ *for all* $A \in \boldsymbol{F}(D)$.

(g) $(A_q\!:\!B)_r = (A_qB^{-1})_r$ *for all* $A \in \boldsymbol{F}(D)$.

(h) $(A_q\!:\!B^{-1})_r = (A_qB_q)_r$ *for all* $A \in \boldsymbol{F}(D)$.

(i) $(A_v\!:\!B^{-1}) = (A_vB_q)_r$ *for all* $A \in \boldsymbol{F}(D)$.

(j) $(A_r\!:\!B^{-1}) = (AB_q)_r$ *for all* $A \in \boldsymbol{F}(D)$.

(k) $(B_y\!:\!A)_r = (B_qA^{-1})_r$ *for all* $A \in \boldsymbol{F}(D)$.

(l) $[(B\!:\!A)_y]_r = (B_qA^{-1})_r$ *for all* $A \in \boldsymbol{F}(D)$.

(m) $(B_v\!:\!A^{-1}) = (B_qA_v)_r$ *for all* $A \in \boldsymbol{F}(D)$.

Proof. (a) $\Rightarrow$ (b) If B_q is r-invertible, then $B_q = (B_q)_v = B_v$, and $(B_qB^{-1})_r = D$. Hence B^{-1} is r-invertible.

(b) $\Rightarrow$ (c) If $A \in \boldsymbol{F}(D)$ and $A_r \subset B_q$, then $C = (A_rB_q^{-1})_r \in \mathcal{F}_r(D)$, and $(B_qC)_r = (B_qB_q^{-1}A_r)_r = [(B_qB_q^{-1})_rA]_r = A_r = A$, since B^{-1} is r-invertible and thus $(B_qB_q^{-1})_r = (B_vB^{-1})_r = [(B^{-1})^{-1}B^{-1}]_r = D$.

(c) $\Rightarrow$ (a) If $a \in B_q^\bullet$, then $aD = (aD)_r \subset B_q$, and thus $aD = (B_qC)_r$ for some $C \in \mathcal{F}_r(D)$. Hence $D = [B_q(a^{-1}C)]_r$, and thus B_q is r-invertible.

(a) $\Rightarrow$ (d) Let $A \in \boldsymbol{F}(D)$. Since $AB_qB^{-1} \subset A(B_qB^{-1})_r = AD = A$, it follows that $AB_q \subset (A:B^{-1})$ and $(AB_q)_r \subset (A:B^{-1})_r$. To prove the reverse inclusion, let $x \in (A:B^{-1})_r$. Then $xB^{-1} \subset (A:B^{-1})_rB^{-1} \subset [(A:B^{-1})B^{-1}]_r \subset A_r$ and $x \in xD = (xB_qB^{-1})_r \subset (A_rB_q)_r = (AB_q)_r$.

(d) $\Rightarrow$ (j) For every $A \in \boldsymbol{F}(D)$, we may apply (d) with A_r instead of A and obtain $(A_r : B^{-1}) = (A_r : B^{-1})_r = (A_r B_q)_r = (AB_q)_r$.

(j) $\Rightarrow$ (i) For every $A \in \boldsymbol{F}(D)$, we may apply (j) with A_v instead of A and obtain $(A_v : B^{-1}) = ((A_v)_r : B^{-1}) = (A_v B_q)_r$.

(i) $\Rightarrow$ (a) With $A = B^{-1} = A_v$, (i) implies $D \supset (B_q B^{-1})_r = (B^{-1} : B^{-1}) \supset D$ and therefore $(B_q B^{-1})_r = D$.

(d) $\Rightarrow$ (h) For every $A \in \boldsymbol{F}(D)$, we apply (d) with A_q instead of A.

(h) $\Rightarrow$ (i) For every $A \in \boldsymbol{F}(D)$, we may apply (h) with A_v instead of A and obtain $(A_v : B^{-1}) = ((A_v)_q : B^{-1})_r = ((A_v)_q B_q)_r = (A_v B_q)_r$.

(b) $\Rightarrow$ (e) By (a) $\Rightarrow$ (d), applied with B^{-1} instead of B. In doing so observe that $(B^{-1})_q = B^{-1}$ and $(B^{-1})^{-1} = B_v = B_q$.

(e) $\Rightarrow$ (g) For every $A \in \boldsymbol{F}(D)$, we may apply (e) with A_q instead of A and obtain $(A_q : B)_r = (A_q : B_q)_r = (A_q B^{-1})_r$.

(a) $\Rightarrow$ (f) Let $A \in \boldsymbol{F}(D)$. Then $AB^{-1}B \subset A$ implies $AB^{-1} \subset (A : B)$ and thus $(A_q B^{-1})_r \subset [(A : B)_q]_r$. For the reverse inclusion, it suffices to show that $(A : B)_q \subset (A_q B^{-1})_r$. If $x \in (A : B)_q$, then $xB_q \subset (A : B)_q B_q \subset [(A : B)B]_q \subset A_q$, and consequently $x \in xD = (xB_q B^{-1})_r \subset (A_q B^{-1})_r$.

(f) $\Rightarrow$ (a) and (g) $\Rightarrow$ (a) In both cases, we set $A = B_q$, observe that $(B_q : B) \supset D$ and obtain $(B^{-1} B_q)_r \subset D$, whence $(B^{-1} B_q)_r = D$.

(a) $\Rightarrow$ (k) and (a) $\Rightarrow$ (l) Let $A \in \boldsymbol{F}(D)$. Since $BA^{-1}A \subset B$, it follows that $BA^{-1} \subset (B : A) \subset (B_y : A)$, hence $B_q A^{-1} \subset (BA^{-1})_q \subset (B : A)_q \subset (B : A)_y$ and $(BA^{-1})_q \subset (B_y : A)_q = (B_y : A)$. Thus we obtain $(B_q A^{-1})_r \subset (B_y : A)_r$ and $(B_q A^{-1})_r \subset [(B : A)_y]_r$.
For the reverse inclusions it suffices to show that $(B_y : A) \subset (B_q A^{-1})_r$ and $(B : A)_y \subset (B_q A^{-1})_r$. Thus assume that either $x \in (B_y : A)$ or $x \in (B : A)_y$. Since $(B : A)_y \subset (B_y : A)_y = (B_y : A)$, we obtain $xA \subset B_y$ in both cases. Now it follows that $xAB^{-1} \subset B_y B^{-1} \subset D$, hence $xB^{-1} \subset A^{-1}$ and $x \in (xB_q B^{-1})_r \subset (B_q A^{-1})_r$.

(k) $\Rightarrow$ (a) and (l) $\Rightarrow$ (a) With $A = B$ we obtain $(B_q B^{-1})_r = (B_y : B)_r \supset D$ from (k) and $(B_q B^{-1})_r = [(B : B)_y]_r \supset D$ from (l). Hence $(B_q B^{-1})_r = D$ follows in both cases.

(k) $\Rightarrow$ (m) Let $A \in \boldsymbol{F}(D)$. By the equivalence of (a) and (k) it follows that (k) holds with $y = v$. We apply (k) with $y = v$ and with A^{-1} instead of A. Then we obtain $(B_v : A^{-1}) = (B_v : A^{-1})_r = (B_q A_v)_r$.

(m) $\Rightarrow$ (a) With $A = B^{-1}$ we obtain $D \supset (B_q B^{-1})_r = (B_v : B_v) \supset D$ and thus $(B_q B^{-1})_r = D$. □

Remark 3.2. Let assumptions be as in Theorem 3.1, and assume moreover that $r \le q$. Then the conditions (f), (g), (h), (k) and (l) simplify by the relations $[(A : B)_q]_r = (A : B)_q$, $(A_q : B)_r = (A_q : B)$, $(A_q : B^{-1})_r = (A_q : B^{-1})$, $(B_y : A)_r = (B_y : A)$ and $[(B : A)_y]_r = (B : A)_y$.

Moreover, condition (g) is obviously equivalent to

(g)′ $(A_q : B_q) = (A_q B^{-1})_r$ for all $A \in \boldsymbol{F}(D)$ (compare [2, Remark 1.6]).

Corollary 3.3. *Let r, q and x be ideal systems on D, $x \ge r$ and $B \in \boldsymbol{F}(D)$. Then B_q is r-invertible if and only if $(A_x : B^{-1}) = (A_x B_q)_r$ for all $A \in \boldsymbol{F}(D)$.*

Proof. Let first B_q be r-invertible and $A \in \boldsymbol{F}(D)$. By Theorem 3.1(j), applied with A_x instead of A, we obtain $(A_x : B^{-1}) = ((A_x)_r : B^{-1}) = (A_x B_q)_r$.

To prove the converse, we assume that $(A_x : B^{-1}) = (A_x B_q)_r$ for all $A \in \boldsymbol{F}(D)$. For any $A \in \boldsymbol{F}(D)$, we apply this relation with A_v instead of A, and then we obtain $(A_v : B^{-1}) = ((A_v)_x : B^{-1}) = ((A_v)_x B_q)_r = (A_v B_q)_r$. Hence B_q is r-invertible by Theorem 3.1(i). □

Corollary 3.4. *Let r be an ideal system on D and $B \in \boldsymbol{F}(D)$. Then B_v is r-invertilbe if and only if $(AB)^{-1} = (A^{-1}B^{-1})_r$ for all $A \in \boldsymbol{F}(D)$.*

Proof. Note that $(XY)^{-1} = (X^{-1} : Y)$ for all $X,\, Y \in \boldsymbol{F}(D)$ [11, Corollary 11.7 ii)].

Let first B_v be r-invertible and $A \in \boldsymbol{F}(D)$. By Theorem 3.1(f), applied with $q = r$ and A^{-1} instead of A, we obtain

$$(A^{-1}B^{-1})_r = [(A^{-1})_v B^{-1}]_r = [(A^{-1} : B)_v]_r = (A^{-1} : B) = (AB)^{-1}\,.$$

Assume now that $(A^{-1}B^{-1})_r = (AB)^{-1}$ for all $A \in \boldsymbol{F}(D)$. For every $A \in \boldsymbol{F}(D)$, we apply this relation with A^{-1} instead of A and obtain

$$(A_v B^{-1})_r = [(A^{-1})^{-1} B^{-1}]_r = (A^{-1}B)^{-1} = ((A^{-1})^{-1} : B) = (A_v : B)_r\,.$$

Hence B_v is r-invertible by Theorem 3.1(g), applied with $q = v$. □

4 (r, q)-Dedekind and (r, q)-Prüfer monoids

We use the notions of r-Prüfer monids and r-Dedekind monoids (resp. domains) as in [11, §17 and §23]. For any property **P** of monoids we say that an integral domain D is a **P**-domain if its multiplicative monoid is a **P**-monoid.

Definition 4.1. Let r and q be ideal systems on D such that $r \le q$.

1. D is called an (r, q)*-Dedekind monoid* if $\mathcal{F}_q(D)^\bullet \subset \mathcal{F}_r(D)^\times$ [that is, every non-zero fractional q-ideal is r-invertible, or, equivalently, $(B_q B^{-1})_r = D$ for all $B \in \boldsymbol{F}(D)$].
2. D is called an (r, q)*-Prüfer monoid* if $\mathcal{F}_{q,\mathrm{f}}(D)^\bullet \subset \mathcal{F}_r(D)^\times$ [that is, every non-zero fractional q-finite q-ideal is r-invertible, or, equivalently, $(F_q F^{-1})_r = D$ for all $F \in \boldsymbol{f}(D)$].

By definition, D is an r-Dedekind monoid [an r-Prüfer monoid] if and only if D is an (r, r)-Dedekind monoid [an (r, r)-Prüfer monoid].

A v-Dedekind monoid is a completely integrally closed monoid [11, Theorem 14.1], a t-Dedekind monoid is a Krull monoid [11, Theorem 23.4], and an (r, v)-Prüfer monoid is an r-GCD-monoid [11, Def. 17.6]. Consequently, a v-Dedekind domain is a completely integrally closed domain, a t-Dedekind domain is a Krull domain, and a d-Dedekind domain is just a Dedekind domain. A v-Prüfer domain is a v-domain (that is, a regularly integrally closed domain in the sense of [5, Ch. VII, §1, Ex. 30, 31]), a t-Prüfer domain is a PvMD (that is, a pseudo-Prüfer domain in the sense of [5, Ch. VII, §2, Ex. 19]), and a d-Prüfer domain is just a Prüfer domain. A (d,v)-Prüfer domain is a GGCD-domain (generalized GCD-domain, see [11, Def. 17.6]).

In [2], r-Dedekind domains are called r-CICDs (r-completely integrally closed domains) and (r,v)-Dedekind domains are called (r,v)-CICDs (note that [2, Proposition 1.1] follows from the equivalence of 1. and 3. in Theorem 3.1).

The definition of r-Dedekind domains given in [2] coincides with ours if r is finitary. In general, an r-Dedekind domain in the sense of [2] is an r_{f}-Dedekind domain according to our definition.

Lemma 4.2. *Let r, p, q be ideal systems on D such that $r \le p \le q$.*

If D is an (r,p)-Dedekind monoid, then D is an (r,q)-Dedekind monoid, and if D is an (r,q)-Dedekind monoid, then D is a (p,q)-Dedekind monoid. In particular, if D is an r-Dedekind monoid, then D is an (r,q)-Dedekind monoid, and if D is an (r,q)-Dedekind monoid, then D is a q-Dedekind monoid.

The same assertions hold true if "Dedekind" is replaced by "Prüfer". Moreover, if $r \le q_{\mathsf{f}}$, then D is an (r,q)-Prüfer monoid if and only if D is an (r,q_{f})-Prüfer monoid.

Proof. The statements concerning Dedekind-like monoids follow from the containments $\mathcal{F}_q(D)^\bullet \subset \mathcal{F}_p(D)^\bullet$ and $\mathcal{F}_r(D)^\times \subset \mathcal{F}_p(D)^\times$.

For the proof of the statements concerning Prüfer-like monoids, assume first that D is an (r,p)-Prüfer monoid, and let $F \in \boldsymbol{f}(D)$. Then $D = (F_pF^{-1})_r \subset (F_qF^{-1})_r \subset D$, hence $(F_qF^{-1})_r = D$, and D is an (r,q)-Prüfer monoid. If D is an (r,q)-Prüfer monoid and $F \in \boldsymbol{f}(D)$, then $D = (F_qF^{-1})_r \subset (F_qF^{-1})_p \subset D$ implies that also $(F_qF^{-1})_p = D$, and thus D is a (p,q)-Prüfer monoid.

The last statement follows since $\mathcal{F}_{q,\mathsf{f}}(D) = \mathcal{F}_{q_{\mathsf{f}},\mathsf{f}}(D)$. □

The statements of Theorem 3.1 provide a wealth of criteria for a monoid to be an (r,q)-Dedekind monoid or an (r,q)-Prüfer monoid. In the case of integral domains, most of them are already in [2] (in different arrangements and with different proofs). A detailed identification is left to the reader. The following two propositions highlight two special cases.

Proposition 4.3. *Let r and q be ideal systems on D such that $r \le q$. Then the following assertions are equivalent:*

(a) *D is an (r,q)-Dedekind monoid.*

(b) *D is an (r,v)-Dedekind monoid and $q = v$.*

(c) *For all A, $B \in \boldsymbol{F}(D)$ we have $(AB)^{-1} = (A^{-1}B^{-1})_r$, and $q = v$.*

(d) *D is a q-Dedekind monoid, and $(AB)_v = (A_vB_v)_r$ for all A, $B \in \boldsymbol{F}(D)$.*

Proof. (a) $\Rightarrow$ (b) Since $\mathcal{F}_q(D) \subset \mathcal{F}_r(D)^\times \subset \mathcal{F}_v(D)$, it follows that $q = v$.

(b) $\Rightarrow$ (c) If $A,\, B \in \boldsymbol{F}(D)$, then B_v is r-invertible, and thus Corollary 3.4 implies $(AB)^{-1} = (A^{-1}B^{-1})_r$.

(c) $\Rightarrow$ (d) For every $B \in \boldsymbol{F}(D)$, Corollary 3.4 implies that $B_q = B_v$ is r-invertible, hence q-invertible, and thus D is a q-Dedekind monoid. For any $A,\, B \in \boldsymbol{F}(D)$, then we apply (c) twice and obtain

$$(AB)_v = ((AB)^{-1})^{-1} = ((A^{-1}B^{-1})_r)^{-1} = (A^{-1}B^{-1})^{-1} = (A_vB_v)_r\,.$$

(d) $\Rightarrow$ (a) If $A \in \mathcal{F}_q(D)^\bullet$, then A is q-invertible. Hence $D = (A^{-1}A)_q$, and since $A,\, A^{-1} \in \mathcal{F}_v(D)$, we obtain $D = (A^{-1}A)_v = (A^{-1}A)_r$. □

Proposition 4.4. *Let r be an ideal system on D. Then the following assertions are equivalent*:

(a) *D is an (r,v)-Prüfer monoid.*

(b) *$(AF)^{-1} = (A^{-1}F^{-1})_r$ for all $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$.*

(c) *D is a v-Prüfer monoid, and $(AF)_v = (A_vF_v)_r$ holds for all $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$.*

Proof. (a) $\Rightarrow$ (b) If $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$, then F_v is r-invertible. Hence Corollary 3.4 implies $(AF)^{-1} = (A^{-1}F^{-1})_r$.

(b) $\Rightarrow$ (c) If $F \in \boldsymbol{f}(D)$, then F_v is r-invertible and thus v-invertible by Corollary 3.4. Hence D is a v-Prüfer monoid. For $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$, we apply (b) twice and obtain $(AF)_v = ((AF)^{-1})^{-1} = ((A^{-1}F^{-1})_r)^{-1} = (A^{-1}F^{-1})^{-1} = (A_vF_v)_r$.

(c) $\Rightarrow$ (a) If $F \in \boldsymbol{f}(D)$, then F is v-invertible and $F^{-1} \in \boldsymbol{F}(D)$. Hence we obtain $D = (F^{-1}F)_v = (F^{-1}F_v)_r$, and therefore F_v is r-invertible. □

5 Characterization of r-Prüfer monoids

Most assertions of the following Theorem 5.1 is well known in the context of finitary ideal systems (see [11, §17] which is modeled after the antetpye of [15, Theorem 6.6]). For star operations which are not necessarily of finite type such results was first proved in [2].

Theorem 5.1. *Let r and y be ideal systems on D such that $y \le r$. Then the following assertions are equivalent*:

(a) *D is an r-Prüfer monoid.*

(b) *For all $a,\, b \in D^\bullet$, the r-ideal $\{a,b\}_r$ is r-invertible.*

(c) *$[(A_y \cap B_y)(A \cup B)]_r = (AB)_r$ for all $A,\, B \in \boldsymbol{F}(D)$.*

(d) *$[(A_y \cap B_y)(A \cup B)]_r = (AB)_r$ for all $A,\, B \in \boldsymbol{f}(D)$.*

(e) *$[F(A_r \cap B_r)]_r = (FA)_r \cap (FB)_r$ for all $A,\, B,\, F \in \boldsymbol{f}(D)$.*

(f) $[F(A_r \cap B_r)]_r = (FA)_r \cap (FB)_r$ *for all* $F \in \boldsymbol{f}(D)$ *and* $A,\, B \in \boldsymbol{F}(D)$.

(g) *For all* $I,\, J \in \mathcal{F}_r(D)^\times$ *we have* $I \cap J \in \mathcal{F}_r(D)^\times$ *and* $(I \cup J)_r \in \mathcal{F}_r(D)^\times$.

(h) *For all* $I,\, J \in \mathcal{F}_r(D)^\times$ *we have* $(I \cup J)_r \in \mathcal{F}_r(D)^\times$.

(i) *For every family* $(A_i)_{i \in I}$ *in* $\boldsymbol{F}(D)$ *and all* $F \in \boldsymbol{f}(D)$ *we have*

$$\Big(\big(\bigcup_{i \in I} A_i\big)_y : F\Big)_r = \Big(\bigcup_{i \in I} ((A_i)_y : F)\Big)_r \,.$$

(j) $\big((A \cup B)_y : F\big)_r = [(A_y : F) \cup (B_y : F)]_r$ *for all* $A,\, B \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.

(k) $\big((A \cup B)_y : F\big)_r = [(A_y : F) \cup (B_y : F)]_r$ *for all* $A,\, B,\, F \in \boldsymbol{f}(D)$.

(l) $\big(A_y : (F_r \cap G_r)\big)_r = [(A_y : F_r) \cup (A_y : G_r)]_r$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F,\, G \in \boldsymbol{f}(D)$.

(m) $[(a^{-1}bD \cap D) \cup (ab^{-1}D \cap D)]_r = D$ *for all* $a,\, b \in D^\bullet$.

Proof. (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (d), (f) $\Rightarrow$ (e), (g) $\Rightarrow$ (h) and (i) $\Rightarrow$ (j) $\Rightarrow$ (k) Obvious.

(b) $\Leftrightarrow$ (m) Let $a,\, b \in D^\bullet$. Then $\{a,\, b\}^{-1} = a^{-1}D \cap b^{-1}D$ and therefore

$$\begin{aligned}(\{a,b\}\{a,b\}^{-1})_r &= (a\{a,b\}^{-1} \cup b\{a,b\}^{-1})_r \\ &= [a(a^{-1}D \cap b^{-1}D) \cup b(a^{-1}D \cap b^{-1}D)]_r = [(a^{-1}bD \cap D) \cup (ab^{-1}D \cap D)]_r \,.\end{aligned}$$

Hence $\{a,b\}_r$ is r-invertible if and only if $[(a^{-1}bD \cap D) \cup (ab^{-1}D \cap D)]_r = D$.

(b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (c) Let $A,\, B \in \boldsymbol{F}(D)$. Then obviously

$$(A_y \cap B_y)(A \cup B) \subset A_yB \cap AB_y \subset (AB)_r \,,$$

which implies $[(A_y \cap B_y)(A \cup B)]_r \subset (AB)_r$.

For the reverse inclusion, it suffices to prove that $AB \subset [(A_y \cap B_y)(A \cup B)]_r$. Thus let $a \in A$ and $b \in B$. Since $ab\{a,\, b\}^{-1} = ab(a^{-1}D \cap b^{-1}D) = aD \cap bD$, (b) implies that

$$\begin{aligned}ab \in [ab\{a,\, b\}\{a,\, b\}^{-1}]_r &= [a(aD \cap bD) \cup b(aD \cap bD)]_r \\ &= [aD \cup bD)(aD \cap bD)]_r \subset [(A_y \cap B_y)(A \cup B)]_r \,.\end{aligned}$$

By (d), it follows that

$$ab \in [(aD)(bD)]_r = [aD \cup bD)(aD \cap bD)]_r \subset [(A_y \cap B_y)(A \cup B)]_r \,,$$

and thus we obtain $AB \subset [(A_y \cap B_y)(A \cup B)]_r$ in both cases.

(c) $\Rightarrow$ (g) If $I,\, J \in \mathcal{F}_r(D)^\times$, then

$$[(I \cap J)(I \cup J)_r]_r = [(I \cap J)(I \cup J)]_r = (IJ)_r \in \mathcal{F}_r(D)^\times \,,$$

which implies $I \cap J \in \mathcal{F}_r(D)^\times$ and $(I \cup J)_r \in \mathcal{F}_r(D)^\times$.

(h) $\Rightarrow$ (a) We must prove that $E_r \in \mathcal{F}_r(D)^\times$ for every finite non-empty subset $E \subset D^\bullet$, and we proceed by induction on $|E|$. The assertion is obvious if $|E| = 1$. If $|E| \geq 1$, $E_r \in \mathcal{F}_r(D)^\times$ and $a \in D^\bullet$, then $(E \cup \{a\})_r = (E_r \cup aD)_r \in \mathcal{F}_r(D)^\times$, and thus the assertion follows by induction on $|E|$.

(a) $\Rightarrow$ (f) Let $F \in \boldsymbol{f}(D)$ and $A,\, B \in \boldsymbol{F}(D)$. Then

$$(FA)_r \cap (FB)_r = (FF^{-1})_r[(FA)_r \cap (FB)_r] \\ \subset \big(F[(F^{-1}FA)_r \cap (F^{-1}FB)_r]\big)_r = [F(A_r \cap B_r)]_r\,.$$

Since $F(A_r \cap B_r) \subset FA_r \cap FB_r \subset (FA)_r \cap (FB)_r$, the reverse inclusion is obvious.

(e) $\Rightarrow$ (d) As we have already proved the equivalence of (d) and (a), it suffices to show that (d) holds with $y = r$. Thus let $A,\, B \in \boldsymbol{f}(D)$, and set $F = (A \cup B)_\delta$. Then $F \in \boldsymbol{f}(D)$, and $[(A_r \cap B_r)(A \cup B)]_r = [(A_r \cap B_r)F]_r = (FA)_r \cap (FB)_r \supset (AB)_r$. On the other hand, we obviously have $(A_r \cap B_r)(A \cup B) \subset A_rB \cup AB_r \subset (AB)_r$, which implies the reverse inclusion.

(a) $\Rightarrow$ (i) Let $(A_i)_{i\in I}$ be a family in $\boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$. Since F_r is r-invertible, Theorem 3.1(f) (applied with $q = r$) implies

$$\Big(\big(\bigcup_{i\in I} A_i\big)_y : F\Big)_r = \Big(\big(\bigcup_{i\in I} A_i\big)_r F^{-1}\Big)_r = \Big(\bigcup_{i\in I} A_iF^{-1}\Big)_r \\ \subset \Big(\bigcup_{i\in I} (A_i)_rF^{-1}\Big)_r = \Big(\bigcup_{i\in I}((A_i)_y : F)\Big)_r\,,$$

and the reverse inclusion is obvious.

(k) $\Rightarrow$ (b) Let $a,\, b \in D^\bullet$ and apply (k) with $A = aD$, $B = bD$ and $F = \{a,b\}_\delta$. Then we obtain

$$D \subset (\{a,b\}_y : \{a,b\})_r = [(aD : \{a,b\}) \cup (bD : \{a,b\})]_r \\ = (a\{a,b\}^{-1} \cup b\{a,b\}^{-1})_r = (\{a,b\}\{a,b\}^{-1})_r = (\{a,b\}_r\{a,b\}^{-1})_r \subset D\,.$$

Hence equality holds, and $\{a,b\}_r$ is r-invertible.

(a) $\Rightarrow$ (l) Let $A \in \boldsymbol{F}(D)$ and $F,\, G \in \boldsymbol{f}(D)$. Then the fractional r-ideals F_r, G_r, F^{-1} and G^{-1} are r-invertible, and since we have already proved that (a) implies (g), it follows that $F_r \cap G_r$ and $(F^{-1} \cup G^{-1})_r$ are also r-invertible. Observe now that $F_r = (F^{-1})^{-1}$, $G_r = (G^{-1})^{-1}$ and

$$(F^{-1} \cup G^{-1})_r = [(F^{-1} \cup G^{-1})^{-1}]^{-1} = [(F^{-1})^{-1} \cap (G^{-1})^{-1}]^{-1} = (F_r \cap G_r)^{-1}\,.$$

We apply Theorem 3.1(f) with $q = r$ and obtain

$$\big(A_y : (F_r \cap G_r)\big)_r = [A_r(F_r \cap G_r)^{-1}]_r = [A_r(F^{-1} \cup G^{-1})_r]_r \\ = [(A_rF^{-1})_r \cup (A_rG^{-1})_r]_r = [(A_y : F_r) \cup (A_y : G_r)]_r$$

(l) $\Rightarrow$ (m) Let $a,\, b \in D^\bullet$, and apply 12. with $A = aD \cap bD$, $F = aD$ and $G = bD$. Then we obtain

$$D \subset ((aD \cap bD) \colon (aD \cap bD))_r = \big[\big((aD \cap bD) \colon aD\big) \cup \big((aD \cap bD) \colon bD\big)\big]_r$$
$$= [(D \cap a^{-1}bD) \cup (D \cap ab^{-1}D)]_r \subset D\,,$$

and consequently $[(a^{-1}bD \cap D) \cup (ab^{-1}D \cap D)]_r = D$. □

Remark 5.2. The presence of the ideal system y in Theorem 5.1 makes the criteria more flexible. The extremal cases $y = \delta$ and $y = r$ are the most interesting ones. Indeed, for $y = r$ the criteria become most transparent, for $y = d$ in the domain case they become comparable with criteria usually formulated in the literature, while the case $y = \delta$ is suitable for the monoid case.

Corollary 5.3. *Let r and q be ideal systems on D such that $r \le q$, and let D be an (r,q)-Prüfer monoid. Then the following assertions are equivalent*:

(a) *D is an r-Prüfer monoid.*

(b) $(A \cup B)_r = (A \cup B)_q$ *for all* $A,\, B \in \mathcal{F}_r(D)^\times$.

(c) $(A \cup B)_r = (A \cup B)_q$ *for all* $A,\, B \in \mathcal{F}_{r,\mathfrak{f}}(D) \cap \mathcal{F}_r(D)^\times$.

(d) $\{a,b\}_r = \{a,b\}_q$ *for all* $a,\, b \in D^\bullet$.

Proof. (a) $\Rightarrow$ (b) By Theorem 5.1(h) we have $(A \cup B)_r \in \mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)$ and therefore $(A \cup B)_r = [(A \cup B)_r]_q = (A \cup B)_q$.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) Obvious.

(d) $\Rightarrow$ (a) For all $a,\, b \in D^\bullet$ we have $\{a,b\}_r = \{a,b\}_q \in \mathcal{F}_{q,\mathfrak{f}}(D)^\bullet \subset \mathcal{F}_r(D)^\times$, since D is an (r,q)-Prüfer monoid. Thus it follows that D is an r-Prüfer monoid by Theorem 5.1(b). □

References

[1] D.D. Anderson and D.F. Anderson, Generalized GCD domains, *Comment. Math. Univ. St. Pauli* **28** (1979), 215 – 221.

[2] D.D. Anderson, D.F. Anderson, M. Fontana and M. Zafrullah, On v-domains and star operations, Manuscript, 2008.

[3] D.D. Anderson and A. Kang, Pseudo-Dedekind domains and the divisorial ideals in $R[X]_T$, *J. Algebra* **122** (1989), 323 –336.

[4] D.D. Anderson, J. Mott and M. Zafrullah, Some quotient based characterizations of domains of multiplicative ideal theory, *Bull. Mat. Ital.* **3-B** (1989), 455 – 476.

[5] N. Bourbaki, Commutative Algebra, Addison-Wesley, 1972.

[6] M. Fontana, J.A. Huckaba and I.J. Papick, Prüfer domains, Marcel Dekker, 1997.

[7] M. Fontana, P. Jara and E. Santos, Prüfer star-multiplication domains and semistar operations, *J. Algebra Appl.* **2003**, 21 – 50.

[8] M. Fontana and G. Picozza, Prüfer $*$-multiplication domains and $*$-coherence, *Ricerche Math.* **55** (2006), 145 – 170.

[9] R. Gilmer, Multiplicative ideal theory, Marcel Dekker, 1972.

[10] M. Griffin, Some results on v-multiplication rings, *Canad. J. Math.* **19** (1967), 710 – 722.

[11] F. Halter-Koch, Ideal Systems. An Introduction to Multiplicative Ideal Theory, Marcel Dekker, 1998.

[12] F. Halter-Koch, Characterization of Prüfer Multiplication Monoids and Domains by Means of Spectral Module Systems, *Monatsh. Math.* **139**, 19 – 31.

[13] E. Houston, S. Malik and J. Mott, Characterizations of $*$-multiplication domains, *Canad. Math. Bull.* **27** (1984), 48 –52.

[14] P. Jaffard, Les Systemes d'Ideaux, Dunot, 1960.

[15] M.D. Larsen, P.J. McCarthy, Multiplicative theory of ideals, Academic Press, 1971.

[16] P. Lorenzen, Abstrakte Begründung der multiplikativen Idealtheorie, *Math. Z.* **45** (1939), 533 – 553.

[17] M. Zafrullah, On generalized Dedekind domains, *Mathematika* **33** (1986), 285 – 296.

[18] M. Zafrullah, Ascending chain condition and star operations, *Comm. Alg.* **17** (1989), 1523 – 1533.

[19] M. Zafrullah, A Question-Answer Session on v-domains, http://www.lohar.com.

Author information

F. Halter-Koch, Institute for Mathematics and Scientific Computing, Heinrichstraße 36, A-8010 Graz, Austria.
E-mail: `franz.halterkoch@aon.at`