Idealtheorie kommutativer Ringe und Monoide

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Contents

CHAPTER 1

Generalities on Monoids

For a set X, we denote by $\mathbb{P}(X)$ the power set and by $\mathbb{P}_{\mathfrak{f}}(X)$ the set of all finite subsets of X. If A, B are sets, then $A \subset B$ or $B \supset A$ means that A is a subset of B which may be equal to B. If A is a proper subset of B, we write $A \subsetneq B$ or $B \supsetneq A$.

As usual, we denote by $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ the sets of integers, rational numbers, real numbers and complex numbers. We denote by $\mathbb{N} = \{x \in \mathbb{Z} \mid x > 0\}$ the set of positive integers, and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $x,\, y \in \mathbb{Z}$ and $x \le y$, we set $[x, y] = \{z \in \mathbb{Z} \mid x \le z \le y\}$. For a set X, we denote by $|X| \in \mathbb{N}_0 \cup \{\infty\}$ its cardinality.

Let X be a set. A subset $\Sigma \subset \mathbb{P}(X)$ is called

- *directed* if, for any $A,\, B \in \Sigma$, there is some $C \in \Sigma$ such that $A \cup B \subset C$;
- a *chain* if, for any $A,\, B \in \Sigma$, we have $A \subset B$ or $B \subset A$.

A family $(A_\lambda)_{\lambda \in \Lambda}$ of subsets of X is called *directed* or a *chain* if the set $\{A_\lambda \mid \lambda \in \Lambda\}$ has this property. If $(A_\lambda)_{\lambda \in \Lambda}$ is directed and E is a finite set, then

$$E \subset \bigcup_{\lambda \in \Lambda} A_\lambda \quad \text{implies} \quad E \subset A_\lambda \quad \text{for some } \lambda \in \Lambda.$$

We shall frequently use Zorn's Lemma in the following form:

> *Let X be a set, $\emptyset \ne \Sigma \subset \mathbb{P}(X)$, and suppose that the union of every chain in Σ belongs to Σ. Then Σ contains maximal elements (with respect to the inclusion).*

A *partial ordering* on a set X is a binary relation $\le$ such that the following assertions hold for all $x,\, y \in X$:

- $x \le x$;
- $x \le y$ and $y \le x$ implies $x = y$.
- $x \le y$ and $y \le z$ implies $x \le z$.

If $\le$ is a partial ordering on X, we call $(X, \le)$ a *partially ordered set.* We call $\le$ a *total ordering* and $(X, \le)$ a *totally ordered set* if, for all $x,\, y \in X$ we have either $x \le y$ or $y \le x$.

Let $(X, \le)$ be a partially ordered set. Then every subset of X is again a partially ordered set with the induced order. A totally ordered subset of X is called a *chain.* Sometimes we will use the abstract form of Zorn's Lemma as follows:

> *Let $(X, \le)$ be a non-empty partially ordered set, and assume that every non-empty chain in X has an upper bound. Then X contains maximal elements.*

For a partially ordered set $(X, \le)$, the following assertions are equivalent:

- *For every sequence $(a_n)_{n \ge 0}$ in X satisfying $a_n \le a_{n+1}$ for all $n \ge 0$, there exists some $m \ge 0$ such that $a_n = a_m$ for all $n \ge m$* [*in other words, every ascending sequence in X becomes ultimately stationary*].
- *Every non-empty subset of X contains a maximal element.*

If these conditions are fulfilled, then $(X, \le)$ is said to be *noetherian* or to satisfy the ACC (the *ascending chain condition*).

1.1. Preliminaries on Monoids

Let K be a multiplicative semigroup. An element $n \in K$ is called a *zero element* if $na = n$ for all $a \in K$. An element $e \in K$ is called a *unit element* if $ea = a$ for all $a \in K$. Plainly, K possesses at most one zero element, denoted by $0 = 0_K$ and at most one unit element, denoted by $1 = 1_K$. For subsets $X, Y \subset K$ and $a \in K$, we define $XY = \{xy \mid x \in X,\, y \in Y\}$ and $aX = \{a\}X$. For $n \in \mathbb{N}$, we define X^n recursively by $X^1 = 1$ and $X^{n+1} = X^n X$, and we set $X^{(n)} = \{x^n \mid x \in X\}$.

By a *monoid* we mean a multiplicative semigroup K containing a zero element $0 = 0_K$ and a unit element $1 = 1_K$. Clearly, $0_K = 1_K$ if and only if $|K| = 1$, and in this case K is called a *trivial monoid*. A *monoid without zero* is a multiplicative semigroup K which is either trivial or does not contain a zero element. Thus the trivial monoid is both a monoid and a monoid without zero. A subset $S \subset K$ is called *multiplicatively closed* if $1 \in S$ and $SS \subset S$.

Let K be a monoid. An element $a \in K$ is called *cancellative* if $ab = ac$ implies $b = c$ for all $b, c \in K$. For a subset $X \subset K$, we set $X^\bullet = X \setminus \{0\}$, and we denote by X^* the set of all cancellative elements of X. If K is non-trivial, then $K^* \subset K^\bullet$. K is called *cancellative* if $K^\bullet \subset K^*$. Hence K is cancellative if and only if either K is trivial or $K^\bullet = K^*$.

An element $u \in K$ is called *invertible* if there exists some $u' \in K$ such that $uu' = 1$. In this case, u' is uniquely determined by u, it is called the *inverse* of u and denoted by u^{-1}. We denote by $K^\times$ the set of all invertible elements of K. Endowed with the induced multiplication, $K^\times$ is a group, and $K^\times \subset K^*$. The monoid K is called

- *reduced* if $K^\times = \{1\}$;
- *divisible* if $K^\bullet \subset K^\times$.

By definition, K is divisible if and only if either K is trivial or $K^\bullet = K^\times$. If K is divisible, then K is cancellative.

The most important example of a monoid is the multiplicative monoid $D = (D, \cdot)$ of a ring D (throughout this volume, rings are assumed to be commutative and unitary, and modules and ring homomorphisms are assumed to be unitary). Note that D is a trivial monoid if and only if D is a zero ring, and $D \setminus D^*$ is the set of zero divisors of D. If D is non-trivial, then D is cancellative if and only if D is a domain, and D is divisible if and only if D is a field.

Let D be a monoid. A subset $Q \subset D$ is called

- *multiplicatively closed* if $1 \in Q$ and $QQ \subset Q$ (then $QQ = Q$);
- a *submonoid* if it is multiplicatively closed and $0 \in Q$;
- a (*semigroup*) *ideal* of D if $0 \in Q$ and $DQ \subset Q$ (then $DQ = Q$);
- a *principal ideal* of D if $Q = Da$ for some $a \in D$.
- a *prime ideal* of D if Q is an ideal and $D \setminus Q$ is multiplicatively closed.

By definition, $\{0, 1\}$ is the smallest submonoid of D, $\{0\} = D0$ and $D = D1$ are principal ideals of D, and $D \setminus D^\times$ is a prime ideal of D.
If D is cancellative [reduced], then every submonoid of D is also cancellative [reduced].

For $a, b \in D$ we define $a \,|\, b$ if $bD \subset aD$. If $b = au$ for some $u \in D^\times$, then $aD = bD$. Conversely, if D is cancellative and $aD = bD$, then $b = au$ for some $u \in D^\times$.

Lemma 1.1.1. *Let D be a monoid.*

1. *If $J \subset D$ is an ideal, then $J = D$ if and only if $J \cap D^\times \neq \emptyset$.*

2. *D is divisible if and only if $\{0\}$ and D are the only ideals of D.*
3. *If D is cancellative and not trivial, then $D^\bullet$ is a multiplicatively closed subset and $\{0\}$ is a prime ideal of D.*
4. *Let $(Q_\lambda)_{\lambda\in\Lambda}$ be a family of subsets of D,*

$$Q^* = \bigcup_{\lambda\in\Lambda} Q_\lambda \quad \textit{and} \quad Q_* = \bigcap_{\lambda\in\Lambda} Q_\lambda\,.$$

   (a) *If $(Q_\lambda)_{\lambda\in\Lambda}$ is a family of ideals of D, then Q^* and Q_* are ideals of D.*
   (b) *If $(Q_\lambda)_{\lambda\in\Lambda}$ is a family of prime ideals of D, then Q^* is a prime ideal of D, and if $(Q_\lambda)_{\lambda\in\Lambda}$ is a chain, then Q_* is also a prime ideal of D.*
   (c) *If $(Q_\lambda)_{\lambda\in\Lambda}$ is a family of submonoids of D, then Q_* is a submonoid of D, and if $(Q_\lambda)_{\lambda\in\Lambda}$ is directed, then Q^* is also a submonoid of D.*

PROOF. 1. Let $J \subset D$ be an ideal. If $J = D$, then $J \cap D^\times = D^\times \ne \emptyset$. If $u \in J \cap D^\times$ and $a \in D$, then $a = (au^{-1})u \in J$ and therefore $J = D$.

2. Let D be divisible and $J \subset D$ an ideal of D. If $a \in J^\bullet$, then $1 = a^{-1}a \in J$ and therefore $J = D$. If D is not divisible and $a \in D^\bullet \setminus D^\times$, then $1 \notin aD$, and therefore aD is a non-zero ideal distinct from D.

3. If D is cancellative and not trivial, then $D^\bullet = D^*$ is multiplicatively closed, and therefore $\{0\}$ is a prime ideal of D.

4. (a) If $a \in D$ and $x \in Q^*$, then $x \in Q_\lambda$ for some $\lambda \in \Lambda$ and therefore $ax \in Q_\lambda \subset Q^*$. If $a \in D$ and $x \in Q_*$, then $x \in Q_\lambda$ and thus $ax \in Q_\lambda$ for all $\lambda \in \Lambda$, and therefore $ax \in Q_*$.

(b) If $a,\, b \in D \setminus Q^*$, then $a,\, b \in D \setminus Q_\lambda$ and therefore $ab \in D \setminus Q_\lambda$ for all $\lambda \in \Lambda$. Hence it follows that $ab \in D \setminus Q^*$, and therefore Q^* is a prime ideal of D.

Let now $(Q_\lambda)_{\lambda\in\Lambda}$ be a chain and $a,\, b \in D \setminus Q_*$. Then there exist $\lambda,\, \mu \in \Lambda$ such that $a \notin Q_\lambda$ and $b \notin Q_\mu$, and we may assume that $Q_\lambda \subset Q_\mu$. Then it follows that $a,\, b \notin Q_\lambda$, hence $ab \notin Q_\lambda$ and therefore $ab \notin Q_*$. Hence Q_* is a prime ideal of D.

(c) Let $(Q_\lambda)_{\lambda\in\Lambda}$ be a family of submonoids of D. Then $0 \in Q_* \subset Q^*$. If $a,\, b \in Q_*$, then $a,\, b \in Q_\lambda$ and therefore $ab \in Q_\lambda$ for all $\lambda \in \Lambda$. Hence $ab \in Q_*$, and therefore Q_* is a submonoid of D.

Let now $(Q_\lambda)_{\lambda\in\Lambda}$ be directed and $a,\, b \in Q^*$. Then there exists some $\lambda \in \Lambda$ such that $a,\, b \in Q_\lambda$. Hence $ab \in Q_\lambda \subset Q^*$, and therefore Q^* is a submonoid of D. □

Let K and L be a monoids. A map $f\colon K \to L$ is called a (*monoid*) *homomorphism* if

$$f(1_K) = 1_L\,, \quad f(0_K) = 0_L\,, \quad \text{and} \quad f(xy) = f(x)f(y) \quad \text{for all } x,\, y \in K.$$

As usual, a homomorphism is called a *monomorphism* [an *epimorphism*, an *isomorphism*] if it is injective [surjective, bijective]. The monoids K and L are called *isomorphic* if there exists an isomorphism $f\colon K \to L$, and in this case we write $f\colon K \xrightarrow{\sim} L$.

Let $f\colon K \to L$ be a monoid homomorphism. Then $f(K^\times) \subset L^\times$, and $f \,|\, K^\times\colon K^\times \to L^\times$ is a group homomorphism. If $J \subset L$ is an ideal, then $f^{-1}(J) \subset K$ is also an ideal [indeed, if $x \in f^{-1}(J)$ and $a \in K$, then $f(ax) = f(a)f(x) \in LJ = J$ and therefore $ax \in f^{-1}(J)$].

Let K be a monoid and $G \subset K^\times$ a subgroup. Then we set $K/G = \{aG \mid a \in K\}$, and we define a multiplication on K/G by means of $(aG)(bG) = abG$ for all $a,\, b \in K$. This definition does not depend on the representatives, it makes K/G into a monoid, and $\pi\colon K \to K/G$, defined by $\pi(a) = aG$ for all $a \in K$, is a monoid epimorphism, called *canonical*. By definition, $(K/G)^\bullet = \{aG \mid a \in K^\bullet\}$, $(K/G)^* = \{aG \mid a \in K^*\}$, and $(K/G)^\times = K^\times/G$ (the factor group). Consequently, K/G is cancellative [divisible] if and only if K is cancellative [divisible].

If $G \subset K^\times$ is a subgroup, then the canonical epimorphism $\pi\colon K \to K/G$ is an isomorphism if and only if $G = \{1\}$, and in this case we identify K with $K/\{1\}$ by means of π and set $K = K/\{1\}$. The monoid $K/K^\times$ is reduced. It is called the *associated reduced monoid* of K.

Let $f\colon K \to L$ be a monoid homomorphism, and let $G \subset K^\times$ and $H \subset L^\times$ be subgroups such that $f(G) \subset H$. Then there is a unique homomorphism $f^*\colon K/G \to L/H$ such that $f^*(aG) = f(a)H$ for all $a \in K$. We say that f^* is *induced by* f.

Let K and L be divisible monoids. A map $f\colon K \to L$ is a monoid homomorphism if and only if $f(0_K) = 0_L$, and $f \mid K^\times\colon K^\times \to L^\times$ is a group homomorphism. In this case, $f^{-1}(1) = \operatorname{Ker}(f \mid K^\times)$ is a subgroup of $K^\times$, and f induces a monomorphism $f^*\colon K/f^{-1}(1) \to L$.

Let K be a monoid. For subsets $X, Y \subset K$ and $y \in K$, we define

$$(X\!:\!Y) = (X\!:_K\!Y) = \{z \in K \mid zY \subset X\} \quad \text{and} \quad (X\!:\!y) = (X\!:\!\{y\})\,.$$

Lemma 1.1.2. *Let K be a monoid and $X, X', Y, Y' \subset K$.*

1. *If $X \subset X'$ and $Y \subset Y'$, then $(X\!:\!Y') \subset (X'\!:\!Y)$.*
2. $(X\!:\!YY') = ((X\!:\!Y)\!:\!Y')$.
3. $(X\!:\!X)$ *is a submonoid of K.*
4. *If $a \in K^\times$, then $(aX\!:\!Y) = a(X\!:\!Y)$ and $(X\!:\!aY) = a^{-1}(X\!:\!Y)$.*
5. *If $(Y_\lambda)_{\lambda\in\Lambda}$ is a family of subsets of K, then*

$$\Big(X\!:\bigcup_{\lambda\in\Lambda} Y_\lambda\Big) = \bigcap_{\lambda\in\Lambda}(X\!:\!Y_\Lambda)\,, \quad \textit{and if} \quad Y \subset K^\times\,, \quad \textit{then} \quad (X\!:\!Y) = \bigcap_{y\in Y} y^{-1}X\,.$$

Proof. 1. If $z \in (X\!:\!Y')$, then $zY \subset zY' \subset X \subset X'$, and therefore $z \in (X'\!:\!Y)$.

2. If $z \in K$, then

$$z \in (X\!:\!YY') \iff zYY' = (zY')Y \subset X \iff zY' \subset (X\!:\!Y) \iff z \in ((X\!:\!Y)\!:\!Y')\,.$$

3. Clearly, $0 \in (X : X)$, and if $x, y \in (X : X)$, then $xyX = x(yX) \subset xX \subset X$, and therefore $xy \in (X\!:\!Y)$.

4. Let $a \in K^\times$ and $z \in K$. Then

$$z \in (aX\!:\!Y) \iff zY \subset aX \iff a^{-1}zY \subset X \iff a^{-1}z \in (X\!:\!Y) \iff z \in a(X\!:\!Y)$$

and

$$z \in (X\!:\!aY) \iff zaY \subset X \iff za \in (X\!:\!Y) \iff z \in a^{-1}(X\!:\!Y)\,.$$

5. Let $(Y_\lambda)_{\lambda\in\Lambda}$ be a family of subsets of K and $z \in K$. Then

$$z \in \Big(X\!:\bigcup_{\lambda\in\Lambda} Y_\lambda\Big) \iff zY_\lambda \subset X \text{ for all } \lambda \in \Lambda \iff z \in (X\!:\!Y_\lambda) \text{ for all } \lambda \in \Lambda \iff z \in \bigcap_{\lambda\in\Lambda}(X\!:\!Y_\Lambda)\,.$$

If $Y \subset K^\times$, then

$$(X\!:\!Y) = \Big(X\!:\bigcup_{y\in Y}\{y\}\Big) = \bigcap_{y\in Y}(X\!:\!\{y\}) = \bigcap_{y\in Y} y^{-1}(X\!:\!\{1\}) = \bigcap_{y\in Y} y^{-1}X\,. \qquad \square$$

1.2. Quotient Monoids

Remarks and Definition 1.2.1. Let K be a monoid and $T \subset K$ a multiplicatively closed subset. For (x,t), $(x',t') \in K \times T$, we define

$$(x,t) \sim (x',t') \quad \text{if} \quad st'x = stx' \quad \text{for some} \quad s \in T\,.$$

Then $\sim$ is an equivalence relation on $K \times T$.

Proof. Obviously, $\sim$ is reflexive and symmetric. To prove transitivity, let (x,t), (x',t'), $(x'',t'') \in K \times T$ be such that $(x,t) \sim (x',t')$ and $(x',t') \sim (x'',t'')$. Then there exist s, $s' \in T$ such that $st'x = stx'$ and $s't''x' = s't'x''$. Then it follows that $s'st' \in T$ and $(s'st')t''x = s't''stx' = (s'st')tx''$, hence $(x,t) \sim (x'',t'')$. □

We define the *quotient monoid* $T^{-1}K$ of K with respect to T by $T^{-1}K = K \times T/\sim$. For $(x,t) \in K \times T$, we denote by

$$\frac{x}{t} \in T^{-1}X \quad \text{the equivalence class of } (x,t)\,, \quad \text{and we define} \quad j_T \colon K \to T^{-1}K \quad \text{by} \quad j_T(x) = \frac{x}{1}\,.$$

The map j_T is called the *natural embedding* (although it need not be injective). By definition, if (x,t), $(x',t') \in K \times T$, then

$$\frac{x}{t} = \frac{x'}{t'} \quad \text{if and only if} \quad st'x = stx' \quad \text{for some} \quad s \in T\,,$$

and if $T \subset K^*$, then

$$\frac{x}{t} = \frac{x'}{t'} \quad \text{if and only if} \quad t'x = tx'\,.$$

If $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in T^{-1}K$, then $z_1, \ldots, z_n$ have a common denominator, that is, there exist $x_1, \ldots, x_n \in K$ and $t \in T$ such that

$$z_i = \frac{x_i}{t} \quad \text{for all} \quad i \in [1,n]\,.$$

For x, $x' \in K$ and t, $t' \in T$, we define

$$\frac{x}{t} \cdot \frac{x'}{t'} = \frac{xx'}{tt'}\,.$$

This definition does not depend on the choice of the representatives. Endowed with this multiplication, $T^{-1}K$ becomes a monoid with unit element $\frac{1}{1}$ and zero element $\frac{0}{1}$, and j_T is a monoid homomorphism. If $0 \in T$, then $T^{-1}K$ is a trivial monoid.

Proof. Suppose that (x,t), (x_1,t_1), (x',t'), $(x_1',t_1') \in K \times T$, $(x,t) \sim (x_1,t_1)$ and $(x',t') \sim (x_1',t_1')$. We must prove that $(xx',tt') \sim (x_1x_1', t_1t_1')$. Let s, $s' \in T$ be such that $st_1x = stx_1$ and $s't_1'x' = s't'x_1'$. Then it follows that $ss' \in T$ and $ss't_1t_1'xx' = ss'tt'x_1x_1'$, which implies $(xx',tt') \sim (x_1x_1', t_1t_1')$. Now it is obvious that this multiplication is associative and commutative, $\frac{1}{1}$ is a unit element and $\frac{0}{1}$ is a zero element. If x, $y \in K$, then

$$j_T(xy) = \frac{xy}{1} = \frac{x}{1}\frac{y}{1} = j_T(x)j_T(y)\,, \quad j_T(0) = \frac{0}{1} \quad \text{and} \quad j_T(1) = \frac{1}{1}\,.$$

Hence j_T is a monoid homomorphism. If $0 \in T$, then $(x,t) \sim (x',t')$ for all (x,t), $(x',t') \in K \times T$, and therefore $|T^{-1}K| = 1$. □

For every subset $X \subset K$, we set

$$T^{-1}X = \Big\{ \frac{x}{t} \;\Big|\; x \in X\,,\ t \in T \Big\} \subset T^{-1}K\,.$$

If $X' \subset X \subset K$, then $T^{-1}X' \subset T^{-1}X \subset T^{-1}K$. Hence it follows that $T^{-1}(X \cap Y) \subset T^{-1}X \cap T^{-1}Y$ for any subsets X, $Y \subset K$.

Theorem 1.2.2. *Let K be a monoid, $T \subset K$ a multiplicatively closed subset and $j_T \colon K \to T^{-1}K$ the natural embedding.*

1. *If $X, Y \subset K$, then $T^{-1}(XY) = (T^{-1}X)(T^{-1}Y)$, and if additionally $TX = X$ and $TY = Y$, then $T^{-1}(X \cap Y) = T^{-1}X \cap T^{-1}Y$.*
2. *If $(X_\lambda)_{\lambda \in \Lambda}$ is a family of subsets of K, then*
$$T^{-1}\Big(\bigcup_{\lambda \in \Lambda} X_\lambda\Big) = \bigcup_{\lambda \in \Lambda} T^{-1}X_\lambda \,.$$
3. *If J is an ideal of K, then $T^{-1}J$ is an ideal of $T^{-1}K$, $J \subset j_T^{-1}(T^{-1}J)$, and $T^{-1}J = T^{-1}K$ if and only if $J \cap T \neq \emptyset$.*
4. *If V is an ideal of $T^{-1}J$, then $J = j_T^{-1}(V)$ is an ideal of K, and $V = T^{-1}J$.*

Proof. 1. Let $X, Y \subset K$. If $z \in T^{-1}(XY)$, then $z = \frac{xy}{t}$ for some $x \in X$, $y \in Y$ and $t \in T$, and therefore $z = \frac{x}{t}\frac{y}{1} \in (T^{-1}X)(T^{-1}Y)$. Conversely, if $z \in (T^{-1}X)(T^{-1}Y)$, then $z = \frac{x}{t}\frac{y}{s}$ for some $x \in X$, $y \in Y$ and $s, t \in T$. Hence $z = \frac{xy}{st} \in T^{-1}(XY)$.

Assume now that $TX = X$ and $TY = Y$. Clearly, $T^{-1}(X \cap Y) \subset T^{-1}X \cap T^{-1}Y$. If $z \in T^{-1}X \cap T^{-1}Y$, then $z = \frac{x}{t} = \frac{y}{s}$, where $x \in X$, $y \in Y$ and $s, t \in T$. Then there is some $w \in T$ such that $wsx = xty$. Since $wsx = wty \in TX \cap TY = X \cap Y$ it follows that
$$z = \frac{wsx}{wst} \in T^{-1}X \cap T^{-1}Y \,.$$

2. If $\alpha \in \Lambda$, then
$$X_\alpha \subset \bigcup_{\lambda \in \Lambda} X_\lambda \quad \text{implies} \quad T^{-1}X_\alpha \subset T^{-1}\Big(\bigcup_{\lambda \in \Lambda} X_\lambda\Big), \quad \text{and therefore} \quad \bigcup_{\lambda \in \Lambda} T^{-1}X_\lambda \subset T^{-1}\Big(\bigcup_{\lambda \in \Lambda} X_\lambda\Big).$$
Conversely, if
$$z \in T^{-1}\Big(\bigcup_{\lambda \in \Lambda} X_\lambda\Big),, \quad \text{then} \quad z = \frac{x}{t}\,, \quad \text{where } t \in T \text{ and } x \in X_\alpha \text{ for some } \alpha \in \Lambda$$
and therefore
$$z \in T^{-1}X_\alpha \subset \bigcup_{\lambda \in \Lambda} T^{-1}X_\lambda \,.$$

3. Obviously, $T^{-1}J$ is an ideal of $T^{-1}K$, and $J \subset j_T^{-1}(T^{-1}J)$. If $T^{-1}K = T^{-1}J$, then $\frac{1}{1} \in T^{-1}J$. Hence $\frac{1}{1} = \frac{a}{t}$ for some $a \in J$ and $t \in T$, and there exists some $s \in T$ such that $st = sa \in T \cap J$. Conversely, if $s \in T \cap J$, then $\frac{1}{1} = \frac{s}{s} \in T^{-1}J$, which implies $T^{-1}J = T^{-1}K$.

4. Since j_T is a monoid homomorphism, it follows that $J = j_T^{-1}(V)$ is an ideal of K. If $a \in J$ and $t \in T$, then $\frac{a}{1} \in V$ and therefore $\frac{a}{t} = \frac{1}{t}\frac{a}{1} \in V$. Hence $T^{-1}J \subset V$. To prove the converse, let $\frac{a}{t} \in V$, where $a \in K$ and $t \in T$. Then $\frac{a}{1} = \frac{t}{1}\frac{a}{t} \in V$, hence $a \in J$ and $\frac{a}{t} \in T^{-1}J$. □

Theorem 1.2.3. *Let K and L be a monoids, $T \subset K$ a multiplicatively closed subset and $\varphi \colon K \to L$ be a homomorphism such that $\varphi(T) \subset L^\times$. Then there exists a unique homomorphism $\Phi \colon T^{-1}K \to L$ such that $\Phi \circ j_T = \varphi$. It is given by*
$$\Phi\Big(\frac{a}{t}\Big) = \varphi(t)^{-1}\varphi(a) \quad \textit{for all } \ a \in K \ \textit{ and } \ t \in T.$$

Proof. Let $\Phi \colon T^{-1}K \to L$ be a homomorphism satisfying $\Phi \circ j_T = \varphi$. For $a \in K$ and $t \in T$ we have $\varphi(t) \in L^\times$,
$$\varphi(t)\Phi\Big(\frac{a}{t}\Big) = \Phi\Big(\frac{t}{1}\Big)\Phi\Big(\frac{a}{t}\Big) = \Phi\Big(\frac{at}{t}\Big) = \Phi\Big(\frac{a}{1}\Big) = \varphi(a)\,, \quad \text{and therefore} \quad \Phi\Big(\frac{a}{t}\Big) = \varphi(t)^{-1}\varphi(a)\,.$$

This proves uniqueness and the formula for Φ. To prove existence we define Φ by the formula above and prove that this definition does not depend on the choice of representatives.

If $a,\, a' \in K$ and $t,\, t' \in T$ are such that $\frac{a}{t} = \frac{a'}{t'}$, then there is some $s \in T$ such that $st'a = sta'$, hence $\varphi(s)\varphi(t')\varphi(a) = \varphi(s)\varphi(t)\varphi(a')$ and therefore $\varphi(t)^{-1}\varphi(a) = \varphi(t')^{-1}\varphi(a')$.

By the very definition, $\Phi \circ j_T = \varphi$, $\Phi\bigl(\frac{0}{1}\bigr) = \varphi(1)^{-1}\varphi(0) = 0$ and $\Phi\bigl(\frac{1}{1}\bigr) = \varphi(1)^{-1}\varphi(1) = 1$. If $a,\, a' \in K$ and $t,\, t' \in T$, then

$$\Phi\Bigl(\frac{a}{t}\,\frac{a'}{t'}\Bigr) = \Phi\Bigl(\frac{aa'}{tt'}\Bigr) = \varphi(tt')^{-1}\varphi(aa') = \varphi(t)^{-1}\varphi(a)\varphi(t')^{-1}\varphi(a') = \Phi\Bigl(\frac{a}{t}\Bigr)\Phi\Bigl(\frac{a'}{t'}\Bigr)\,.$$

Hence Φ is a homomorphism. □

Theorem und Definition 1.2.4. *Let K be a monoid, $T \subset K$ a multiplicatively closed subset, and*

$$\overline{T} = \{ s \in K \mid sK \cap T \neq \emptyset \}\,.$$

$\overline{T}$ is called the *divisor-closure* of T, and T is called *divisor-closed* if $T = \overline{T}$.

1. *Let $\mathcal{J}_T$ be the set of all ideals $J \subset K$ such that $J \cap T = \emptyset$ and*
$$P = \bigcup_{J \in \mathcal{J}_T} J\,.$$
Then $\overline{T} = K \setminus P$ is multiplicatively closed, $T \subset \overline{T} = \overline{\overline{T}}$, and if $\overline{T} \neq K$, then P is a prime ideal, and it is the greatest ideal of K such that $P \cap T = \emptyset$.
2. $(T^{-1}K)^\times = T^{-1}\overline{T}$*, and there is an isomorphism*
$$\iota\colon T^{-1}K \ \xrightarrow{\sim}\ \overline{T}^{-1}K\,, \quad \textit{given by} \quad \iota\Bigl(\frac{x}{t}\Bigr) = \frac{x}{t} \quad \textit{for all} \ \ x \in K \ \ \textit{and} \ \ t \in T\,.$$
Note that ι is not the identity map, since the two fractions appearing in its description denote different equivalence classes. However, we shall identify them: $T^{-1}K = \overline{T}^{-1}K$.
3. *Let $S \subset T$ be a multiplicatively closed subset. Then $ST \subset K$ and $T^{-1}S \subset T^{-1}K$ are multiplicatively closed subsets, and there is an isomorphism*
$$\Phi\colon (T^{-1}S)^{-1}(T^{-1}K) \ \xrightarrow{\sim}\ (ST)^{-1}K\,, \quad \textit{given by} \quad \Phi\Bigl(\frac{\frac{x}{t}}{\frac{s}{t'}}\Bigr) = \frac{t'x}{st}$$
for all $\in K$, $t,\, t' \in T$ and $s \in S$.
4. *If $X, Y \subset K$, then $(T^{-1}X :_{T^{-1}K} T^{-1}Y) = (T^{-1}X : T^{-1}Y) = (T^{-1}X : j_T(Y)) \supset T^{-1}(X : Y)$, and equality holds if $TX = X$ and Y is finite.*

Proof. 1. Suppose that $s \in \overline{T}$, and let $J \in \mathcal{J}_T$. If $a \in K$ is such that $sa \in T$, then $sa \notin J$ and thus $s \notin J$. Hence $\overline{T} \subset K \setminus P$. Conversely, if $s \in K \setminus P$, then $sK \notin \mathcal{J}_T$, hence $sK \cap T \neq \emptyset$ and $s \in \overline{T}$.

Clearly $T \subset \overline{T} \subset \overline{\overline{T}}$. If $s \in \overline{\overline{T}}$, then $ts \in \overline{T}$ for some $t \in K$, hence $t'ts \in T$ for some $t' \in K$, and therefore $s \in \overline{T}$. Hence $\overline{T} = \overline{\overline{T}}$. If $s_1,\, s_2 \in \overline{T}$, there exist $t_1,\, t_2 \in K$ such that $s_1t_1,\, s_2t_2 \in T$, which implies $s_1s_2t_1t_2 \in T$ and thus $s_1s_2 \in \overline{T}$. Hence $\overline{T}$ is multiplicatively closed. If $\overline{T} \neq K$, then P is an ideal of K by Lemma 1.1.1. By definition, P is the greatest ideal of K such that $P \cap T = \emptyset$, and it is a prime ideal since $\overline{T}$ is multiplicatively closed.

2. Let $x \in K$ and $t \in T$. We shall prove that $\frac{x}{t} \in (T^{-1}K)^\times$ if and only if $t \in \overline{T}$.

If $\frac{x}{t} \in (T^{-1}K)^\times$, then there exist $x' \in K$ and $t' \in T$ such that $\frac{x}{t}\,\frac{x'}{t'} = \frac{1}{1}$. Hence there is some $w \in T$ such that $wxx' = wtt'$, and $wtt' \in T$ implies $x \in \overline{T}$. Conversely, if $x \in \overline{T}$ and $t \in T$, let $w \in K$ be such that $xw \in T$. Then $\frac{tw}{xw} \in T^{-1}K$ and $\frac{x}{t}\,\frac{tw}{xw} = \frac{1}{1}$, and therefore $\frac{x}{t} \in (t^{-1}K)^\times$.

Let $j_{\overline{T}}\colon K \to \overline{T}^{-1}K$ be the natural embedding. Since $j_{\overline{T}}(T) \subset \overline{T}^{-1}T \subset (\overline{T}^{-1}K)^\times$, Theorem 1.2.3 implies the existence of some homomorphism $\iota\colon T^{-1}K \to \overline{T}^{-1}K$ satisfying $\iota\bigl(\frac{x}{t}\bigr) = \frac{x}{t}$ for all $x \in K$ and $t \in T$.

ι is injective: Let $x,\, x' \in K$ and $t,\, t' \in T$ be such that $\frac{x}{t} = \frac{x'}{t'}$ in $\overline{T}^{-1}K$. Then there exists some $s \in \overline{T}$ such that $st'x = stx'$. If $w \in K$ is such that $ws \in T$, then $(ws)t'x = (ws)tx'$ and therefore $\frac{x}{t} = \frac{x'}{t'}$ in $T^{-1}K$.

ι is surjective: Let $z \in \overline{T}^{-1}K$, say $z = \frac{x}{s}$, where $x \in K$ and $s \in \overline{T}$. If $t \in K$ is such that $st \in T$, then $y = \frac{xt}{st} \in T^{-1}K$, and $\iota(y) = z$.

3. Clearly, $ST \subset K$ and $T^{-1}S \subset T^{-1}K$ are multiplicatively closed, and the homomorphism $j_{ST} \colon K \to (ST)^{-1}K$ satisfies $j_{ST}(T) \subset (ST)^{-1}T \subset ((ST)^{-1}K)^\times$. Hence Theorem 1.2.3 implies the existence of some homomorphism $\varphi \colon T^{-1}K \to (ST)^{-1}K$ satisfying $\varphi\big(\frac{x}{t}\big) = \big(\frac{t}{1}\big)^{-1}\frac{x}{1} = \frac{x}{t}$ for all $x \in K$ and $t \in T$. Since $\varphi(T^{-1}S) \subset (ST)^{-1}S \subset ((ST)^{-1}K)^\times$, again Theorem 1.2.3 implies the existence of a homomorphism $\Phi \colon (T^{-1}S)^{-1}(T^{-1}K) \to (ST)^{-1}K$ satisfying

$$\Phi\Big(\frac{\frac{x}{t}}{\frac{s}{t'}}\Big) = \varphi\Big(\frac{s}{t'}\Big)^{-1}\varphi\Big(\frac{x}{t}\Big) = \frac{t'x}{st} \quad \text{for all} \quad x \in K\,,\ s \in S \ \text{ and } \ t,\, t' \in T\,.$$

Φ is injective: Let $x,\, x_1 \in K$, $s,\, s_1 \in S$ and $t,\, t_1,\, t',\, t_1' \in T$ be such that

$$\Phi\Big(\frac{\frac{x}{t}}{\frac{s}{t'}}\Big) = \Phi\Big(\frac{\frac{x_1}{t_1}}{\frac{s_1}{t_1'}}\Big) \in (ST)^{-1}K\,, \quad \text{that is,} \quad \frac{t'x}{st} = \frac{t_1x_1}{s_1t_1}\,.$$

Then there exist some $v \in S$ and $w \in T$ such that $vws_1t_1t'x = vwstt_1'x_1$. Hence

$$\frac{v}{w}\,\frac{x}{t}\,\frac{s_1}{t_1'} = \frac{v}{w}\,\frac{x_1}{t_1}\,\frac{s}{t'} \in T^{-1}K\,, \quad \text{and therefore} \quad \frac{\frac{x}{t}}{\frac{s}{t'}} = \frac{\frac{x_1}{t_1}}{\frac{s_1}{t_1'}} \in (T^{-1}S)^{-1}(T^{-1}K)\,.$$

Φ is surjective: Let $z = \frac{x}{st} \in (ST)^{-1}K$, where $s \in S$, $t \in T$ and $x \in K$. Then

$$y = \frac{\frac{x}{t}}{\frac{s}{1}} \in (T^{-1}S)^{-1}(T^{-1}K) \quad \text{and} \quad \Phi(y) = z\,.$$

4. We may assume that $Y \ne \emptyset$. Since $[\,T^{-1}(X:Y)\,](T^{-1}Y) = T^{-1}[\,(X:Y)Y\,] \subset T^{-1}X$ and $j_T(Y) \subset T^{-1}Y$, we obtain

$$T^{-1}(X\!:\!Y) \subset (T^{-1}X\!:\!T^{-1}Y) \subset (T^{-1}X\!:\!j_T(Y)\,.$$

For the proof of $(T^{-1}X\!:\!j_T(Y) \subset (T^{-1}X\!:\!T^{-1}Y)$, let $z \in K$ and $s \in T$ be such that $\frac{z}{s} \in (T^{-1}X\!:\!j_T(Y))$. If $\frac{y}{t} \in T^{-1}Y$ (where $y \in Y$ and $t \in T$), then $\frac{y}{1} \in j_T(Y)$, and therefore $\frac{y}{1}\,\frac{z}{s} = \frac{yz}{s} \in T^{-1}X$, say $\frac{yz}{s} = \frac{x}{w}$ for some $x \in X$ and $w \in T$, which implies that $\frac{z}{s}\,\frac{y}{t} = \frac{x}{wt} \in T^{-1}X$.

Assume now that $TX = X$ and $Y = \{y_1, \ldots, y_m\}$ for some $m \in \mathbb{N}$, and let $\frac{z}{t} \in (T^{-1}X\!:\!T^{-1}Y)$, where $z \in K$ and $t \in T$. For $j \in [1,m]$, it follows that $\frac{z}{t}\,\frac{y_j}{1} \in T^{-1}X$, and thus there exist $x_1, \ldots, x_m \in X$ and $s \in T$ such that, for all $j \in [1,m]$, we have $\frac{z}{t}\,\frac{y_j}{1} = \frac{x_j}{s}$ and therefore $w_jszy_j = w_jtx_j$ for some $w_j \in T$. Then $w = w_1 \cdot \ldots \cdot w_m \in T$ and $wszy_j = wtx_j \in TX = X$ for all $j \in [1,m]$. Hence we obtain $wsz \in (X\!:\!Y)$, and $\frac{z}{t} = \frac{wsz}{wst} \in T^{-1}(X\!:\!Y)$. □

Theorem und Definition 1.2.5. *Let K and L be monoids, $T \subset K$ a multiplicatively closed subset and $\varphi \colon K \to L$ be a homomorphism. Then $\varphi(T) \subset L$ is a multiplicatively closed subset, and there exists a unique homomorphism $T^{-1}\varphi \colon T^{-1}K \to \varphi(T)^{-1}L$ such that $(T^{-1}\varphi)\circ j_T = j_{\varphi(T)}\circ\varphi$. It is given by*

$$(T^{-1}\varphi)\Big(\frac{x}{t}\Big) = \frac{\varphi(x)}{\varphi(t)} \quad \textit{for all} \quad x \in K \quad \textit{and} \quad t \in T\,.$$

$T^{-1}\varphi$ is called the *quotient homomorphism* of φ with respect to T.

Proof. Clearly, $1 = \varphi(1) \in \varphi(T)$, and $\varphi(T)\varphi(T) = \varphi(TT) = \varphi(T)$, and therefore $\varphi(T) \subset L$ is multiplicatively closed.

By Theorem 1.2.4.2 we have $j_{\varphi(T)}(\varphi(T)) \subset (\varphi(T)^{-1}L)^\times$, and by Theorem 1.2.3 there exists a monoid homomorphism $T^{-1}\varphi\colon T^{-1}K \to \varphi(T)^{-1}L$ such that $(T^{-1}\varphi)\circ j_T = j_{\varphi(T)}\circ\varphi$.

It remains to prove uniqueness and the formula. Thus let $\Phi\colon T^{-1}K \to \varphi(T)^{-1}L$ be a homomorphism such that $\Phi\circ j_T = j_{\varphi(T)}\circ\varphi$. If $x \in K$ and $t \in T$, then

$$\Phi\Big(\frac{x}{t}\Big) = \Phi\big(j_T(t)^{-1}j_T(x)\big) = \Phi\circ j_T(t)^{-1}\,\Phi\circ j_T(x) = \Big(\frac{\varphi(t)}{1}\Big)^{-1}\Big(\frac{\varphi(x)}{1}\Big) = \frac{\varphi(x)}{\varphi(t)}\,. \qquad \square$$

Theorem 1.2.6. *Let K be a monoid and $T \subset K^*$ a multiplicatively closed subset.*

1. *The natural embedding $j_T\colon K \to T^{-1}K$ is a monomorphism, and $(T^{-1}K)^\bullet = T^{-1}K^\bullet$.*
2. *If $a \in K$ and $s \in T$, then $\frac{a}{s} \in (T^{-1}K)^*$ if and only if $a \in K^*$. In particular, $(T^{-1}K)^* = T^{-1}K^*$, and $T^{-1}K$ is cancellative if and only if K is cancellative.*

Proof. 1. If $x, y \in K$ are such that $j_T(x) = j_T(y)$, then $sx = sy$ for some $s \in T$ and consequently $x = y$. In particular, if $j_T(x) = \frac{0}{1}$, then $x = 0$, and therefore $(T^{-1}K)^\bullet = T^{-1}K^\bullet$.

2. Let $a \in K$ and $s \in T$. If $a \in K^*$ and

$$\frac{a}{s}\frac{x}{t} = \frac{a}{s}\frac{x'}{t'} \quad \text{for some} \quad x, x' \in K \quad \text{and} \quad t, t' \in T, \quad \text{then} \quad st'ax = stax'\,, \quad \text{hence} \quad t'x = tx' \quad \text{and} \quad \frac{x}{t} = \frac{x'}{t'}\,,$$

since $sa \in K^*$. If $a \notin K^*$, then there exist $x, x' \in K$ such that $x \ne x'$ and $ax = ax'$. But then it follows that

$$\frac{a}{s}\frac{x}{1} = \frac{a}{s}\frac{x'}{1} \quad \text{and} \quad \frac{x}{1} \ne \frac{x'}{1}\,, \quad \text{hence} \quad \frac{a}{s} \notin (T^{-1}K)^*\,.$$

Hence it follows that $(T^{-1}K)^* = T^{-1}K^*$.

If K is cancellative, then $K^\bullet \subset K^*$, hence $(T^{-1}K)^\bullet = T^{-1}K^\bullet \subset T^{-1}K^* = (T^{-1}K)^*$, and thus $T^{-1}K$ is cancellative. If K is not cancellative, then there is some $a \in K^\bullet \setminus K^*$. Since $\frac{a}{1} \in (T^{-1}K)^\bullet \setminus (T^{-1}K)^*$, it follows that also $T^{-1}K$ is not cancellative. $\square$

Remarks and Definition 1.2.7. Let K be a monoid and $T \subset K^*$ a multiplicatively closed subset. Then we identify K with $j_T(K) \subset T^{-1}K$ by means of j_T. Hence

$$K \subset T^{-1}K\,, \quad a = \frac{a}{1} \ \text{ for all } \ a \in K\,, \quad T \subset (T^{-1}K)^\times, \quad \text{and} \quad \frac{a}{t} = t^{-1}a \quad \text{for all } \ a \in K \ \text{ and } \ t \in T\,.$$

In particular, it follows that $T^{-1}K = K$ if and only if $T \subset K^\times$.

Let $K \subset K_1$ be a submonoid and $T \subset K \cap K_1^\times$ a multiplicatively closed subset. Then $T \subset K^*$ and $T^{-1}K \subset T^{-1}K_1 = K_1$. Hence we obtain $K \subset T^{-1}K = \{t^{-1}x \mid x \in K\,,\ t \in T\} \subset K_1$.

The monoid $\mathsf q(K) = K^{*-1}K$ is called the *total quotient monoid* of K. By Theorem 1.2.6 it follows that

$$\mathsf q(K)^\bullet = K^{*-1}K^\bullet \quad \text{and} \quad \mathsf q(K)^\times = \mathsf q(K)^* = K^{*-1}K^*\,.$$

In particular, $K^* \subset \mathsf q(K)^\times$, and therefore $K \subset T^{-1}K \subset \mathsf q(K)$ for every multiplicatively closed subset $T \subset K^*$.

If $\varphi\colon K \to L$ is a monoid homomorphism satisfying $\varphi(K^*) \subset L^*$, then $\mathsf q(\varphi) = K^{*-1}\varphi\colon \mathsf q(K) \to \mathsf q(L)$ is called the *quotient homomorphism* of φ.

Theorem 1.2.8. *Let D be a monoid and $K = \mathsf q(D)$.*

1. *K is divisible if and only if D is cancellative.*
2. *If $G \subset D^\times$ is a subgroup, then $K/G = \mathsf q(D/G)$. In particular, $K/D^\times = \mathsf q(D/D^\times)$.*

3. *The following assertions are equivalent*:

$$\text{(a)}\quad D = K\,. \qquad \text{(b)}\quad zD = K \;\; \textit{for some} \;\; z \in K\,. \qquad \text{(c)}\quad D^* \cap \bigcap_{a \in D^*} aD \ne \emptyset\,.$$

PROOF. 1. If D is cancellative, then then $K^\bullet = D^{*-1}D^\bullet \subset D^{*-1}D^* = K^\times$, and therefore K is divisible. The converse is obvious, since $D \subset K$ is a submonoid.

2. By definition, $D/G \subset K/G$, and we assert that $(D/G)^* \subset (K/G)^\times$. Indeed, if $aG \in (D/G)^*$ for some $a \in D$, then $a \in D^* \subset K^\times$ and $aG \in (K/G)^\times$. Consequently, $\mathsf{q}(D/G) \subset K/G$, and if $z \in K/G$, say $z = a^{-1}bG$, where $a \in D^*$ and $b \in D$, then $z = (aG)^{-1}(bG) \in \mathsf{q}(D/G)$. Hence $\mathsf{q}(D/G) = K/G$.

3. (a) $\Rightarrow$ (b) Obvious.

(b) $\Rightarrow$ (c) Let $z \in K$ be such that $zD = K$. Then $z \in K^\times$, say $z = b^{-1}c$, where $b,\, c \in D^*$, and $b^{-1}D = c^{-1}K = K$. We assert that $b \in aD$ for all $a \in D^*$. Indeed, if $a \in D^*$, then $a^{-1} = b^{-1}u$ for some $u \in D$ and therefore $b = au \in aD$.

(c) $\Rightarrow$ (a) Let $b \in D^*$ be such that $b \in aD$ for all $a \in D^*$. If $x = a^{-1}c \in K$, where $a \in D^*$ and $c \in D$, then $x = b^{-1}c(a^{-1}b) \in b^{-1}D$. Hence $K = b^{-1}D$, and therefore $D = bK = K$. □

Remark 1.2.9. Let K be a ring and $T \subset K$ a multiplicatively closed subset.
For $z,\, z' \in T^{-1}K$, let $x,\, x' \in K$ and $t \in T$ be such that

$$z = \frac{x}{t}\,, \quad z' = \frac{x'}{t}\,, \quad \text{and define} \quad z + z' = \frac{x + x'}{t}\,.$$

This definition does not depend on the choice of representatives. Endowed with this addition, $T^{-1}K$ is the usual quotient ring of commutative ring theory. In particular, $\mathsf{q}(K)$ is the total quotient ring, and if K is a domain, then $\mathsf{q}(K)$ is the quotient field of K.

1.3. Prime and primary ideals

Throughout this section, let D be a monoid, and for $X,\, Y \subset D$, we set $(X\!:\!Y) = (X :_D Y)$.

Lemma 1.3.1. *Let $Q \subset D$ be an ideal.*

1. *If $Q \ne D$, then Q is a prime ideal if and only if, for all $A,\, B \subset D$, $AB \subset Q$ implies $A \subset Q$ or $B \subset Q$.*
2. *Let Q be a prime ideal, $n \in \mathbb{N}$, and let $J_1, \ldots, J_n \subset D$ be ideals such that either $J_1 \cdot \ldots \cdot J_n \subset Q$ or $J_1 \cap \ldots \cap J_n \subset Q$. Then there exists some $i \in [1, n]$ such that $J_i \subset Q$.*

PROOF. 1. Let $Q \ne D$ be a prime ideal, $A,\, B \subset D$, $AB \subset Q$ and $A \not\subset Q$. If $a \in A \setminus Q$ and $b \in B$, then $ab \in AB \subset Q$ and therefore $b \in Q$. Hence it follows that $B \subset Q$.

2. Since $J_1 \cdot \ldots \cdot J_n \subset J_1 \cap \ldots \cap J_n$, it suffices to prove the assertion for the product. But this follows from 1. by induction on n. □

Theorem und Definition 1.3.2. Let $J \subset D$ be an ideal. We call

$$\sqrt{J} = {}_D\sqrt{J} = \{x \in D \mid x^n \in J \text{ for some } n \in \mathbb{N}\}$$

the *radical* of J (in D), and we call J a *radical ideal* of D if $J = \sqrt{J}$. We denote by $\Sigma(J) = \Sigma_D(J)$ the set of all prime ideals $P \subset D$ such that $J \subset P$, and we denote by $\mathcal{P}(J) = \mathcal{P}_D(J)$ the set of minimal elements of $\Sigma(J)$. The elements of $\mathcal{P}(J)$ are called *prime divisors* of J.

1. *Let $I \subset D$ be another ideal of D.*

(a) *$I \subset \sqrt{I} = \sqrt{\sqrt{I}}$, and $I \subset J$ implies $\sqrt{I} \subset \sqrt{J}$.*

(b) *$\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.*

2. *If $J \ne D$, then $\mathcal{P}(J) \ne \emptyset$, and for every $P \in \Sigma(J)$ there exists some $P_0 \in \mathcal{P}(J)$ such that $P_0 \subset P$.*
3. *If $J \ne D$, then $\sqrt{J} \ne D$,*

$$\sqrt{J} = \bigcap_{P \in \mathcal{P}(J)} P\,,$$

and $\sqrt{J}$ is a prime ideal if and only if it is the only prime divisor of J.

Proof. 1. (a) Clearly, $I \subset \sqrt{I}$, and $I \subset J$ implies $\sqrt{I} \subset \sqrt{J}$. If $x \in \sqrt{\sqrt{I}}$, then $x^n \in \sqrt{I}$ for some $n \in \mathbb{N}$, hence $x^{nm} = (x^n)^m \in I$ for some $m \in \mathbb{N}$, and therefore $x \in \sqrt{I}$.

(b) Since $IJ \subset I \cap J \subset I,\ J$, we obtain $\sqrt{IJ} \subset \sqrt{I \cdot_r J} \subset \sqrt{I \cap J} \subset \sqrt{I} \cap \sqrt{J}$. If $a \in \sqrt{I} \cap \sqrt{J}$, then there exist $m,\ n \in \mathbb{N}$ such that $a^m \in I$ and $a^n \in J$. Hence $a^{m+n} = a^m a^n \in IJ$, and $a \in \sqrt{IJ}$.

2. If $J \ne P$, then $D \setminus D^\times \in \Sigma(J)$. For $P \in \Sigma(J)$, let $\Omega_P = \{P' \in \Sigma(J) \mid P' \subset P\}$. The intersection of every family in Ω_P belongs to Ω_P, and by Zorn's Lemma, applied for the partially ordered set $(\Omega_P, \supset)$, it follows that Ω_P has a minimal element P_0 with respect to the inclusion. Then $P_0 \in \mathcal{P}(J)$ and $P_0 \subset P$.

3. If $\sqrt{J} = D$, then $1 \in \sqrt{J}$ implies $1 \in J$ and thus $J = D$. Clearly, $\sqrt{J} \subset P$ for all $P \in \mathcal{P}(J)$. We prove that for every $a \in D \setminus \sqrt{J}$ there exists some $P_0 \in \mathcal{P}(J)$ such that $a \notin P_0$. Thus suppose that $a \in D \setminus \sqrt{J}$. Then $T = \{a^n \mid n \in \mathbb{N}_0\}$ is a multiplicatively closed subset of D satisfying $T \cap J = \emptyset$. If $\overline{T}$ denotes the divisor-closure of T, then Theorem 1.2.4 implies $P = D \setminus \overline{T}$ is a prime ideal, and it is the greatest ideal of D such that $P \cap T = \emptyset$. Hence $J \subset P$, and by 2. there exists some $P_0 \in \mathcal{P}(J)$ such that $P_0 \subset P$ and therefore $a \notin P_0$. □

Theorem und Definition 1.3.3. An ideal $Q \subset D$ is called *primary* if $Q \ne D$ and, for all $a,\ b \in D$, if $ab \in Q$ and $a \notin Q$, then $b \in \sqrt{Q}$.

1. *Let $Q \subset D$ be an ideal.*
   (a) *Q is a prime ideal if and only if Q it is a primary ideal, and $\sqrt{Q} = Q$.*
   (b) *If Q is a primary ideal, then $\sqrt{Q}$ is the only prime divisor of Q.*
   If Q is a primary ideal and $P = \sqrt{Q}$, then Q is called *P-primary.*
2. *For ideals $Q,\ P \subsetneq D$ the following assertions are equivalent:*
   (a) *Q is P-primary.*
   (b) *$Q \subset P \subset \sqrt{Q}$, and for all $a,\ b \in D$, if $ab \in Q$ and $a \notin Q$, then $b \in P$.*
   (c) *$Q \subset P \subset \sqrt{Q}$, and for all $A,\ B \subset D$, if $AB \subset Q$ and $A \not\subset Q$, then $B \subset P$.*
3. *Let $P \subset D$ be a prime ideal.*
   (a) *If Q and Q' are P-primary ideals, then $Q \cap Q'$ is also P-primary.*
   (b) *If Q is a P-primary ideal and $B \subset D$ is any subset such that $B \not\subset Q$, then $(Q :_D B)$ is also P-primary.*
4. *Let $\varphi\colon D \to D'$ be a monoid homomorphism and $Q' \subset D'$ an ideal. Then $\varphi^{-1}(Q') \subset D$ is an ideal, $\sqrt{\varphi^{-1}(Q')} = \varphi^{-1}(\sqrt{Q'})$. If Q' is primary [a prime ideal], then so is $\varphi^{-1}(Q')$.*

Proof. 1. Suppose that $a,\ b \in D$, $ab \in \sqrt{Q}$ and $a \notin \sqrt{Q}$. Then there is some $n \in \mathbb{N}$ such that $(ab)^n = a^n b^n \in Q$ and $a^n \notin Q$. Since Q is primary, we obtain $b^n \in \sqrt{Q}$ and therefore $b \in \sqrt{\sqrt{Q}} = \sqrt{Q}$. Hence $\sqrt{Q}$ is a prime ideal, and we must prove that $\sqrt{Q}$ is the smallest prime ideal containing Q. Indeed, if $P \subset D$ is a prime ideal and $Q \subset P$, then $\sqrt{Q} \subset \sqrt{P} = P$.

2. (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (b) Obvious.

(b) $\Rightarrow$ (c) Suppose that $Q \subset P \subset \sqrt{Q}$, $A, B \subset D$, $AB \subset Q$ and $A \not\subset Q$. Let $a \in A \setminus Q$. For all $b \in B$, we have $ab \in AB \subset Q$ and therefore $b \in P$. Hence $B \subset P$.

(b) $\Rightarrow$ (a) If $Q \subset P \subset \sqrt{Q}$, then $P = \sqrt{Q}$ by 1. Hence Q is P-primary.

3.(a) If $\sqrt{Q} = \sqrt{Q'} = P$, then $\sqrt{Q \cap Q'} = \sqrt{Q} \cap \sqrt{Q'} = P$. Suppose that $a, b \in D$, $ab \in Q \cap Q'$ and $a \notin Q \cap Q'$, say $a \notin Q$. Then it follows that $b \in P$, and thus $Q \cap Q'$ is P-primary.

(b) Note that $Q \subset (Q\!:\!B) \subsetneq D$, since $B \not\subset Q$. Hence $P = \sqrt{Q} \subset \sqrt{(Q\!:\!B)}$, and by 2. it suffices to prove that, for all $a, b \in D$, if $ab \in (Q\!:\!B)$ and $a \notin (Q\!:\!B)$, then $b \in P$.

If $a, b \in D$, $ab \in (Q\!:\!B)$ and $a \notin (Q\!:\!B)$, then $abB \subset Q$, $aB \not\subset Q$ and hence $b \in P$, again by 2.

4. Obviously, $\varphi^{-1}(Q') \subset D$ is an ideal. If $a \in D$, then

$$a \in \sqrt{\varphi^{-1}(Q')} \iff a^n \in \varphi^{-1}(Q') \text{ for some } n \in \mathbb{N} \iff \varphi(a)^n \in Q' \text{ for some } n \in \mathbb{N}$$
$$\iff \varphi(a) \in \sqrt{Q'} \iff a \in \varphi^{-1}(\sqrt{Q'})\,. \quad \text{Hence } \sqrt{\varphi^{-1}(Q')} = \varphi^{-1}(\sqrt{Q'}).$$

Now let Q' be primary, $a, b \in D$, $ab \in \varphi^{-1}(Q')$ and $a \notin \varphi^{-1}(Q')$. Then $\varphi(a)\varphi(b) \in Q'$ and $\varphi(a) \notin Q'$, hence $\varphi(b) \in \sqrt{Q'}$ and therefore $b \in \varphi^{-1}(\sqrt{Q'}) = \sqrt{\varphi^{-1}(Q')}$. If Q' is a prime ideal, then it is primary and $\sqrt{Q'} = Q'$. Hence the same holds for $\varphi^{-1}(Q')$. □

Definition 1.3.4. Let $J \subset D$ be an ideal, $n \in \mathbb{N}_0$, $Q_1, \ldots, Q_n \subset D$ distinct primary ideals, and $\mathfrak{Q} = \{Q_1, \ldots, Q_n\}$.

1. $\mathfrak{Q}$ is called a *primary decomposition* of J if $J = Q_1 \cap \ldots \cap Q_n$.
2. $\mathfrak{Q}$ is called *reduced* if $\sqrt{Q}_1, \ldots, \sqrt{Q}_n$ are distinct, and $Q_1 \cap \ldots \cap Q_{i-1} \cap Q_{i+1} \cap \ldots \cap Q_n \not\subset Q_i$ for all $i \in [1, n]$.

Theorem 1.3.5. *Let $J \subset D$ be an ideal..*

1. *If $\mathfrak{Q}$ is a primary decomposition of J for which $|\mathfrak{Q}|$ is minimal, then $\mathfrak{Q}$ is reduced. In particular, if J possesses a primary decomposition, then it also possesses a reduced one.*
2. *Let $\mathfrak{Q}$ be a reduced primary decomposition of J. For a prime ideal $P \subset D$, the following conditions are equivalent:*
   (a) *$P = \sqrt{Q}$ for some $Q \in \mathfrak{Q}$.*
   (b) *There exists some $z \in D \setminus J$ such that $P = \sqrt{(J\!:\!z)}$.*
3. *Let $\mathfrak{Q}$ and $\mathfrak{Q}'$ be reduced primary decompositions of J. Then there is a bijective map $\sigma \colon \mathfrak{Q} \to \mathfrak{Q}'$ such that $\sqrt{\sigma(Q)} = \sqrt{Q}$ for all $Q \in \mathfrak{Q}$, and if $\sqrt{Q}_1$ is minimal in $\{\sqrt{Q} \mid Q \in \mathfrak{Q}\}$, then $\sigma(Q_1) = Q_1$.*

Proof. 1. Assume to the contrary that $|\mathfrak{Q}| = n$ and $\mathfrak{Q} = \{Q_1, \ldots, Q_n\}$ is not reduced. Then $n \geq 2$ and after renumbering (if necessary) we may assume that either $\sqrt{Q}_1 = \sqrt{Q}_2$ or $Q_2 \cap \ldots \cap Q_n \subset Q_1$. We set $\mathfrak{Q}_1 = \{Q_1 \cap Q_2, Q_3, \ldots, Q_n\}$ if $\sqrt{Q_1} = \sqrt{Q_2}$, and $\mathfrak{Q}_1 = \{Q_2, \ldots, Q_n\}$ if $Q_2 \cap \ldots \cap Q_n \subset Q_1$. Then $\mathfrak{Q}_1$ is a primary decomposition of J satisfying $|\mathfrak{Q}_1| = n - 1$, a contradiction.

2. Suppose that $\mathfrak{Q} = \{Q_1, \ldots, Q_n\}$, where $n \in \mathbb{N}_0$ and $Q_1, \ldots, Q_n$ are distinct. If $n = 0$, then $J = D$, and there is nothing to do. If $n = 1$, then $\mathfrak{Q} = \{J\}$, and the assertion follows by Theorem 1.3.3. Thus we may assume that $n \geq 2$.

(a) $\Rightarrow$ (b) Assume that $P = \sqrt{Q_1}$. If $z \in (Q_2 \cap \ldots \cap Q_n) \setminus Q_1$, then $(Q_i\!:\!z) = D$ for all $i \in [2, n]$, and

$$(J\!:\!z) = (Q_1 \cap \ldots \cap Q_n\!:\!z) = \bigcap_{i=1}^{n} (Q_i\!:\!z) = (Q_1\!:\!z) \quad \text{is } P\text{-primary by Theorem 1.3.3.3(b).}$$

(b) $\Rightarrow$ (a) Let $z \in D \setminus J$ be such that $P = \sqrt{(J:z)}$. Then

$$P = \sqrt{(Q_1 \cap \ldots \cap Q_n : z)} = \bigcap_{i=1}^{n} \sqrt{(Q_i : z)} = \bigcap_{\substack{i=1 \\ z \notin Q_i}}^{n} \sqrt{Q_i}\,,$$

and therefore $P = Q_i$ for some $i \in [1, n]$.

3. By 2. it follows that $\{\sqrt{Q} \mid Q \in \mathfrak{Q}\} = \{\sqrt{Q} \mid Q \in \mathfrak{Q}'\}$ consists of all prime ideals of the form $(J:z)$ for some $z \in D \setminus J$. Therefore there exists a bijective map $\sigma \colon \mathfrak{Q} \to \mathfrak{Q}'$ such that $\sqrt{\sigma(Q)} = \sqrt{Q}$ for all $Q \in \mathfrak{Q}$.

Assume now that $\mathfrak{Q} = \{Q_1, \ldots, Q_n\}$, where $n \in \mathbb{N}_0$, $Q_1, \ldots, Q_n$ are distinct, and let $\sqrt{Q_1}$ be minimal in the set $\{\sqrt{Q_1}, \ldots \sqrt{Q_n}\}$. By symmetry, it suffices to prove that $\sigma(Q_1) \subset Q_1$. Assume the contrary, and consider the ideal $B = \sigma(Q_2) \cap \ldots \cap \sigma(Q_n)$. Since $Q_1 \supset J = B \cap \sigma(Q_1) \supset B\sigma(Q_1)$, it follows that $B \subset \sqrt{Q_1}$ and thus $\sqrt{Q_i} = \sqrt{\sigma(Q_i)} \subset \sqrt{Q_1}$ for some $i \in [2, n]$, a contradiction, since $\sqrt{Q_1}$ was minimal and $\sqrt{Q_1} \neq \sqrt{Q_i}$. $\square$

Theorem 1.3.6. *Let $T \subset D^\bullet$ a multiplicatively closed subset and $j_T \colon D \to T^{-1}D$ the natural embedding.*

1. *If $J \subset D$ is an ideal, then $T^{-1}\sqrt{J} = \sqrt{T^{-1}J}$.*
2. *The assignment $Q \mapsto T^{-1}Q$ defines an inclusion-preserving bijective map*

$$j_T^* \colon \{Q \subset D \mid Q \text{ is a primary ideal, } Q \cap T = \emptyset\} \;\to\; \{\overline{Q} \in\subset T^{-1}D \mid \overline{Q} \text{ is a primary ideal}\}\,.$$

Its inverse is given by $\overline{Q} \mapsto j_T^{-1}(\overline{Q})$, and if $Q \subset D$ is a primary ideal, then $T^{-1}\sqrt{Q} = \sqrt{T^{-1}Q}$. In particular:

- *j_T^* induces an inclusion-preserving bijective map from the set of all prime ideals $P \subset D$ such that $P \cap T = \emptyset$ onto the set of all prime ideals of $T^{-1}D$.*
- *If $P \subset D$ is a prime ideal and $P \cap T = \emptyset$, then j_T^* induces an inclusion-preserving bijective map from the set of all P-primary ideals of D onto the set of all $T^{-1}P$-primary ideals of $T^{-1}D$.*

3. *Let $J \subset D$ be an ideal and $\mathfrak{Q}$ is a reduced primary decomposition of J. Then*

$$\mathfrak{Q}_T = \{T^{-1}Q \mid Q \in \mathfrak{Q}\,,\ Q \cap T = \emptyset\}$$

is a reduced primary decomposition of $T^{-1}J$.

Proof. 1. Let $J \subset D$ be an ideal. If $x \in T^{-1}\sqrt{J}$, then $x = \frac{a}{t}$, where $a \in \sqrt{J}$ and $t \in T$. If $n \in \mathbb{N}$ is such that $a^n \in J$, then

$$x^n = \frac{a^n}{t^n} \in T^{-1}J \quad \text{and} \quad x \in \sqrt{T^{-1}J}\,.$$

Conversely, suppose that $x = \frac{a}{t} \in \sqrt{T^{-1}J}$, where $a \in D$ and $t \in T$, and let $n \in \mathbb{N}$ be such that $x^n \in T^{-1}J$. Then

$$x^n = \frac{a^n}{t^n} = \frac{c}{s} \quad \text{for some} \quad c \in J \ \text{ and } \ s \in T\,.$$

Let $w \in T$ be such that $wsa^n = wct^n \in J$. Then $(wsa)^n = (ws)^{n-1}wsa^n \in J$, hence $wsa \in \sqrt{J}$ and

$$x = \frac{wsa}{wst} \in T^{-1}\sqrt{J}\,.$$

2. It suffices to prove the following assertion:

A. If $Q \subset D$ is a primary ideal and $Q \cap T = \emptyset$, then $T^{-1}Q$ is primary, and $Q = j_T^{-1}(T^{-1}Q)$.

Indeed, suppose that **A** holds. If $\overline{Q} \subset T^{-1}D$ is a primary ideal, then $j_T^{-1}(\overline{Q}) \subset D$ is primary and $\overline{Q} = T^{-1}j_T^{-1}(\overline{Q})$ by the Theorems 1.3.3.4 and 1.2.2.4. Moreover, for every ideal $Q \subset D$ we have $T^{-1}\sqrt{Q} = \sqrt{T^{-1}Q}$ by 1., and the assertions follow.

Proof of **A.** Let $Q \subset D$ be a primary ideal and $Q \cap T = \emptyset$. Let $x,\, y \in T^{-1}D$ be such that $xy \in T^{-1}Q$ and $x \notin T^{-1}Q$. We set

$$x = \frac{a}{t}\,, \quad y = \frac{b}{s} \quad \text{and} \quad xy = \frac{c}{w}\,, \quad \text{where } \ a,\, b \in D\,, \quad c \in Q\,, \quad t,\, s,\, w \in T \ \text{ and } \ a \notin Q.$$

Then there exists some $v \in T$ such that $vwab = vtsc \in Q$, and as $a \notin Q$, we obtain $vwb \in \sqrt{Q}$. If $n \in \mathbb{N}$ is such that $(vwb)^n \in Q$, then

$$y^n = \frac{(vwb)^n}{(vws)^n} \in T^{-1}Q\,. \qquad \text{Hence } \ T^{-1}Q \ \text{ is primary.}$$

Obviously, $j_T^{-1}(T^{-1}Q) \supset Q$. To prove the reverse inclusion, let $c \in j_T^{-1}(T^{-1}Q)$. Then $\frac{c}{1} = \frac{a}{t}$ for some $a \in Q$ and $t \in T$, and there exists some $s \in T$ such that $cst = sa \in Q$. If $c \notin Q$, then there is some $n \in \mathbb{N}$ such that $(st)^n \in Q \cap T$, a contradiction.

3. By 1. and 2., $\mathfrak{Q}_T$ is a primary decomposition of $T^{-1}J$, since

$$J = \bigcap_{Q \in \mathfrak{Q}} Q \qquad \text{implies} \qquad T^{-1}J = \bigcap_{Q \in \mathfrak{Q}} T^{-1}Q = \bigcap_{Q \in \mathfrak{Q}_T} T^{-1}Q\,.$$

We must prove that $\mathfrak{Q}_T$ is reduced. Assume first that $Q,\, Q' \in \mathfrak{Q}_T$ are such that $\sqrt{T^{-1}Q} = \sqrt{T^{-1}Q'}$. Then $\sqrt{Q} = j_T^{-1}(T^{-1}\sqrt{Q}) = j_T^{-1}(\sqrt{T^{-1}Q}) = j_T^{-1}(\sqrt{T^{-1}Q'}) = j_T^{-1}(T^{-1}\sqrt{Q'}) = \sqrt{Q'}$ and therefore $Q = Q'$. If $Q_1 \in \mathfrak{Q}_T$, then

$$\bigcap_{\substack{Q \in \mathfrak{Q}_T \\ Q \ne Q_1}} T^{-1}Q \subset T^{-1}Q_1 \quad \text{implies} \quad \bigcap_{\substack{Q \in \mathfrak{Q}_T \\ Q \ne Q_1}} Q = \bigcap_{\substack{Q \in \mathfrak{Q}_T \\ Q \ne Q_1}} j_T^{-1}(T^{-1}Q) = j_T^{-1}\Big(\bigcap_{\substack{Q \in \mathfrak{Q}_T \\ Q \ne Q_1}} T^{-1}Q\Big) \subset j_T^{-1}(T^{-1}Q_1) = Q_1\,,$$

which is impossible. Hence $\mathfrak{Q}_T$ is reduced. □

Definition 1.3.7. Let $P \subset D$ be a prime ideal and $K \supset D$ an overmonoid. Then the monoid $K_P = (D \setminus P)^{-1}K$ is called the *localization* of K at P. We denote by $j_P = j_{D \setminus P} \colon K \to K_P$ the natural embedding, and for $X \subset K$, we set $X_P = (D \setminus P)^{-1}X \subset K_P$.

Theorem 1.3.8. *Let $P \subset D$ be a prime ideal, $T \subset D^\bullet$ a multiplicatively closed subset and $P \cap T = \emptyset$. If $a \in D$ and $s \in T$, then $\frac{a}{s} \in T^{-1}P$ if and only if $a \in P$. In particular, $T^{-1}(D \setminus P) = T^{-1}D \setminus T^{-1}P$, and there is an isomorphism*

$$\Phi \colon (T^{-1}D)_{T^{-1}P} \ \stackrel{\sim}{\to} \ D_P\,, \quad \textit{given by} \quad \Phi\Big(\frac{\frac{a}{s}}{\frac{c}{t}}\Big) = \frac{at}{cs} \quad \textit{for all} \quad a \in D\,,\ c \in D \setminus P \ \ \textit{and} \ \ s,\, t \in T\,.$$

In particular, if D is cancellative, then $(T^{-1}D)_{T^{-1}P} = D_P \subset \mathsf{q}(D)$.

Proof. Clearly, $a \in P$ and $s \in T$ implies $\frac{a}{s} \in T^{-1}P$. Conversely, if $a \in D$ and $s \in T$ are such that $\frac{a}{s} \in T^{-1}P$, then $\frac{a}{1} = \frac{s}{1}\frac{a}{s} \in T^{-1}P$ and thus $a \in P$ by Theorem 1.3.6. Hence $T^{-1}(D \setminus P) = T^{-1}D \setminus T^{-1}P$, and Theorem 1.2.4.3, applied with $S = D \setminus P$, gives the asserted isomorphism. □

Theorem 1.3.9. *Let D be a cancellative monoid, $K = \mathsf{q}(D)$, and let $P,\, Q \subset D$ be prime ideals.*

1. *If $Q \not\subset P$, then $(D \colon Q) \subset D_P$.*
2. *If $P \not\subset Q$, then $D_P \subsetneq (D_P)_Q$.*
3. *If $I \subset D$ is an ideal such that $I = \sqrt{I} \subset P$, then $(P \colon P) \subset (I \colon I)$.*

Proof. 1. If $x \in (D{:}Q)$ and $y \in Q \setminus P$, then $xy \in D$, and $x = y^{-1}(xy) \in D_P$.

2. By definition, $D_P \subset (D_P)_Q$. If $x \in P \setminus Q$, then $x \in P_P = D_P \setminus D_P^\times$, and therefore it follows that $x^{-1} \in (D_P)_Q \setminus D_P$.

3. Let $x \in (P{:}P)$ and $y \in I$. We must prove that $xy \in I$. Since $I = \sqrt{I}$, Theorem 1.3.2.3 shows that it suffices to prove that $xy \in Q$ for all $Q \in \mathcal{P}(I)$. If $Q = P \in \mathcal{P}(I)$, then $xy \in (P{:}P)I \subset (P{:}P)P \subset P$. If $Q \in \mathcal{P}(I) \setminus \{P\}$, then $P \not\subset Q$, and $xyP \subset I(P{:}P)P \subset IP \subset I \subset Q$ implies $xy \in Q$. □

1.4. Fractional subsets

Definition 1.4.1. Let D be a monoid, $K = \mathsf{q}(D)$ its total quotient monoid and $X \subset K$.

1. X is called *D-fractional* if there exists some $a \in D^*$ such that $aX \subset D$.

   Every finite subset of K is D-fractional, and every subset of a D-fractional set is D-fractional.
2. X is called a *fractional* (*semigroup*) *ideal* of D if X is D-factional, $0 \in X$ and $DX \subset X$ (then $DX = X$).
3. X is called a *fractional principal ideal* of D if $X = Da$ for some $a \in K$.

   By definition, if $X \subset D$, then X is a fractional [principal] ideal of D if and only if X is a [principal] ideal of D.

Theorem 1.4.2. *Let D be a monoid, $K = \mathsf{q}(D)$ its total quotient monoid and $X, Y \subset K$.*

1. *If $c \in K$ and X is D-fractional, then cX is D-fractional.*
2. *X is D-fractional if and only if there exists some $c \in K^\times$ such that $cX \subset D$.*
3. *If $X, Y \subset K$ are D-fractional, then $X \cup Y$, $X \cap Y$ and XY are also D-fractional.*
4. *If X is D-fractional and $Y \cap K^\times \neq \emptyset$, then $(X{:}Y)$ is D-fractional.*
5. *Let $T \subset D^*$ be a multiplicatively closed subset [and $T^{-1}D \subset \mathsf{q}(D)$]. Then X is $T^{-1}D$-fractional if and only if $cX \subset T^{-1}D$ for some $c \in D^*$. In particular, if $Y \subset K$ is D-fractional, then $T^{-1}Y$ is $T^{-1}D$-fractional.*
6. *Let C be a monoid such that $D \subset C \subset K$. If C is D-fractional, then every C-fractional subset $X \subset K$ is D-fractional.*

Proof. 1. Let $c = b^{-1}d \in K$ (where $b \in D^*$ and $d \in D$). If X is D-fractional and $a \in D^*$ is such that $aX \subset D$, then $ba \in D^*$ and $ba(cX) = daX \subset dD \subset D$. Hence cX is D-fractional.

2. If X is D-fractional, then there exists some $c \in D^* \subset K^\times$ such that $cX \subset D$. Conversely, let $c = b^{-1}d \in K^\times$ (where $b, d \in D^*$) be such $cX \subset D$. Then $dX \subset bcX \subset bD \subset D$, and thus X is D-fractional.

3. Let $a, b \in D^*$ be such that $aX \subset D$ and $bY \subset D$. Then $a(X \cap Y) \subset D$, $ab(X \cup Y) \subset D$ and $abXY \subset D$. Hence $X \cap Y$, $X \cup Y$ and XY are D-fractional.

4. If $y \in Y \cap K^\times$, then $y^{-1}X$ is D-fractional by 1., and since $(X{:}Y) \subset y^{-1}X$, it follows that $(X{:}Y)$ is D-fractional.

5. Let X be $T^{-1}D$-fractional and $z = (T^{-1}D)^* = T^{-1}D^*$ such that $zX \subset T^{-1}D$. Then $z = t^{-1}c$, where $t \in T$ and $c \in D^*$, and $cX = tzX \subset T^{-1}D$. The converse is obvious, since $D^* \subset (T^{-1}D)^*$. If $Y \subset K$ is D-fractional and $c \in D^*$ is such that $cY \subset D$, then $cT^{-1}Y = T^{-1}cY \subset T^{-1}D$, and thus $T^{-1}Y$ is $T^{-1}D$-fractional.

6. Let $a \in K^\times$ be such that $aC \subset D$. If $X \subset K$ is C-fractional and $c \in K^\times$ is such that $cX \subset C$, then $ac \in K^\times$ and $acX \subset D$. □

1.5. Free monoids, factorial monoids and GCD-monoids

Throughout this section, let D be a cancellative monoid and $K = \mathsf{q}(D)$.

Definition 1.5.1.

1. Let $X \subset D$. An element $d \in D$ is called a *greatest common divisor* of D if dD is the smallest principal ideal containing X [equivalently, $d \,|\, x$ for all $x \in X$, and if $e \in D$ and $e \,|\, x$ for all $x \in D$, then $e \,|\, d$]. We denote by $\mathrm{GCD}(X) = \mathrm{GCD}_D(X)$ the set of all greatest common divisors of X.

   By definition, $\mathrm{GCD}(X) = \{0\}$ if and only if $X^\bullet = \emptyset$, and $\mathrm{GCD}(X \cup \{0\}) = \mathrm{GCD}(X)$. If $d \in \mathrm{GCD}(X)$, then $\mathrm{GCD}(X) = dD^\times$. Consequently, if D is reduced, then $|\mathrm{GCD}(X)| \le 1$, and we write $d = \gcd(X)$ instead of $\mathrm{GCD}(X) = \{d\}$. If $X = \{a_1, \ldots, a_n\}$ for some $n \in \mathbb{N}$ and $a_1, \ldots a_n \in D$, we set $\mathrm{GCD}(a_1, \ldots, a_n) = \mathrm{GCD}(X)$ resp. $\gcd(a_1, \ldots, a_n) = \gcd(X)$. In particular, $\mathrm{GCD}(a) = aD^\times$ for all $a \in D$. Two elements $a, b \in D$ are called *coprime* if $\mathrm{GCD}(a, b) = D^\times$.

   If $X \subset D$, $d \in \mathrm{GCD}(X)$ and $\varepsilon \colon D \to D/D^\times$ denotes the reduction homomorphism, then $\varepsilon(d) = dD^\times = \gcd(\pi(X))$.
2. D is called a GCD-*monoid* if $\mathrm{GCD}(E) \ne \emptyset$ for all $E \in \mathbb{P}_{\mathsf{f}}(D)$. Hence D is a GCD-monoid if and only if $D/D^\times$ is a GCD-monoid. Every divisible monoid is a GCD-monoid.
3. A homomorphism $\varphi \colon D \to D'$ of GCD-monoids is called a GCD-*homomorphism* if
$$\varphi(\mathrm{GCD}(E)) \subset \mathrm{GCD}(\varphi(E)) \quad \text{for every} \quad E \in \mathbb{P}_{\mathsf{f}}(D)\,.$$
   We denote by $\mathrm{Hom}_{\mathrm{GCD}}(D, D')$ the set of all GCD-homomorphisms $\varphi \colon D \to D'$.

Theorem 1.5.2.

1. *Let $(X_\lambda)_{\lambda \in \Lambda}$ be a family of subsets of D, $b_\lambda \in \mathrm{GCD}(X_\lambda)$ for every $\lambda \in \Lambda$, and $B = \{b_\lambda \mid \lambda \in \Lambda\}$. Then*
$$X = \bigcup_{\lambda \in \Lambda} X_\lambda \quad \textit{implies} \quad \mathrm{GCD}(X) = \mathrm{GCD}(B)\,.$$
   In particular, D is a GCD*-monoid if and only if $\mathrm{GCD}(a, b) \ne \emptyset$ for all $a, b \in D^\bullet$.*
2. *If $X \subset D$, $a \in D$ and $\mathrm{GCD}(aX) \ne \emptyset$, then $\mathrm{GCD}(aX) = a\,\mathrm{GCD}(X)$.*

Proof. 1. It suffices to prove that X and B are contained in the same principal ideals of D. If $b \in D$, then
$$X \subset bD \iff X_\lambda \in bD \ \text{ for all } \ \lambda \in \Lambda \iff b_\lambda D \subset bD \ \text{ for all } \ \lambda \in \Lambda \iff B \in bD\,.$$
If D is a GCD-monoid, then $\mathrm{GCD}(a, b) \ne \emptyset$ for all $a, b \in D$. Conversely, suppose that $\mathrm{GCD}(a, b) \ne \emptyset$ for all $a, b \in D^\times$, and let $E \in \mathbb{P}_{\mathsf{f}}(D)$. We must prove that $\mathrm{GCD}(E) \ne \emptyset$, and since $\mathrm{GCD}(E) = \mathrm{GCD}(E \setminus \{0\})$, we may assume that $E \subset D^\bullet$. We use induction on $|E|$. If $|E| \le 2$, there is nothing to do. Thus assume that $|E| \ge 3$ and $a \in E$. If $b \in \mathrm{GCD}(E \setminus \{a\})$ and $d \in \mathrm{GCD}(a, b)$, then $d \in \mathrm{GCD}(E)$.

2. It suffices to prove that $\mathrm{GCD}(aX) \subset a\,\mathrm{GCD}(X)$. For $a = 0$, this is obvious. Thus suppose that $a \in D^\bullet$, and let $c \in \mathrm{GCD}(aX)$. Then $aX \subset aD$ implies $cD \subset aD$, hence $c = ab$ for some $b \in D$, and $X \subset bD$. If $b' \in D$ is such that $X \subset b'D$, then $aX \subset ab'D$, hence $cD = abD \subset ab'D$ and therefore $bD \subset b'D$. Consequently, bD is the smallest principal ideal containing X, $b \in \mathrm{GCD}(X)$, and $c = ab \in a\,\mathrm{GCD}(X)$. □

Theorem 1.5.3. *Let D be a* GCD*-monoid.*

1. *If $E, F \in \mathbb{P}_{\mathsf{f}}(D)$ and $b \in D$, then $\mathrm{GCD}(EF) = \mathrm{GCD}(E)\,\mathrm{GCD}(F)$ and $\mathrm{GCD}(bE) = b\,\mathrm{GCD}(E)$.*
2. *Let $a, b, c \in D$ be such that $a \,|\, bc$. Then there exist $b', c' \in D$ such that $a = b'c'$, $b' \,|\, b$ and $c' \,|\, c$. In particular, if $\mathrm{GCD}(a, b) = D^\times$, then $a \,|\, c$.*

3. *Every* $z \in K$ *has a representation in the form* $z = a^{-1}b$ *with* $a \in D^\bullet$ *and* $b \in D$ *such that* $\mathrm{GCD}(a,b) = D^\times$. *In this representation* $aD^\times$ *and* $bD^\times$ *are uniquely determined by* z.

Proof. We use Theorem 1.5.2.

1. Suppose that $e \in \mathrm{GCD}(E)$ and $f \in \mathrm{GCD}(F)$, and observe that

$$EF = \bigcup_{b \in E} bF\,.$$

For every $b \in E$, we have $bf \in \mathrm{GCD}(bF)$, and since $\{bf \mid b \in E\} = Ef$, we obtain $ef \in \mathrm{GCD}(EF)$.

2. Let $b' \in \mathrm{GCD}(a,b)$ and $c' \in D$ such that $a = b'c'$. Then it follows that $b'c \in \mathrm{GCD}(ac,bc)$, and $b'\,\mathrm{GCD}(c',c) = \mathrm{GCD}(b'c',b'c) = \mathrm{GCD}(a,ac,bc) = \mathrm{GCD}(a,bc) = aD^\times = b'c'D^\times$, which implies that $\mathrm{GCD}(c',c) = c'D^\times$ and therefore $c \,|\, c'$. In particular, if $\mathrm{GCD}(a,b) = D^\times$, we may assume that $b' = 1$, and then $a = c' \,|\, c$.

3. If $z \in K$, then $z = a_1^{-1}b_1$, where $a_1 \in D^\bullet$ and $b_1 \in D$. If $d \in \mathrm{GCD}(a_1,b_1)$, then $a_1 = ad$ and $b_1 = bd$, where $a,\, b \in D$, and $d = \mathrm{GCD}(ad,bd) = d\,\mathrm{GCD}(a,b)$. Hence $\mathrm{GCD}(a,b) = D^\times$ and $z = a^{-1}b$. To prove uniqueness, suppose that $z = a'^{-1}b'$, where $a' \in D^\bullet$, $b' \in D$ and $\mathrm{GCD}(a',b') = D^\times$. Then $a'b = ab'$, and since $\mathrm{GCD}(a,b) = \mathrm{GCD}(a',b') = D^\times$, it follows that $a \,|\, a'$, $b \,|\, b'$, $a' \,|\, a$ and $a \,|\, a'$. Hence $aD = a'D$ and $bD = b'D$. □

Definition 1.5.4.

1. An element $q \in D^\bullet$ is called
   - an *atom* if $q \notin D^\times$ and, for all $a,\, b \in D$, $q = ab$ implies $a \in D^\times$ or $b \in D^\times$ [equivalently, qD is maximal in the set $\{aD \mid a \in D \setminus D^\times\}$];
   - a *prime element* if $q \notin D^\times$ and, for all $a,\, b \in D^\bullet$, $q \,|\, ab$ implies $q \,|\, a$ or $q \,|\, b$ [equivalently, qD is a prime ideal].
2. D is called
   - *atomic* if every $a \in D^\bullet \setminus D^\times$ is a product of atoms;
   - *factorial* if every $a \in D^\bullet \setminus D^\times$ is a product of prime elements.
3. D is said to satisfy the ACCP (ascending chain condition for principal ideals) if there is no sequence $(a_nD)_{n\ge 0}$ of principal ideals of D such that $a_nD \subsetneq a_{n+1}D$ for all $n \in \mathbb{N}$ [equivalently, every non-empty set of principal ideals of D contains a maximal element].
4. D is called *free* with basis $P \subset D$ if the map

$$\chi_P \colon \mathbb{N}_0^{(P)} \;\to\; D^\bullet\,, \quad \text{defined by} \quad \chi\big((n_p)_{p\in P}\big) = \prod_{p \in P} p^{n_p}\,, \quad \text{is bijective.}$$

5. A subset $P \subset D$ is called a *complete set of primes* if every $p \in P$ is a prime element and, for every prime element $p \in D$ there is a unique $p_0 \in P$ such that $pD = p_0D$ [equivalently, $p = p_0u$ for some $u \in D^\times$].

Theorem 1.5.5.

1. *If* D *satisfies the* ACCP, *then* D *is atomic.*
2. *Every prime element of* D *is an atom, and if* D *is a* GCD*-monoid, then every atom is a prime element.*
3. D *is factorial if and only if* D *is atomic and every atom is a prime element.*

Proof. 1. Let Ω be the set of all principal ideals aD, where $a \in D^\bullet \setminus D^\times$ is not a product of atoms. Assume that, contrary to the assertion, $\Omega \neq \emptyset$. Since D satisfies the ACCP, Ω contains a maximal element aD, and since a is not an atom, it has a factorization $a = bc$, where $b,\, c \in D \setminus D^\times$. In particular, it follows that $aD \subsetneq bD$ and $aD \subsetneq cD$, and therefore $bD,\, cD \notin \Omega$. Hence both b and c are products of atoms, and therefore $a = bc$ is also a product of atoms, a contradiction.

2. Let $p \in D$ be a prime element and $a \in D \setminus D^\times$ such that $pD \subset aD$. We must prove that $pD = aD$. Since $p = au$ for some $u \in D$ and therefore $p \,|\, au$, it follows that $p \,|\, a$ or $p \,|\, u$. If $p \,|\, a$, then $aD = pD$ and we are done. If $p \,|\, u$, then $u = pv$ for some $v \in D$, hence $p = apv$, and from $1 = av$ it follows that $a \in D^\times$, a contradiction.

Assume now that D is a GCD-monoid, and let $q \in D$ be an atom. If $a,\, b \in D$ and $q \,|\, ab$, then Theorem 1.5.3.2 implies that there exist $a',\, b' \in D$ such that $a' \,|\, a$, $b' \,|\, b$ and $q = a'b'$. Hence it follows that $a' \in D^\times$ or $b' \in D^\times$, say $a' \in D^\times$. But then $b' \,|\, b$ implies $q \,|\, b$.

3. If D is atomic and every atom is a prime element, then every $a \in D \setminus D^\times$ is product of prime elements and thus D is factorial.

If D is factorial, then D is atomic, since every prime element is an atom. If $q \in D$ is an atom, then $q = p_1 \cdot \ldots \cdot p_r$, where $r \in \mathbb{N}$ and $p_1, \ldots, p_r$ are prime elements. But then it follows that $r = 1$ and $q = p_1$ is a prime element. $\square$

Theorem und Definition 1.5.6.

1. *For a subset $P \subset D$, the following assertions are equivalent:*
   (a) *D is factorial and P is a complete set of primes.*
   (b) *Every $a \in D^\bullet$ has a unique representation*
   $$a = u \prod_{p \in P} p^{\mathsf{v}_p(a)}\,, \quad \text{where} \quad u \in D^\times, \quad \mathsf{v}_p(a) \in \mathbb{N}_0 \;\; \text{and} \;\; \mathsf{v}_p(a) = 0 \;\; \text{for almost all} \;\; p \in P\,.$$
   (c) *$D/D^\times$ is free with basis $\varepsilon(P)$, where $\varepsilon \colon D \to D/D^\times$ denotes the canonical epimorphism.*

   For $a \in D^\bullet$, we call $\mathsf{v}_p(a)$ the *p-adic exponent* of a, and we set $\mathsf{v}_p(0) = \infty$.
2. *D is free with basis P if and only if D is factorial and reduced and P is the set of prime elements of D.*
3. *Let D be factorial, P a complete set of primes and $\emptyset \neq X \subset D^\bullet$. Then*
   $$d = \prod_{p \in P} p^{\min\{\mathsf{v}_p(x) \mid x \in X\}} \in \mathrm{GCD}(X)\,,$$
   and there exists some $E \in \mathbb{P}_{\mathsf{f}}(X)$ such that $d \in \mathrm{GCD}(E)$.
4. *D is factorial if and only if D is an atomic* GCD-*monoid.*

Proof. 1. (a) $\Rightarrow$ (b) Let $a \in D^\bullet$. Then $a = u'p_1' \cdot \ldots \cdot p_r'$, where $r \in \mathbb{N}_0$, $u' \in D^\times$, and $p_1', \ldots, p_r' \in D$ are prime elements. For $i \in [1, r]$, let $p_i \in P$ and $u_i \in D^\times$ be such that $p_i' = p_i u_i$. Then $u = u'u_1 \cdot \ldots \cdot u_r \in D^\times$, and $a = up_1 \cdot \ldots \cdot p_r$. For $p \in P$, let $n_p = |\{i \in [1, r] \mid p_i = p\}| \in \mathbb{N}_0$. Then $n_p = 0$ for almost all $p \in P$, and
$$a = u \prod_{p \in P} p^{n_p}\,.$$
We must prove uniqueness. Thus assume that
$$a = u \prod_{p \in P} p^{n_p} = u' \prod_{p \in P} p^{n_p'}\,,$$

where $u,\, u' \in D^\times$, $n_p,\, n_p' \in \mathbb{N}_0$ for all $p \in P$, and $n_p = n_p' = 0$ for almost all $p \in P$. Then we obtain

$$u^{-1}u' \prod_{\substack{p\in P\\ n_p' > n_p}} p^{n_p'-n_p} = \prod_{\substack{p\in P\\ n_p' < n_p}} p^{n_p-n_p'} \,.$$

Assume now that there is some $q \in P$ such that $n_q' > n_q$. Then it follows that

$$\prod_{\substack{p\in P\\ n_p' < n_p}} p^{n_p-n_p'} \in qP\,, \quad \text{and therefore} \quad p \in qP \ \text{ for some } \ p \in P \ \text{ such that } \ n_p' < n_p\,,$$

a contradiction. Hence there is no $p \in P$ such that $n_p' > n_p$, and for the same reason there is no $p \in P$ such that $n_p' > n_p$. Hence it follows that $n_p = n_p'$ for all $p \in P$, and consequently $u = u'$.

(b) $\Leftrightarrow$ (c) By definition, $\varepsilon \,|\, P \colon P \to \varepsilon(P)$ is bijective, and if $a \in D^\bullet$, $u \in D^\times$ and $(n_p)_{p\in P} \in \mathbb{N}_0^{(P)}$, then

$$a = u \prod_{p\in P} p^{n_p} \quad \text{if and only if} \quad \varepsilon(a) = \prod_{p\in P} \varepsilon(p)^{n_p}\,.$$

(b) $\Rightarrow$ (a) It suffices to prove that P is a complete set of primes. From the uniqueness in (b) we obtain:

- If $a,\, b \in D^\bullet$, then $\mathsf{v}_p(ab) = \mathsf{v}_p(a) + \mathsf{v}_p(b)$.
- If $a \in D^\bullet$ and $p \in P$, then $a \in pD$ if and only if $\mathsf{v}_p(a) > 0$.

Hence every $p \in P$ is a prime element. Indeed, if $p \in P$ and $a,\, b \in D^\bullet$ are such that $ab \in pD$, then $\mathsf{v}_p(ab) = \mathsf{v}_p(a) + \mathsf{v}_p(b) > 0$, hence $\mathsf{v}_p(a) > 0$ or $\mathsf{v}_p(b) > 0$ and therefore $a \in pD$ or $p \in pD$.

If $q \in D$ is a prime element, then $q \in pD$ for every $p \in P$ such that $\mathsf{v}_p(D) > 0$. But if $q \in pD$, then $qD = pD$, since q is an atom and qD is a maximal principal ideal. Hence there is a unique $p \in P$ such that $qD = pD$.

2. Obvious by 1.

3. Clearly, $\min\{\mathsf{v}_p(x) \mid x \in X\} \in \mathbb{N}_0$ for all $p \in P$, and $\min\{\mathsf{v}_p(x) \mid x \in X\} = 0$ for almost all $p \in P$. Hence $d \in D^\bullet$. If $b \in D^\bullet$, then $X \subset bD$ holds if and only if $\mathsf{v}_p(b) \le \mathsf{v}_p(x)$ for all $x \in X$ and $p \in P$. Therefore we obtain $d \in \mathrm{GCD}(X)$.

Let now $b \in X$ be arbitrary. Then $\mathsf{v}_p(d) \le \mathsf{v}_p(b)$, and the set $P_0 = \{p \in P \mid \mathsf{v}_p(b) \ne 0\}$ is finite. For every $p \in P_0$ there is some $x_p \in X$ such that $\mathsf{v}_p(x_p) = \mathsf{v}_p(d_p)$. If $E = \{b\} \cup \{x_p \mid p \in P_0\}$, then $d \in \mathrm{GCD}(E)$.

4. If D is factorial, then D is atomic by Theorem 1.5.5, and D is a GCD-monoid by 3. If D is an atomic GCD-monoid, then every atom is a prime element and therefore D is factorial, again by Theorem 1.5.5. $\square$

CHAPTER 2

The formalism of module and ideal systems

2.1. Weak module and ideal systems

Definition 2.1.1. Let K be a monoid.

1. A *weak module system* on K is a map $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$, $X \mapsto X_r$ such that, for all $c \in K$ and $X, Y \in \mathbb{P}(K)$ the following conditions are fulfilled:

   M1. $X \cup \{0\} \subset X_r$.

   M2. If $X \subset Y_r$, then $X_r \subset Y_r$.

   M3. $cX_r \subset (cX)_r$.

2. A *module system* on K is a weak module system r on K such that equality holds in **M3** for all $c \in K$ and $X \in \mathbb{P}(K)$.
3. Let r be a weak module system on K. A subset $J \subset K$ is called an *r-module* if $J = X_r$ for some subset $X \subset K$ (then $X \cup \{0\} \subset J$ by **M1**). An r-module $J \subset K$ is called *r-finitely generated* if $J = E_r$ for some finite subset $E \subset K$.

   We denote by

   - $\mathcal{M}_r(K)$ the set of all r-modules in K, and by
   - $\mathcal{M}_{r,\mathfrak{f}}(K)$ the set of all r-finitely generated r-modules in K.

   A submonoid $D \subset K$ is called an *r-monoid* if it is an r-module.
4. For two r-modules $J_1, J_2 \subset K$, we define their *r-product* by $J_1 \cdot_r J_2 = (J_1J_2)_r$, and we call $\cdot_r$ the *r-multiplication*.

Theorem 2.1.2. *Let K be a monoid, r be a weak module system on K and $X, Y \subset K$.*

1. $(X_r)_r = X_r$. *In particular, X is an r-module if and only if $X = X_r$.*
2. *If $X \subset Y$, then $X_r \subset Y_r$. In particular,*

$$X_r = \bigcap_{\substack{J \in \mathcal{M}_r(K) \\ J \supset X}} J$$

   is the smallest r-module containing X.

3. $X_r = (X \cup \{0\})_r$, $\emptyset_r = \{0\}_r$, *and if r is a module system, then $\emptyset_r = \{0\}_r = \{0\}$.*
4. *The intersection of any family of r-modules is again an r-module.*
5. *For every family $(X_\lambda)_{\lambda \in \Lambda}$ in $\mathbb{P}(K)$ we have*

$$\bigcup_{\lambda \in \Lambda} (X_\lambda)_r \subset \Big(\bigcup_{\lambda \in \Lambda} X_\lambda \Big)_r = \Big(\bigcup_{\lambda \in \Lambda} (X_\lambda)_r \Big)_r .$$

6. $(XY)_r = (X_rY)_r = (XY_r)_r = (X_rY_r)_r$. *If* $T \subset K$ *and* $1 \in T$, *then the following assertions are equivalent*:

$$\text{(a)} \quad X_r = TX_r \qquad \text{(b)} \quad X_r = (TX)_r\,. \qquad \text{(c)} \quad X_r = T_rX_r\,.$$

In particular, if $TX = X$, *then* $T_rX_r = X_r$.

7. *Equipped with the* r*-multiplication,* $\mathcal{M}_r(K)$ *is a monoid with unit element* $\{1\}_r$, *zero element* $\emptyset_r$, *and* $\mathcal{M}_{r,\mathfrak{f}}(K) \subset \mathcal{M}_r(K)$ *is a submonoid.*

8. *For every family* $(X_\lambda)_{\lambda\in\Lambda}$ *in* $\mathbb{P}(K)$ *we have the distributive law*

$$\Big(\bigcup_{\lambda\in\Lambda} X_\lambda Y\Big)_r = \Big(\bigcup_{\lambda\in\Lambda} X_\lambda\Big)_r \cdot_r Y_r = \Big(\bigcup_{\lambda\in\Lambda} (X_\lambda)_r \cdot_r Y_r\Big)_r\,.$$

9. $(X\!:\!Y)_r \subset (X_r\!:\!Y) = (X_r\!:\!Y_r) = (X_r\!:\!Y)_r$. *In particular, if* X *is an* r*-module, then* $(X\!:\!Y)$ *is also an* r*-module.*

Proof. 1. **M1** implies $X_r \subset (X_r)_r$, and since $X_r \subset X_r$, we obtain $(X_r)_r \subset X_r$ by **M2**. Hence $(X_r)_r = X_r$.

If $X = X_r$, then X is an r-module by definition. Conversely, if X is an r-module, then $X = Z_r$ for some subset $Z \subset K$, and then $X_r = (Z_r)_r = Z_r = X$.

2. If $X \subset Y$, then $X \subset Y_r$ and therefore $X_r \subset Y_r$, again by **M1** and **M2**.

If $J \in \mathcal{M}_r(K)$ and $X \subset J$, then $X_r \subset J_r = J$, and therefore

$$X_r \subset \bigcap_{\substack{J\in\mathcal{M}_r(K)\\ J\supset X}} J\,.$$

Since $X_r = (X_r)_r \in \mathcal{M}_r(K)$, the reverse inclusion is obvious.

3. By **M1** we have $X \cup \{0\} \subset X_r$, hence $(X \cup \{0\})_r \subset X_r$ by **M2**, and since $X_r \subset (X \cup \{0\})_r$ by 2., equality follows. If r is a module system, then $\{0\} = 0\{1\}_r = \{0\}_r$.

4. Let $(J_\lambda)_{\lambda\in\Lambda}$ be a family of r-modules, and

$$X = \bigcap_{\lambda\in\Lambda} J_\lambda\,.$$

Then $\{J_\lambda \mid \lambda \in \Lambda\} \subset \{J \in \mathcal{M}_r(K) \mid J \supset X\}$ and therefore

$$X_r = \bigcap_{\substack{J\in\mathcal{M}_r(K)\\ J\supset X}} J \subset \bigcap_{\lambda\in\Lambda} J_\lambda = X \subset X_r\,, \quad \text{which implies equality.}$$

5. For each $\alpha \in \Lambda$ we have

$$X_\alpha \subset \bigcup_{\lambda\in\Lambda} X_\lambda \subset \Big(\bigcup_{\lambda\in\Lambda} X_\lambda\Big)_r\,, \quad \text{hence} \quad (X_\alpha)_r \subset \Big(\bigcup_{\lambda\in\Lambda} X_\lambda\Big)_r \quad \text{and} \quad \bigcup_{\lambda\in\Lambda} (X_\lambda)_r \subset \Big(\bigcup_{\lambda\in\Lambda} X_\lambda\Big)_r\,.$$

Now it follows by **M2** that

$$\Big(\bigcup_{\lambda\in\Lambda} (X_\lambda)_r\Big)_r \subset \Big(\bigcup_{\lambda\in\Lambda} X_\lambda\Big)_r\,, \quad \text{and} \quad \bigcup_{\lambda\in\Lambda} X_\lambda \subset \bigcup_{\lambda\in\Lambda} (X_\lambda)_r \quad \text{implies} \quad \Big(\bigcup_{\lambda\in\Lambda} X_\lambda\Big)_r \subset \Big(\bigcup_{\lambda\in\Lambda} (X_\lambda)_r\Big)_r\,.$$

6. Using **M3**, we obtain

$$XY_r = \bigcup_{x\in X} xY_r \subset \bigcup_{x\in X} (xY)_r \subset (XY)_r \quad \text{and} \quad X_rY_r = \bigcup_{y\in Y_r} X_ry \subset \bigcup_{y\in Y_r} (Xy)_r \subset (XY_r)_r\,.$$

Hence it follows, using **M2**, that $(X_rY_r)_r \subset (XY_r)_r \subset (XY)_r \subset X_rY_r \subset (X_rY_r)_r$, an thus equality holds throughout.

(a) $\Rightarrow$ (b) From $TX \subset TX_r = X_r$ we obtain $(TX)_r \subset X_r \subset (TX)_r$, and thus $X_r = (TX)_r$.

(b) $\Rightarrow$ (c) From $X_r \subset T_rX_r \subset (T_rX_r)_r = (TX)_r = X_r$ we obtain $X_r = T_rX_r$.

(c) $\Rightarrow$ (a) From $X_r \subset TX_r \subset T_rX_r = X_r$ we obtain $X_r = TX_r$.

If $TX = X$, then $(TX)_r = X_r$ and therefore $T_rX_r = X_r$.

7. Obviously, $\cdot_r$ is commutative, and for every subset $X \subset K$ we have $(1X)_r = X_r$ and $(\emptyset X)_r = \emptyset_r$. If $J_1,\, J_2,\, J_3 \in \mathcal{M}_r(K)$, then $(J_1 \cdot_r J_2) \cdot_r J_3 = ((J_1J_2)_rJ_3)_r = (J_1J_2J_3)_r = (J_1(J_2J_3)_r)_r = J_1 \cdot_r (J_2 \cdot J_3))$. Hence $\cdot_r$ is associative, and $\mathcal{M}_r(K)$ is a monoid with unit element $\{1\}_r$ and zero element $\emptyset_r$.

If $J_1,\, J_2 \in \mathcal{M}_{r,\mathsf{f}}(K)$, then there exist finite subsets $E_1,\, E_2 \subset K$ such that $J_1 = (E_1)_r$ and $J_2 = (E_2)_r$. Hence it follows that $J_1 \cdot_r J_2 = ((E_1)_r(E_2)_r)_r = (E_1E_2)_r \in \mathcal{M}_{r,\mathsf{f}}(K)$.

8. If $(X_\lambda)_{\lambda\in\Lambda}$ is a family in $\mathbb{P}(K)$, then 5. implies that

$$\Big(\bigcup_{\lambda\in\Lambda} X_\lambda Y\Big)_r = \Big(\Big(\bigcup_{\lambda\in\Lambda} X_\lambda\Big)Y\Big)_r = \Big(\bigcup_{\lambda\in\Lambda} X_\lambda\Big)_r \cdot_r Y_r = \Big(\bigcup_{\lambda\in\Lambda} (X_\lambda Y)_r\Big)_r = \Big(\bigcup_{\lambda\in\Lambda} (X_\lambda)_r \cdot_r Y_r\Big)_r .$$

9. Since $(X_r : Y)Y \subset (X_r : Y)_rY \subset (X_r : Y)_rY_r \subset ((X_r : Y)Y)_r \subset (X_r)_r = X_r$, it follows that $(X_r\!:\!Y)_r \subset (X_r\!:\!Y) \subset (X_r\!:\!Y)_r \subset (X_r\!:\!Y_r) \subset (X_r\!:\!Y)$ and therefore $(X_r\!:\!Y) = (X_r\!:\!Y_r) = (X_r\!:\!Y)_r$. Since $(X\!:\!Y) \subset (X_r\!:\!Y) = (X_r\!:\!Y)_r$ it follows that $(X\!:\!Y)_r \subset (X_r\!:\!Y)$.

If X is an r-module, then $(X\!:\!Y)_r = (X_r\!:\!Y)_r = (X_r\!:\!Y) = (X\!:\!Y)$, and therefore $(X\!:\!Y)$ is also an r-module. □

Remarks and Definition 2.1.3. Let K be a monoid and $D \subset K$ a submonoid.

1. A (weak) module system $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$ is called a
   - (*weak*) *D-module system* if $DJ \subset J$ (and thus $DJ = J$) for every $J \in \mathcal{M}_r(K)$.
   - (*weak*) *ideal system* of D if it is a (weak) D-module system and $D_r = D$.

   In this case, we say more precisely that r is a (weak) ideal system of D defined on K. Whenever it does not matter on which overmonoid of D the ideal system r is defined, we say that r is an ideal system of D.

   If r is a (weak) ideal system of D defined on K, then $r\,|\,\mathbb{P}(D)\colon \mathbb{P}(D) \to \mathbb{P}(D)$ is also a (weak) ideal system of D.
2. Let $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$ be a weak ideal system of D. An r-modules $J \in \mathcal{M}_r(K)$ is called an *r-ideal* of D if $J \subset D$. If J is an r-ideal of D, then $0 \in J$ and $DJ = J$, and thus J is a (semigroup) ideal of D. We denote by
   - $\mathcal{I}_r(D) = \{J \in \mathcal{M}_r(K) \mid J \subset D\}$ the set of all r-ideals of D and by
   - $\mathcal{I}_{r,\mathsf{f}}(D) = \mathcal{I}_r(D) \cap \mathcal{M}_{r,\mathsf{f}}(K)$ the set of all r-finitely generated r-ideals of D.

   By definition, $\mathcal{I}_{r,\mathsf{f}}(D) \subset \mathcal{I}_r(D) \subset \mathcal{M}_r(K)$ are submonoids.
3. Let again $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$ be a weak ideal system of D, and assume that $K = \mathsf{q}(D)$. Then an r-module $J \in \mathcal{M}_r(K)$ is called a *fractional r-ideal* of D if J is D-fractional. If J is a fractional r-ideal of D, then $0 \in J$ and $DJ = J$, and thus J is a fractional (semigroup) ideal of D. We denote by
   - $\mathcal{F}_r(D) = \{J \in \mathcal{M}_r(K) \mid J \text{ is } D\text{-fractional}\}$ the set of all fractional r-ideals of D,

   and we assert that $\mathcal{M}_{r,\mathsf{f}}(K) \subset \mathcal{F}_r(D)$ [Proof: If $J \in \mathcal{M}_{r,\mathsf{f}}(K)$, then $J = E_r$ for some $E \in \mathbb{P}_{\mathsf{f}}(K)$. Hence there exists some $a \in D^*$ such that $aE \subset D$, and therefore $aJ = aE_r \subset (aE)_r \subset D_r = D$].

   Consequently, we denote by
   - $\mathcal{F}_{r,\mathsf{f}}(D) = \mathcal{M}_{r,\mathsf{f}}(K)$ the set of all r-finitely generated fractional r-ideals of D.

   By definition, $\mathcal{F}_{r,\mathsf{f}}(D) = \mathcal{M}_{r,\mathsf{f}}(K) \subset \mathcal{F}_r(D) \subset \mathcal{M}_r(K)$, and $\mathcal{M}_{r,\mathsf{f}}(K) \subset \mathcal{M}_r(K)$ is a submonoid. We assert that also $\mathcal{F}_r(D) \subset \mathcal{M}_r(K)$ is a submonoid. [Proof: If $J_1,\, J_2 \in \mathcal{F}_r(D)$, then J_1J_2 is D-fractional by Theorem 1.4.2.3. Hence there exists some $c \in D^*$ such that $cJ_1J_2 \subset D$, and then $c(J_1J_2)_r \subset (cJ_1J_2)_r \subset D_r = D$ implies that $J_1 \cdot_r J_2 = (J_1J_2)_r \in \mathcal{F}_r(D)$].

Consequently, $\mathcal{I}_r(D) \subset \mathcal{F}_r(D)$ and $\mathcal{I}_{r,\mathsf{f}}(D) \subset \mathcal{F}_{r,\mathsf{f}}(D)$ are also submonoids.

Theorem 2.1.4. *Let K be a monoid and $D \subset K$ a submonoid. Assume that $K = \mathsf{q}(D)$, and let $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$ be an ideal system of D. Then*

$$\mathcal{F}_r(D) = \{a^{-1}I \mid I \in \mathcal{I}_r(D)\,,\ a \in D^*\,\} = \{J \in \mathbb{P}(K) \mid aJ \in \mathcal{I}_r(D) \ \textit{for some}\ \ a \in D^*\,\}$$

and

$$\mathcal{F}_{r,\mathsf{f}}(D) = \{a^{-1}I \mid I \in \mathcal{I}_{r,\mathsf{f}}(D)\,,\ a \in D^*\,\} = \{J \in \mathbb{P}(K) \mid aJ \in \mathcal{I}_{r,\mathsf{f}}(D) \ \textit{for some}\ \ a \in D^*\,\}$$

Proof. We show that

$$\mathcal{F}_r(D) \subset \{a^{-1}I \mid I \in \mathcal{I}_r(D)\,,\ a \in D^*\,\} \subset \{J \in \mathbb{P}(K) \mid aJ \in \mathcal{I}_r(D) \ \text{for some}\ \ a \in D^*\,\} \subset \mathcal{F}_r(D)\,.$$

If $J \in \mathcal{F}_r(D)$, then there exists some $a \in D^*$ such that $I = aJ \subset D$, and $I_r = aJ_r = aJ = I$. Hence $I \in \mathcal{I}_r(D)$ and $J = a^{-1}I$. If $I \in \mathcal{I}_r(D)$ and $a \in D^*$, then $J = a^{-1}I \subset K$ and $I = aJ$. If $J \subset K$, $a \in D^*$ and $I = aJ \in \mathcal{I}_r(D)$, then J is D-fractional, and $J_r = (a^{-1}I)_r = a^{-1}I_r = a^{-1}I = J$, hence $J \in \mathcal{F}_r(D)$.

In all arguments above, J is r-finitely generated if and only if I is r-finitely generated, and thus also the second set of equalities holds. □

Examples 2.1.5 (Some (weak) ideal systems).

1. **Trivial systems**. Let K be a monoid. There are two trivial weak ideal systems y, y_1 on K, defined as follows.

$y_1\colon \mathbb{P}(K) \to \mathbb{P}(K)$, defined by $X_{y_1} = K$ for all subsets $X \subset K$.

$y\colon \mathbb{P}(K) \to \mathbb{P}(K)$, defined by $X_y = \{0\}$ if $X \subset \{0\}$, and $X_y = K$ if $X \not\subset \{0\}$.

It is easily checked that y and y_1 are weak ideal systems of K.

Let K be divisible. Then K and $\{0\}$ are the only semigroup ideals of K. Hence y and y_1 are the only weak ideal systems of K, and y is even an ideal system of K.

2. **The semigroup system**. Let K be a monoid and $D \subset K$ a submonoid. The *semigroup system* of D defined on K is the system $s(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$, defined by

$$\emptyset_{s(D)} = \{0\}\,, \quad \text{and} \quad X_{s(D)} = DX = \bigcup_{a \in X} Da \quad \text{if} \quad X \neq \emptyset\,.$$

It is plain that $s(D)$ is an ideal system of D, and $\mathcal{M}_{s(D)}(K) = \{J \subset K \mid 0 \in J \text{ and } DJ = J\}$. In particular, $\mathcal{I}_{s(D)}(D)$ is the set of all semigroup ideals of D. If $c \in K$, then $\{c\}_{s(D)} = cD$, the union of any family of $s(D)$-modules is again an $s(D)$-module, and if $J_1,\ J_2 \in \mathcal{M}_{s(D)}(K)$, then $J_1 \cdot_{s(D)} J_2 = J_1J_2$.

If $K = \mathsf{q}(D)$, then $\mathcal{F}_{s(D)}(D)$ is the set of all fractional (semigroup) ideals of D, and

$$\mathcal{F}_{s(D),\mathsf{f}}(D) = \{c_1D \cup \ldots \cup c_mD \mid m \in \mathbb{N}\,,\ c_1, \ldots, c_m \in K\,\}\,.$$

If K is divisible, then $s(K)$ is the only ideal system of K. In fact, it coincides with the trivial system y considered in Example 1.

3. **The Dedekind system**. Let K be a ring and $D \subset K$ a subring. The *Dedekind system* of D defined on K is the system $d(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$, defined by

$$X_{d(D)} = \{a_1x_1 + \ldots + a_nx_n \mid n \in \mathbb{N}\,,\ x_1, \ldots, x_n \in X\,,\ a_1, \ldots, a_n \in K\} = {}_K(X) \quad \text{for all} \ \ X \in \mathbb{P}(K)\,,$$

[$X_{d(D)}$ is the D-submodule of K generated by X].

It is plain that $d(D)$ is an ideal system of D, and $\mathcal{M}_{d(D)}(K)$ is the set of all D-submodules of K. A D-module $J \in \mathcal{M}_{d(D)}(K)$ is $d(D)$-finitely generated if and only if it is a finitely generated D-module.

$\mathcal{I}_{d(D)}(D)$ is the set of all ideals of D, and if $c \in K$, then $\{c\}_{d(D)} = \{c\}_{s(D)} = cD$. For every family $(J_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{M}_{d(D)}(K)$, we have

$$\Big(\bigcup_{\lambda\in\Lambda} J_\lambda\Big)_{d(K)} = \sum_{\lambda\in\Lambda} J_\lambda\,.$$

If $J_1,\, J_2 \in \mathcal{I}_{d(K)}(K)$, then $J_1 \cdot_{d(K)} J_2$ is the additive abelian group generated by J_1J_2.

If K is a field, then $d(K) = s(K)$ is the only ideal system of K.

4. **The system of homogenous ideals**. Let K be a graded ring with homogeneous components $(K_i)_{i\geq 0}$, that means,

$$K = \bigoplus_{i\geq 0} K_i \quad \text{as an additive abelian group, and} \quad K_iK_j \subset K_{i+j} \quad \text{for all } i,\, j \geq 0\,.$$

An element $x \in K$ is called *homogenous* of degree $i \geq 0$ if $x \in K_i$. Every $x \in K$ has a unique representation

$$x = \sum_{i\geq 0} x_i\,, \quad \text{where} \quad x_i \in K_i \ \text{ and } \ x_i = 0 \ \text{ for almost all } \ i \geq 0.$$

In this representation we call x_i the i-th homogenous component of x. For every subset $X \subset K$ let X^h be the set of all homogeneous components of elements of X. An ideal $J \subset K$ is called *homogenous* if $J^h \subset J$, equivalently

$$J = \sum_{i\geq 0} J \cap K_i\,.$$

Then $X_h = (X^h)_{d(K)}$ is the smallest homogeneous ideal containing X, and

$$h\colon \mathbb{P}(K) \to \mathbb{P}(K)\,, \quad X \mapsto X_h\,, \quad \text{is a weak ideal system of } K.$$

6. **The system of filters**. Let $(K, \leq, 0, 1)$ be a lattice. That means, $(K, \leq)$ is a partially ordered set, $0 = \max(K)$, $1 = \min(K)$, and any two elements $a,\, b \in K$ possess a supremum $ab = a \vee b$ and an infimum $a \wedge b$. Then K is a monoid with unit 1 and zero 0.

If M is a set, then $(K, \leq, 0, 1) = (\mathbb{P}(M), \subset, M, \emptyset)$ is a lattice (the *subset lattice* of M).

Let $(K, \leq, 0, 1)$ be a lattice. A non-empty subset $F \subset K$ is called a *filter* if for all $a,\, b \in K$ the following assertions hold:

- If $a \leq b$ and $a \in F$, then $b \in F$.
- If $a,\, b \in F$, then $ab \in F$.

For a subset $X \subset K$, let X_f be the smallest filter containing X. Then $\emptyset_f = \{0\}$, and if $X \neq \emptyset$, then

$$X_f = \bigcap_{\substack{X\subset F\\ F \text{ is a filter}}} F \;=\; \{x \in K \mid \text{ there exist } x_1, \ldots, x_r \in X \text{ such that } x \geq x_1 \cdot \ldots \cdot x_r\}\,.$$

The map $f\colon \mathbb{P}(X) \to \mathbb{P}(X)$, $X \mapsto X_f$, is a weak ideal system on K, and for every $c \in K$ it follows that $\{c\}_f = \{x \in K \mid x \geq c\} = cK$. [All this is easily checked, observing that, for all $x,\, y \in K$, $x \leq y$ holds if and only if $xy = y$].

Theorem 2.1.6. *Let K be a monoid, $D \subset K$ a submonoid and r a weak module system on K.*

1. *D_r is an r-monoid. In particular, $\{1\}_r$ is the smallest r-monoid in K, and if $D \subset \{1\}_r$, then $\{1\}_r = D_r$.*
2. *Let r be a weak D-module system. Then r is a weak D_r-module system, $\{1\}_r = D_r$, and if $X \subset K$, then $X_r = DX_r = D_rX_r = (DX)_r$ and $D_r \subset (X_r \colon X)$.*
3. *r is a weak D-module system if and only if $D \subset \{1\}_r$. In particular:*
   (a) *r is a weak $\{1\}_r$-module system.*

(b) *If r is a D-module system, then $\{c\}_r = cD_r$ for all $c \in K$.*

(c) *If r is an ideal system of D, then $\{c\}_r = cD$ for all $c \in K$.*

4. *If r is a weak ideal system of D and $I,\, J \in \mathcal{I}_r(D)$, then $I \cdot_r J \subset I \cap J$.*

Proof. 1. By Theorem 2.1.2.6, $DD = D$ implies $D_rD_r = D_r$. Hence $D_r \subset K$ is a submonoid and thus an r-monoid. In particular, $\{1\}_r = \{0,1\}_r$ is an r-monoid, and it is the smallest r-monoid in K.

If $D \subset \{1\}_r$, then $D_r \subset \{1\}_r \subset D_r$, and therefore $\{1\}_r = D_r$.

2. If $X \subset K$, then $X_r = DX_r$ by definition, and therefore $X_r = D_rX_r = (DX)_r$ by Theorem 2.1.2.6. In particular, $D_r \subset D_r\{1\}_r = \{1\}_r$ and therefore $D_r = \{1\}_r$. If $J \in \mathcal{M}_r(K)$, then $J = J_r$ and $J = DJ$ implies $J = D_rJ$, and therefore r is a weak D_r-module system.

If $X \subset K$, then $(X_r\!:\!X)$ is an r-module and $1 \in (X_r\!:\!X)$. Hence it follows that $D_r = \{1\}_r \subset (X_r\!:\!X)$.

3. If r is a weak D-module system, then $\{1\}_r = D\{1\}_r \supset D$. Conversely, if $D \subset \{1\}_r$ and $J \in \mathcal{M}_r(K)$, then $J \subset DJ =\subset \{1\}_rJ \subset J_r = J$, and thus r is a weak D-module system.

(a) Since r is obviously a $\{0,1\}$-module system and $\{0,1\}_r = \{1\}_r$, it is also an $\{1\}_r$-module system.

(b), (c) If r is a D-module system, then $\{c\}_r = c\{1\}_r = cD_r$, and if r is an ideal system of D, then $D_r = D$.

4. Let r be a weak ideal system of D and $I,\, J \in \mathcal{I}_r(D)$. Then $I \cap J \in \mathcal{I}_r(D)$, and since I and J are semigroup ideals, it follows that $IJ \subset I \cap J$, and consequently $I \cdot_r J = (IJ)_r \subset I \cap J$. □

2.2. Finitary and noetherian (weak) module systems

Theorem und Definition 2.2.1. *Let K be a monoid and r a weak module system on K.*

1. *The following assertions are equivalent*:

(a) *For every subset $X \subset K$, we have*

$$X_r \;=\; \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(X)} E_r\,.$$

(b) *For all $X \subset K$ and $a \in X_r$ there exists a finite subset $E \subset X$ such that $a \in E_r$.*

(c) *For every directed family $(X_\lambda)_{\lambda\in\Lambda}$ in $\mathbb{P}(K)$ we have*

$$\Big(\bigcup_{\lambda\in\Lambda} X_\lambda\Big)_r \;=\; \bigcup_{\lambda\in\Lambda} (X_\lambda)_r\,.$$

(d) *The union of every directed family of r-modules is again an r-module.*

(e) *If $X \subset K$, $J \in \mathcal{M}_{r,\mathsf{f}}(K)$ and $J \subset X_r$, then there exists some $E \in \mathbb{P}_{\mathsf{f}}(X)$ such that $J \subset E_r$.*

If r satisfies these equivalent conditions, then r is called *finitary*.

2. *If r is finitary, $X \subset K$ and $X_r \in \mathcal{M}_{r,\mathsf{f}}(K)$, then there exists some $E \in \mathbb{P}_{\mathsf{f}}(X)$ such that $E_r = X_r$.*

3. *If r and q are finitary weak module systems on K, then $r = q$ if and only if $E_r = E_q$ for all $E \in \mathbb{P}_{\mathsf{f}}(X)$.*

Proof. 1. (a) $\Rightarrow$ (b) Obvious.

(b) $\Rightarrow$ (c) If

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda\,, \quad \text{then} \quad X_r \supset \bigcup_{\lambda\in\Lambda} (X_\lambda)_r\,.$$

To prove the converse, let $x \in X_r$ and $E \subset X$ finite such that $x \in E_r$. Since $(X_\lambda)_{\lambda\in\Lambda}$ is directed, there exists some $\alpha \in \Lambda$ such that $E \subset X_\alpha$, hence $E_r \subset (X_\alpha)_r$, and consequently

$$x \in E_r \subset \bigcup_{\lambda\in\Lambda} (X_\lambda)_r \,.$$

(c) $\Rightarrow$ (d) Let $(X_\lambda)_{\lambda\in\Lambda}$ be a directed family of r-modules. Then

$$\Bigl(\bigcup_{\lambda\in\Lambda} X_\lambda\Bigr)_r = \bigcup_{\lambda\in\Lambda} (X_\lambda)_r = \bigcup_{\lambda\in\Lambda} X_\lambda \,.$$

(d) $\Rightarrow$ (a) Obviously,

$$\bigcup_{E\in\mathbb{P}_\mathfrak{f}(X)} E_r \subset X_r \,.$$

For $E,\, F \in \mathbb{P}_\mathfrak{f}(X)$, we have $E_r \cup F_r \subset (E \cup F)_r$. Hence $(E_r)_{E\in\mathbb{P}_\mathfrak{f}(X)}$ is directed, and we obtain

$$X_r = \Bigl(\bigcup_{E\in\mathbb{P}_\mathfrak{f}(X)} E\Bigr)_r \subset \Bigl(\bigcup_{E\in\mathbb{P}_\mathfrak{f}(X)} E_r\Bigr)_r = \bigcup_{E\in\mathbb{P}_\mathfrak{f}(X)} E_r \,.$$

(b) $\Rightarrow$ (e) Suppose that $X \subset K$ and $J = F_r \subset X_r$, where $F \in \mathbb{P}_\mathfrak{f}(K)$. For every $c \in F$, there is some $E(c) \in \mathbb{P}_\mathfrak{f}(X)$ such that $c \in E(c)_r$. Then

$$E = \bigcup_{c\in E} E(c) \in \mathbb{P}_\mathfrak{f}(X)\,, \quad F \subset \bigcup_{c\in E} E(c)_r \subset E_r \quad \text{and thus} \quad J = F_r \subset E_r \,.$$

(e) $\Rightarrow$ (b) If $X \subset K$ and $a \in X_r$, then $\{a\}_r \in \mathcal{M}_{r,\mathfrak{f}}(K)$ and $\{a\}_r \subset X_r$. Hence there exists a finite subset $E \subset X$ such that $a \in \{a\}_r \subset E_r$.

2. If r is finitary, $X \subset K$ and $X_r \in \mathcal{M}_{r,\mathfrak{f}}(K)$, then we apply 1.(e) with $J = X_r \in \mathcal{M}_{r,\mathfrak{f}}$ to obtain $X_r \subset E_r$ for some $E \in \mathbb{P}_\mathfrak{f}(X)$, and thus $X_r = E_r$.

3. By 1.(a), two finitary weak module systems coincide if and only if they coincide on finite sets. $\square$

Theorem und Definition 2.2.2. *Let K be a monoid and $D \subset K$ a submonoid.*

1. *Let $r\colon \mathbb{P}_\mathfrak{f}(K) \to \mathbb{P}(K)$ be a map such that, for all $c \in K$ and $E,\, F \in \mathbb{P}_\mathfrak{f}(K)$ the following conditions are fulfilled:*

   M1$_\mathfrak{f}$. $E \cup \{0\} \subset E_r$.

   M2$_\mathfrak{f}$. *If* $E \subset F_r$, *then* $E_r \subset F_r$.

   M3$_\mathfrak{f}$. $cE_r \subset (cE)_r$.

   Then there exists a unique finitary weak module system $\overline{r}$ on K satisfying $\overline{r}\,|\,\mathbb{P}_\mathfrak{f}(K) = r$. It is given by

   $$X_{\overline{r}} \;=\; \bigcup_{E\in\mathbb{P}_\mathfrak{f}(X)} E_r \quad \textit{for all} \quad X \subset K\,.$$

   $\overline{r}$ is a weak D-module system if and only if $cD \subset \{c\}_r$ for all $c \in K$, and it is a module system if and only if $(cE)_r = cE_r$ for all $c \in K$ and $E \in \mathbb{P}_\mathfrak{f}(K)$.

   $\overline{r}$ is called that *total system* defined by r and is usually again denoted by r.

2. *Let r be a weak module system on K. Then there exists a unique finitary weak module system $r_\mathfrak{f}$ on K such that $E_r = E_{r_\mathfrak{f}}$ for all finite subsets of K. It is given by*

   $$X_{r_\mathfrak{f}} = \bigcup_{E\in\mathbb{P}_\mathfrak{f}(X)} E_r \quad \textit{for all} \quad X \subset K\,,$$

   and it has the following properties:

   (a) *$X_{r_\mathfrak{f}} \subset X_r$ for all $X \in \mathbb{P}(K)$, $\mathcal{M}_r(K) \subset \mathcal{M}_{r_\mathfrak{f}}(K)$, and $\mathcal{M}_{r_\mathfrak{f},\mathfrak{f}}(K) = \mathcal{M}_{r,\mathfrak{f}}(K)$.*

(b) $(r_{\mathsf f})_{\mathsf f} = r_{\mathsf f}$, *and* r *is finitary if and only if* $r = r_{\mathsf f}$.

(c) *If* r *is a module system, then* $r_{\mathsf f}$ *is a module system, too.*

(d) $r_{\mathsf f}$ *is a weak* D*-module system* [*a weak ideal system of* D] *if and only if* r *is a weak* D*-module system* [*a weak ideal system of* D].

$r_{\mathsf f}$ is called the *finitary system associated with* r.

PROOF. 1. Let $\overline{r}\colon \mathbb{P}(K) \to \mathbb{P}(K)$ be defined by

$$X_{\overline{r}} \;=\; \bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} E_r \quad \text{for all} \quad X \subset K\,.$$

We prove that $\overline{r}$ satisfies the properties **M1**, **M2**, **M3** for all $c \in K$ and $X, Y \subset K$. Once this is done, it is obvious that $E_{\overline{r}} = E_r$ for all $E \in \mathbb{P}_{\mathsf f}(X)$. Hence $\overline{r} \,|\, \mathbb{P}_{\mathsf f}(K) = r$, and $\overline{r}$ is finitary.

M1. Since $E \cup \{0\} \subset E_r$ for all $E \in \mathbb{P}_{\mathsf f}(X)$, we obtain $X \cup \{0\} \subset X_{\overline{r}}$.

M2. Suppose that $X \subset Y_{\overline{r}}$, and let $x \in X_r$. There exists some $E \in \mathbb{P}_{\mathsf f}(X)$ such that $x \in E_r$, and

$$E \subset Y_{\overline{r}} \;=\; \bigcup_{F \in \mathbb{P}_{\mathsf f}(Y)} F_r\,.$$

For every $e \in E$, there exists some $F(e) \in \mathbb{P}_{\mathsf f}(Y)$ such that $e \in F(e)_r$, and we obtain

$$F = \bigcup_{e \in E} F(e) \in \mathbb{P}_{\mathsf f}(Y)\,, \quad \text{and} \quad E \subset \bigcup_{e \in E} F(e)_r \subset F_r\,,$$

hence $E_r \subset F_r \subset Y_{\overline{r}}$ and $x \in Y_{\overline{r}}$.

M3. Note that $\mathbb{P}_{\mathsf f}(cX) = \{cE \mid E \in \mathbb{P}_{\mathsf f}(X)\}$. Hence it follows that

$$cX_{\overline{r}} \;=\; \bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} cE_r \;\subset\; \bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} (cE)_r \;=\; \bigcup_{F \in \mathbb{P}_{\mathsf f}(cX)} F_r \;= (cX)_{\overline{r}}\,,$$

and $cX_{\overline{r}} = (cX)_{\overline{r}}$ holds if and only if $cE_r = (cE)_r$ for all $E \in \mathbb{P}_{\mathsf f}(X)$. Consequently, $\overline{r}$ is a module system if and only if $cE_r = (cE)_r$ for all $E \in \mathbb{P}_{\mathsf f}(X)$. By Theorem 2.1.6.3 it follows that $\overline{r}$ is a weak D-module system if and only if $cD \subset \{c\}_r$ for all $c \in K$.

It remains to prove the uniqueness of $\overline{r}$. If $\widetilde{r}$ is any finitary weak module system on K satisfying $\widetilde{r} \,|\, \mathbb{P}_{\mathsf f}(K) = r$, then

$$X_{\widetilde{r}} = \bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} E_{\widetilde{r}} = \bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} E_r = X_{\overline{r}} \quad \text{for all} \;\; X \subset K, \text{ and therefore} \quad \widetilde{r} = \overline{r}\,.$$

2. By 1., applied with $r \,|\, \mathbb{P}_{\mathsf f}(X)$, there exists a unique weak module system $r_{\mathsf f}$ on K such that $E_{r_{\mathsf f}} = E_r$ for all $E \in \mathbb{P}_{\mathsf f}(X)$, and if $X \subset K$, then $X_{r_{\mathsf f}}$ is given as asserted.

(a) If $X \in \mathbb{P}(K)$, then $E_r \subset X_r$ for all $E \in \mathbb{P}_{\mathsf f}(X)$, and therefore $X_{r_{\mathsf f}} \subset X_r$. If $X \in \mathcal{M}_r(K)$, then $X_{r_{\mathsf f}} \subset X_r = X$ and therefore $X = X_{r_{\mathsf f}} \in \mathcal{M}_{r_{\mathsf f}}(K)$. Since $E_r = E_{r_{\mathsf f}}$ for all $E \in \mathbb{P}_{\mathsf f}(K)$, it follows that $\mathcal{M}_{r_{\mathsf f},\mathsf f}(K) = \mathcal{M}_{r,\mathsf f}(K)$.

(b) By the uniqueness of $r_{\mathsf f}$ it follows that $r_{\mathsf f} = r$ if and only if r is finitary, and since $r_{\mathsf f}$ is finitary, we obtain $(r_{\mathsf f})_{\mathsf f} = r_{\mathsf f}$.

(c) If r is a module system, then $(cE)_r = cE_r$ for all $c \in K$ and $E \in \mathbb{P}_{\mathsf f}(K)$, and then $r_{\mathsf f}$ is a module system by 1.

(d) Since $\{1\}_r = \{1\}_{r_{\mathsf f}}$, Theorem 2.1.6.3 implies that $r_{\mathsf f}$ is a weak D-module system if and only if r is a weak D-module system. In this case, $D_r = \{1\}_r = \{1\}_{r_{\mathsf f}} = D_{r_{\mathsf f}}$, and therefore $r_{\mathsf f}$ is a weak ideal system of D if and only if r is a weak ideal system of D. □

Remark 2.2.3.

1. Let K be a monoid, $D \subset K$ a submonoid and $s(D)$ the semigroup system of D defined on K (see Example 2.1.5.2). If $\emptyset \neq X \subset K$, then
$$X_{s(D)} = DX = \bigcup_{a \in X} Da \subset \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(X)} DE = \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(X)} E_{s(D)} \subset DX\,,$$
and therefore $s(D)$ is finitary.
2. Let K be a ring, $D \subset K$ a subring and $d(D)$ the Dedekind system of D defined on K (see Example 2.1.5.3). Since every D-module is the union of its finitely generated submodules, the system $d(D)$ is finitary.

Example 2.2.4. Let K be a topological monoid (that is, a monoid equipped with a topology such that the multiplication $K \times K \to K$, $(x,y) \mapsto xy$, is continuous). Let $c \colon \mathbb{P}(K) \to \mathbb{P}(K)$ be defined by
$$X_c = \overline{X_{s(K)}} = \begin{cases} \overline{\{0\}} & \text{if} \quad X = \emptyset\,, \\ \overline{XK} & \text{if} \quad X \neq \emptyset\,. \end{cases}$$
Then X_c is the smallest closed semigroup ideal of K containing X. If $\emptyset \neq X \subset K$ and $z \in K$, then $z\overline{XK} \subset \overline{zXK} \subset \overline{XK}$, and therefore c is a weak ideal system on K. If $z \in K$ is such that the map $\tau_z \colon K \to K$, defined by $\tau_z(x) = zx$, is closed, then $(zX)_c = zX_c$ for all $X \in \mathbb{P}(K)$. In particular, if τ_z is a closed map for all $z \in K$, then c is an ideal system of K. In particular, this holds if K is compact. In general however, c is not finitary.

We consider the additive monoid $\mathbb{R}_{\geq 0}$. For every $z \in \mathbb{R}_{\geq 0}$, the map $x \mapsto z + x$ is closed, and thus c is an ideal system on $\mathbb{R}_{\geq 0}$. If $\gamma \in \mathbb{R}_{\geq 0}$ and $X = (\gamma, \infty)$, then $X_c = [\gamma, \infty)$, but for every finite subset $E \subset (\gamma, \infty)$, it follows that $E_c = [\min(E), \infty) \subset (\gamma, \infty)$. Hence $X_{c_{\mathsf{f}}} = X$, $c \neq c_{\mathsf{f}}$, and c is not finitary.

Theorem und Definition 2.2.5. *Let K be a monoid, $D \subset K$ a submonoid and $r \colon \mathbb{P}(K) \to \mathbb{P}(K)$ a weak ideal system of D defined on K.*

1. *The following conditions are equivalent:*
   (a) *$\mathcal{I}_r(D)$ satisfies the* ACC*:*
      - *For every sequence $(J_n)_{n \geq 0}$ in $\mathcal{I}_r(D)$ satisfying $J_n \subset J_{n+1}$ for all $n \geq 0$, there exists some $m \geq 0$ such that $J_n = J_m$ for all $n \geq m$.*
      - *Every non-empty set of r-ideals has a maximal element.*

   (b) *For every subset $X \subset D$, there exists some $E \in \mathbb{P}_{\mathsf{f}}(X)$ such that $X \subset E_r$ (and then $X_r = E_r$).*

   (c) *$r \,|\, \mathbb{P}(D)$ is finitary, and $\mathcal{I}_r(D) = \mathcal{I}_{r,\mathsf{f}}(D)$.*

   If these conditions are fulfilled, then r is called a *noetherian* weak ideal system, and D is called *r-noetherian.*
2. *D is r-noetherian if and only if D is r_{f}-noetherian.*
3. *If $K = \mathsf{q}(D)$ and D is r-noetherian, then $\mathcal{F}_r(D) = \mathcal{F}_{r,\mathsf{f}}(D)$ (that is, every fractional r-ideal is r-finitely generated).*

Proof. 1. (a) $\Rightarrow$ (b) Let $X \subset D$ and $\Omega = \{F_r \mid F \in \mathbb{P}_{\mathsf{f}}(X)\}$. By assumption, there exists some $E \in \mathbb{P}_{\mathsf{f}}(X)$ such that E_r is maximal in Ω, and we assert that $E_r = X_r$. Indeed, if $E_r \subsetneq X_r$, then $X \not\subset E_r$, and if $c \in X \setminus E_r$, then $E_r \subsetneq (E \cup \{c\})_r$, which contradicts the maximality of E_r.

Clearly, if $E \in \mathbb{P}_{\mathsf{f}}(X)$, then $X \subset E_r$ if and only if $X_r = E_r$.

(b) $\Rightarrow$ (c) By (b), every r-ideal is r-finitely generated. Hence $\mathcal{I}_r(D) = \mathcal{I}_{r,\mathsf{f}}(D)$, and r is finitary.

(c) $\Rightarrow$ (a) Let $(J_n)_{n\ge0}$ be an ascending sequence in $\mathcal{I}_r(D)$. Then

$$J=\bigcup_{n\ge0}J_n$$

is an r-ideal (since $r\,|\,\mathbb{P}(D)$ is finitary), and there exists some $E\in\mathbb{P}_{\mathsf f}(J)$ such that $J=E_r$. There is some $m\in\mathbb{N}$ such that $E\subset J_m$, and then it follows that $J_n=J_m$ for all $n\ge m$.

2. If D is $r_{\mathsf f}$-noetherian, then D is r-noetherian, since $\mathcal{I}_r(D)\subset\mathcal{I}_{r_{\mathsf f}}(D)$ by Theorem 2.2.2.2 (a). If D is r-noetherian, then $r\,|\,\mathbb{P}(D)=r_{\mathsf f}\,|\,\mathbb{P}(D)$ by 1.(c), and thus D is $r_{\mathsf f}$-noetherian.

3. Since $\mathcal{I}_r(D)=\mathcal{I}_{r,\mathsf f}(D)$, Theorem 2.1.4 implies

$$\mathcal{F}_r(D)=\{a^{-1}I\mid I\in\mathcal{I}_r(D)\,,\ a\in D^\bullet\}=\{a^{-1}I\mid I\in\mathcal{I}_{r,\mathsf f}(D)\,,\ a\in D^\bullet\}=\mathcal{F}_{r,\mathsf f}(D)\,.\qquad\square$$

2.3. Comparison and mappings of module systems

Definition 2.3.1. Let K be a monoid, and let r and q be weak module systems on K. We call q *finer* than r and r *coarser* than q and write $r\le q$ if $X_r\subset X_q$ for all subsets $X\subset K$.

$\le$ is a partial order on the set of all weak module systems on K.

Theorem 2.3.2. *Let K be a monoid, and let r and q be weak module systems on K. Then $r_{\mathsf f}\le r$, and the following assertions are equivalent*:

(a) $r\le q$.

(b) $X_q=(X_r)_q$ *for all subsets* $X\subset K$.

(c) $\mathcal{M}_q(K)\subset\mathcal{M}_r(K)$.

If r is finitary, then there are also equivalent:

(d) $E_r\subset E_q$ *for all finite subsets* $E\subset K$.

(e) $\mathcal{M}_{q_{\mathsf f}}(K)\subset\mathcal{M}_r(K)$.

(f) $\mathcal{M}_{q,\mathsf f}(K)\subset\mathcal{M}_r(K)$.

(g) $r\le q_{\mathsf f}$.

Proof. It follows by Theorem 2.2.2 that $r_{\mathsf f}\le r$.

(a) $\Rightarrow$ (b) If $X\subset K$, then $X_r\subset X_q$ by assumption, hence $(X_r)_q\subset X_q$, and since $X\subset X_r$, it follows that $X_q\subset(X_r)_q$.

(b) $\Rightarrow$ (c) If $J\in\mathcal{M}_q(K)$, then $J_r\subset(J_r)_q=J_q=J\subset J_r$, and therefore $J=J_r\in\mathcal{M}_r(K)$.

(c) $\Rightarrow$ (a) If $X\subset K$, then $X_q\in\mathcal{M}_q(K)\subset\mathcal{M}_r(K)$, and therefore $X_q=(X_q)_r\supset X_r$.

Assume now that r is finitary.

(a) $\Rightarrow$ (d) Obvious.

(d) $\Rightarrow$ (e) If $J\in\mathcal{M}_{q_{\mathsf f}}(K)$, then

$$J=J_{q_{\mathsf f}}=\bigcup_{E\in\mathbb{P}_{\mathsf f}(J)}E_q\supset\bigcup_{E\in\mathbb{P}_{\mathsf f}(J)}E_r=J_r\supset J\quad\text{implies that}\quad J=J_r\in\mathcal{M}_r(K)\,.$$

(e) $\Rightarrow$ (f) $\mathcal{M}_{q,\mathsf f}(K)=\mathcal{M}_{q_{\mathsf f},\mathsf f}(K)\subset\mathcal{M}_{q_{\mathsf f}}(K)\subset\mathcal{M}_r(K)$.

(f) $\Rightarrow$ (g) If $E\in\mathbb{P}_{\mathsf f}(K)$, then $E_q\in\mathsf{M}_{q,\mathsf f}(K)\subset\mathcal{M}_r(K)$, and therefore $E_q=(E_q)_r\supset E_r$. Consequently, if $X\subset K$, then

$$X_r=\bigcup_{E\in\mathbb{P}_{\mathsf f}(X)}E_r\subset\bigcup_{E\in\mathbb{P}_{\mathsf f}(X)}E_q=X_{q_{\mathsf f}}\,,\quad\text{and therefore}\quad r\le q_{\mathsf f}\,.$$

(g) $\Rightarrow$ (a) $r\le q_{\mathsf f}\le q$. $\square$

Theorem 2.3.3. *Let K be a monoid and $D \subset K$ a submonoid.*

1. *Let $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$ be a weak module system on K. Then r is a D-module system if and only if $s(D) \le r$.*
2. *Let r and q be ideal systems of D such that $r \le q$. If D is r-noetherian, then D is q-noetherian.*

PROOF. 1. By definition, r is a D-module system if and only if $\mathcal{M}_r(K) \subset \mathcal{M}_{s(D)}(K)$, and by Theorem 2.3.2 this is equivalent to $s(D) \le r$.

2. If $r \le q$, then $\mathcal{I}_q(D) \subset \mathcal{I}_r(D)$. □

Definition 2.3.4. Let $\varphi\colon K \to L$ be a monoid homomorphism, r a weak module system on K and q a weak module system on L.

1. Let $\varphi^*q\colon \mathbb{P}(K) \to \mathbb{P}(K)$ be defined by $X_{\varphi^*q} = \varphi^{-1}(\varphi(X)_q)$. φ^*q is called the *pullback* of q under φ.
2. φ is called an *(r,q)-homomorphism* if $\varphi(X_r) \subset \varphi(X)_q$ for all subsets $X \subset K$. We denote by $\mathrm{Hom}_{(r,q)}(K,L)$ the set of all (r,q)-homomorphisms $\varphi\colon K \to L$.

Remarks 2.3.5. Let $\varphi\colon K \to L$ and $\psi\colon L \to M$ be monoid homomorphisms, r a weak module system on K, q a weak module system on L and y a weak module system on M.

1. Let r be finitary. Then φ is an (r,q)-homomorphism if and only if $\varphi(E_r) \subset \varphi(E)_r$ for all $E \in \mathbb{P}_{\mathfrak{f}}(K)$.
2. $(\psi\circ\varphi)^*y = \varphi^*(\psi^*y)$.
3. If φ is an (r,q)-homomorphism and ψ is a (q,y)-homomorphism, then $\psi\circ\varphi$ is an (r,y)-homomorphism.

   In particular, monoids together with weak module systems form a category.

Theorem 2.3.6. *Let $\varphi\colon K \to L$ a monoid homomorphism, r a weak module system on K and q a weak module system on L.*

1. *φ^*q is a weak module system on K, $\mathcal{M}_{\varphi^*q}(K) = \{\varphi^{-1}(J) \mid J \in \mathcal{M}_q(L)\}$, and if q is finitary, then φ^*q is also finitary.*

   If $B \subset L$ is a submonoid and q is a weak B-module system, then φ^*q is a weak $\varphi^{-1}(B)$-module system.
2. *φ is an (r,q)-homomorphism if and only if $r \le \varphi^*q$ [that is, if and only if $\varphi^{-1}(J) \in \mathcal{M}_r(K)$ for all $J \in \mathcal{M}_q(L)$].*

PROOF. 1. We check the properties **M1**, **M2** and **M3** for φ^*q. Let $X, Y \subset K$ and $c \in K$.

M1. $X_{\varphi^*q} = \varphi^{-1}(\varphi(X)_q) \supset \varphi^{-1}(\varphi(X) \cup \{0\}) \supset X \cup \{0\}$.

M2. If $X \subset Y_{\varphi^*q} = \varphi^{-1}(\varphi(Y)_q)$, then $\varphi(X) \subset (\varphi(Y)_q$, hence $\varphi(X)_q \subset (\varphi(Y)_q$, and therefore $X_{\varphi^*q} = \varphi^{-1}(\varphi(X)_q) \subset \varphi^{-1}(\varphi(Y)_q) = Y_{\varphi^*q}$.

M3. $\varphi(cX_{\varphi^*q}) = \varphi(c)\varphi(X_{\varphi^*q}) \subset \varphi(c)\varphi(X)_q \subset [\varphi(c)\varphi(X)]_q = \varphi(cX)_q$. Hence it follows that $cX_{\varphi^*q} \subset \varphi^{-1}(\varphi(cX)_q) = (cX)_{\varphi^*q}$.

Let q be finitary and $X \subset K$. Then $\mathbb{P}_{\mathfrak{f}}(X) = \{\varphi(E) \mid E \in \mathbb{P}_{\mathfrak{f}}(X)\}$ and therefore

$$X_{\varphi^*q} = \varphi^{-1}(\varphi(X)_q) = \varphi^{-1}\Big(\bigcup_{E\in\mathbb{P}_{\mathfrak{f}}(X)} \varphi(E)_q\Big) = \Big(\bigcup_{E\in\mathbb{P}_{\mathfrak{f}}(X)} \varphi^{-1}(\varphi(E)_q)\Big) = \bigcup_{E\in\mathbb{P}_{\mathfrak{f}}(X)} E_{\varphi^*q}\,.$$

Hence φ^*q is finitary.

Let $B \subset L$ be a submonoid such that q is a weak B-module homomorphism. If $X \subset K$, then $\varphi^{-1}(B)X_{\varphi^*q} = \varphi^{-1}(B)\varphi^{-1}(\varphi(X)_q) \subset \varphi^{-1}(B\varphi(X)_q) = \varphi^{-1}(\varphi(X)_q) = X_{\varphi^*q}$, and therefore φ^*q is a weak $\varphi^{-1}(B)$-module system.

It remains to prove that $\mathcal{M}_{\varphi^*q}(K) = \{\varphi^{-1}(J) \mid J \in \mathcal{M}_q(L)\}$. If $I \in \mathcal{M}_{\varphi^*q}(K)$, then $\varphi(I)_q \in \mathcal{M}_q(L)$, and $I = I_{\varphi^*q} = \varphi^{-1}(\varphi(I)_q)$. Conversely, if $J \in \mathcal{M}_q(L)$, then

$$\varphi^{-1}(J)_{\varphi^*q} = \varphi^{-1}\big(\varphi(\varphi^{-1}(J))_q\big) \subset \varphi^{-1}(J_q) = \varphi^{-1}(J) \subset \varphi^{-1}(J)_{\varphi^*q}\,.$$

Hence equality holds, and $\varphi^{-1}(J) \in \mathcal{M}_{\varphi^*q}(K)$.

2. If $X \subset K$, then $\varphi(X_r) \subset \varphi(X)_q$ holds if and only if $X_r \subset \varphi^{-1}(\varphi(X)_q) = X_{\varphi^*q}$. Consequently, φ is an (r,q)-homomorphism if and only if $r \le \varphi^*q$. □

Theorem und Definition 2.3.7. *Let $\varepsilon\colon K \to K'$ be a surjective monoid homomorphism, $D \subset K$ a submonoid, $D' = \varepsilon(D)$, and $G \subset D^\times$ a subgroup such that $\varepsilon^{-1}(\varepsilon(x)) = xG$ for all $x \in K$. If $\pi\colon K \to K/G$ denotes the natural epimorphism, defined by $\pi(a) = aG$, then ε factorizes in the form*

$$\varepsilon\colon K \overset{\pi}{\to} K/G \overset{\sim}{\to} K' \quad \text{and induces an isomorphism} \quad D/G \overset{\sim}{\to} D'\,.$$

For a weak D-module system r on K we define

$$\varepsilon(r)\colon \mathbb{P}(K') \to \mathbb{P}(K') \quad \text{by} \quad X'_{\varepsilon(r)} = \varepsilon\big[\varepsilon^{-1}(X')_r\big] \quad \text{for all} \quad X' \subset K'\,.$$

1. *$\varepsilon(r)$ is a weak D'-module system on K'. If $X \subset K$, then $\varepsilon(X)_{\varepsilon(r)} = \varepsilon(X_r)$, and $\varepsilon^*\varepsilon(r) = r$. $\varepsilon(r)$ is a module system if and only if r is an module system, and $\varepsilon(r)_{\mathfrak{f}} = \varepsilon(r_{\mathfrak{f}})$.*

   $\varepsilon(r)$ is called the *weak D'-module system induced by* r. In particular, if $K' = K/G$ and $\varepsilon = \pi$, then $\pi(r)$ is called the *reduction of r* modulo G.
2. *The assignment $r \mapsto \varepsilon(r)$ defines a bijective map from the set of all weak D-module systems on K onto the set of all weak D'-module systems on K'. If r' is a weak D'-module system on K', then ε^*r' is a weak D-module system on K, and $r' = \varepsilon(\varepsilon^*r')$.*
3. *If r is a weak D-module system on K, then the maps*

$$\mathcal{M}_r(K) \to \mathcal{M}_{\varepsilon(r)}(K')\,, \quad J \mapsto \varepsilon(J) \qquad \text{and} \qquad \mathcal{M}_{\varepsilon(r)}(K') \to \mathcal{M}_r(K)\,, \quad J' \mapsto \varepsilon^{-1}(J')$$

   are inclusion-preserving, bijective and inverse to each other. In particular, if r is a weak ideal system of D, then D is r-noetherian if and only if D' is $\varepsilon(r)$-noetherian.

Proof. By definition, ε factors as asserted and induces isomorphisms $K/G \to K'$ and $D/G \to D'$. For every subset $X \subset K$, we have $\varepsilon^{-1}(\varepsilon(X)) = XG$, and $X_r = GX_r = (GX)_r$ [indeed, since r is a D-module system, we have $X_r \subset GX_r \subset (GX)_r \subset (DX)_r = X_r$].

1. If $X \subset K$, then $\varepsilon(X)_{\varepsilon(r)} = \varepsilon\big([\,\varepsilon^{-1}(\varepsilon(X)\,]_r\big) = \varepsilon\,[\,(XG)_r\,] = \varepsilon(X_r)$. We prove that $\varepsilon(r)$ satisfies the properties **M1**, **M2**, **M3** for all $c' \in K'$ and X', $Y' \subset K'$. We may assume that $c' = \varepsilon(c)$, $X' = \varepsilon(X)$ and $Y' = \varepsilon(Y)$, where $c \in K$ and X, $Y \subset K$.

M1. $X'_{\varepsilon(r)} = \varepsilon(X)_{\varepsilon(r)} = \varepsilon(X_r) \supset \varepsilon(X \cup \{0\}) = X' \cup \{0\}$.

M2. If $X' \subset Y_{\varepsilon(r)}$, then $\varepsilon(X) \subset \varepsilon(Y_r)$, hence $X \subset Y_rG = Y_r$, $X_r \subset Y_r$, and therefore we obtain $X'_{\varepsilon(r)} = \varepsilon(X)_r \subset \varepsilon(Y)_r = Y'_{\varepsilon(r)}$.

M3. Since $c'X' = \varepsilon(cX)$, we obtain $(c'X')_{\varepsilon(r)} = \varepsilon\,[\,(cX)_r\,] \supset \varepsilon(cX_r) = \varepsilon(c)\varepsilon(X_r) = c'X'_{\varepsilon(r)}$, and equality holds if and only if $(cX)_rG = cX_rG$, that is, if and only if $(cX)_r = cX_r$.

Hence $\varepsilon(r)$ is a weak module system, it is a module system if and only if r is a module system, and it is a D'-module system since $D'X'_{\varepsilon(r)} = \varepsilon(D)\varepsilon(X_r) = \varepsilon(DX_r) = \varepsilon(X_r) = X'_{\varepsilon(r)}$.

If $X' = \varepsilon(X) \subset K'$, then $\mathbb{P}_{\mathfrak{f}}(X') = \{\varepsilon(E) \mid E \in \mathbb{P}_{\mathfrak{f}}(X)\}$, and therefore

$$X'_{\varepsilon(r)_{\mathfrak{f}}} = \bigcup_{E' \in \mathbb{P}_{\mathfrak{f}}(X')} E'_{\varepsilon(r)} = \bigcup_{E \in \mathbb{P}_{\mathfrak{f}}(X)} \varepsilon(E_r) = \varepsilon\Big(\bigcup_{E \in \mathbb{P}_{\mathfrak{f}}(X)} E_r\Big) = \varepsilon(X_{r_{\mathfrak{f}}}) = X'_{\varepsilon(r_{\mathfrak{f}})}\,,$$

Hence $\varepsilon(r)_{\mathsf{f}} = \varepsilon(r_{\mathsf{f}})$. If $X \subset K$, then $X_{\varepsilon^*\varepsilon(r)} = \varepsilon^{-1}\,[\,\varepsilon(X)_{\varepsilon(r)}\,] = \varepsilon^{-1}\,[\,\varepsilon(X_r)\,] = X_rG = X_r$, and therefore $\varepsilon^*\varepsilon(r) = r$.

2. Since $\varepsilon^*\varepsilon(r) = r$ for every weak D-module system r on K, the assignment $r \mapsto \varepsilon(r)$ defines an injective map from the set of all weak D-module systems on K onto the set of all weak D'-module systems on K'.

Let now r' be a weak D'-module system on K'. Since $D = \varepsilon^{-1}(D')$, Theorem 2.3.6 implies that ε^*r' is a weak D-module system on K, and it suffices to prove that $r' = \varepsilon(\varepsilon^*r')$. If $X' = \varepsilon(X) \subset K'$, then

$$X'_{\varepsilon(\varepsilon^*r')} = \varepsilon(X_{\varepsilon^*r'}) = \varepsilon\,[\,\varepsilon^{-1}(\varepsilon(X)_{r'})\,] = \varepsilon\,[\varepsilon^{-1}(X'_{r'})\,] = X'_{r'}\,.$$

3. Let r be a weak D-module system on K. If $J \in \mathcal{M}_r(K)$, then $\varepsilon(J)_{\varepsilon(r)} = \varepsilon(J_r) = \varepsilon(J)$, hence $\varepsilon(J) \in \mathcal{M}_{\varepsilon(r)}(K')$, and $\varepsilon^{-1}(\varepsilon(J)) = JG = J$. If $J' \in \mathcal{M}_{\varepsilon(r)}(K')$, then $J' = J'_{\varepsilon(r)} = \varepsilon\,[\,\varepsilon^{-1}(J')_r\,]$, and therefore $\varepsilon^{-1}(J') = \varepsilon^{-1}(J')_rG = \varepsilon^{-1}(J')_r$. Hence $\varepsilon^{-1}(J') \in \mathcal{M}_r(K)$, and $J' = \varepsilon(\varepsilon^{-1}(J'))$. $\square$

2.4. Quotient monoids and module systems

Theorem 2.4.1. *Let K be a monoid, $D \subset K$ a submonoid and $T \subset D$ a multiplicatively closed subset. Let $j_T \colon K \to T^{-1}K$ be the natural embedding and r a finitary weak D-module system on K.*

1. *There exists a unique finitary weak $T^{-1}D$-module system $T^{-1}r$ on $T^{-1}K$ such that*
$$j_T(E)_{T^{-1}r} = T^{-1}E_r \quad \textit{for all finite subsets } E \subset K.$$
On finite subsets of $T^{-1}K$ is given by
$$\Big\{\frac{a_1}{t_1},\ldots,\frac{a_m}{t_m}\Big\}_{T^{-1}r} = T^{-1}\{a_1,\ldots,a_m\}_r \quad \textit{for all } m\in\mathbb{N},\ a_1,\ldots,a_m \in K \textit{ and } t_1,\ldots,t_m \in T\,.$$
If r is a weak ideal system of D, then $T^{-1}r$ is a weak ideal system of $T^{-1}D$, and if r is a module system, then $T^{-1}r$ is a module system, too.
2. *If $X \subset K$, then $T^{-1}X_r = (T^{-1}X)_{T^{-1}r} = j_T(X)_{T^{-1}r}$.*
3. *If $V \in \mathcal{M}_{T^{-1}r}(T^{-1}K)$, then $J = j_T^{-1}(V) \in \mathcal{M}_r(K)$, and $V = T^{-1}J$.*
4. *The map*
$$j_T^* \colon \mathcal{M}_r(K) \;\to\; \mathcal{M}_{T^{-1}r}(T^{-1}K)\,, \quad \textit{defined by} \quad j_T^*(J) = T^{-1}J\,,$$
is an inclusion-preserving monoid epimorphism satisfying $j_T^*(\mathcal{M}_{r,\mathsf{f}}(K)) = \mathcal{M}_{T^{-1}r,\mathsf{f}}(T^{-1}K)$ and $T^{-1}(J_1 \cap J_2) = T^{-1}J_1 \cap T^{-1}J_2$ for all $J_1,\,J_2 \in \mathcal{M}_r(K)$.
5. *Let r be a weak ideal system of D. If $V \in \mathcal{I}_{T^{-1}r}(T^{-1}D)$, then $J = j_T^{-1}(V) \cap D \in \mathcal{I}_r(D)$, and $V = T^{-1}J$. In particular, $j_T^*(\mathcal{I}_r(D)) = \mathcal{I}_{T^{-1}r}(T^{-1}D)$, $j_T^*(\mathcal{I}_{r,\mathsf{f}}(D)) = \mathcal{I}_{T^{-1}r,\mathsf{f}}(T^{-1}D)$, and if D is r-noetherian, then $T^{-1}D$ is $T^{-1}r$-noetherian.*

Proof. 1. We prove first:

A. $T^{-1}\{a_1,\ldots,a_m\}_r = T^{-1}\{t_1a_1,\ldots,t_ma_m\}_r$ (for $m \in \mathbb{N}$, $a_1,\ldots,a_m \in K$ and $t_1,\ldots,t_m \in T$).

Proof of **A.** By Theorem 2.1.6.2, $\{t_1a_1,\ldots,t_ma_m\}_r \subset (D\{a_1,\ldots,a_m\})_r = \{a_1,\ldots,a_m\}_r$, which implies $T^{-1}\{t_1a_1,\ldots,t_ma_m\}_r \subset T^{-1}\{a_1,\ldots,a_m\}_r$. To prove the reverse inclusion, let $c \in \{a_1,\ldots,a_m\}_r$ and $t \in T$. Since $t_1\cdot\ldots\cdot t_mc \in t_1\cdot\ldots\cdot t_m\{a_1,\ldots,a_m\}_r \subset (D\{t_1a_1,\ldots,t_ma_m\})_r = \{t_1a_1,\ldots,t_ma_m\}_r$, we obtain

$$\frac{c}{t} = \frac{t_1\cdot\ldots\cdot t_mc}{t_1\cdot\ldots\cdot t_mt} \in T^{-1}\{t_1a_1,\ldots,t_ma_m\}_r\,. \qquad \square[\mathbf{A}.]$$

Now we define a map $T^{-1}r \colon \mathbb{P}_{\mathsf{f}}(T^{-1}K) \to \mathbb{P}(T^{-1}K)$ by

$$\Big\{\frac{a_1}{t_1},\ldots,\frac{a_m}{t_m}\Big\}_{T^{-1}r} = T^{-1}\{a_1,\ldots,a_m\}_r \quad \text{for all } m\in\mathbb{N}_0,\ a_1,\ldots,a_m \in K \text{ and } t_1,\ldots,t_m \in T\,,$$

and we must prove that this assignment does not depend on the choice of representatives. We show that, for all $m \in \mathbb{N}$, $a_1, \ldots, a_m, , a_1', \ldots a_m' \in K$ and $t_1, \ldots, t_m,\, t_1', \ldots, t_m' \in T$,

$$\frac{a_j}{t_j} = \frac{a_j'}{t_j'} \quad \text{for all} \quad j \in [1, m] \quad \text{implies} \quad T^{-1}\{a_1, \ldots, a_m\}_r = T^{-1}\{a_1', \ldots, a_m'\}_r\,.$$

For $j \in [1, m]$, let $s_j \in T$ be such that $s_j t_j' a_j = s_j t_j a_j'$. Then **A** implies

$$T^{-1}\{a_1, \ldots, a_m\}_r = T^{-1}\{st_1'a_1, \ldots, st_m'a_m\} = T^{-1}\{st_1'a_1, \ldots, st_m'a_m\}_r = T^{-1}\{a_1', \ldots, a_m'\}_r\,.$$

We shall prove that $T^{-1}r$ satisfies $\mathbf{M1}_\mathfrak{f}$, $\mathbf{M2}_\mathfrak{f}$, $\mathbf{M3}_\mathfrak{f}$ and $\{c\}_{T^{-1}r} \supset cT^{-1}D$ for all $c \in T^{-1}K$ and $E,\, F \in \mathbb{P}_\mathfrak{f}(T^{-1}K)$, and that equality holds in $\mathbf{M3}_\mathfrak{f}$ if r is a module system.

Once this is done, Theorem 2.2.2 implies the existence of a finitary weak $T^{-1}D$-module system on $T^{-1}K$, again denoted by $T^{-1}r$, such that $j_T(E)_{T^{-1}r} = T^{-1}E_r$ for all $E \in \mathbb{P}_\mathfrak{f}(K)$, and that $T^{-1}r$ is a module system if r is a module system. If r is a weak ideal system of D, then $\{1\}_r = D$, hence $\left\{\frac{1}{1}\right\}_{T^{-1}r} = T^{-1}\{1\}_r = T^{-1}D$, and therefore $T^{-1}r$ is a weak ideal system of $T^{-1}D$.

Assume that

$$E = \left\{\frac{a_1}{t_1}, \ldots, \frac{a_m}{t_m}\right\}, \quad F = \left\{\frac{b_1}{s_1}, \ldots, \frac{b_n}{s_n}\right\} \quad \text{and} \quad c = \frac{a}{t}\,,$$

where $m,\, n \in \mathbb{N}_0$, $a_1, \ldots, a_m,\, b_1, \ldots b_n,\, a \in K$ and $t_1, \ldots, t_m,\, s_1, \ldots, s_n,\, t \in T$.

$\mathbf{M1}_\mathfrak{f}$. For $j \in [1, m]$,

$$\frac{a_j}{t_j} \in T^{-1}\{a_1, \ldots, a_m\} \subset T^{-1}\{a_1, \ldots, a_m\}_r = E_{T^{-1}r} \quad \text{implies} \quad E \subset E_{T^{-1}r}\,,$$

and $0 \in \{a_1, \ldots, a_m\}_r$ implies $\frac{0}{1} \in E_{T^{-1}r}$.

$\mathbf{M2}_\mathfrak{f}$. Suppose that $E \subset F_{T^{-1}r} = T^{-1}\{b_1, \ldots, b_n\}_r$, say

$$\frac{a_j}{t_j} = \frac{c_j}{v_j} \quad \text{for all} \quad j \in [1, m], \text{ where } \ c_j \in \{b_1, \ldots, b_n\}_r \ \text{ and } \ v_j \in T.$$

For $j \in [1, m]$, let $w_j \in T$ be such that $w_j v_j a_j = w_j t_j c_j$. Then $w_j v_j a_j \in \{b_1, \ldots, b_n\}_r$, and therefore $E_{T^{-1}r} = T^{-1}\{a_1, \ldots, a_m\}_r = T^{-1}\{w_1v_1a_1, \ldots, w_mv_ma_m\}_r \subset T^{-1}\{b_1, \ldots, b_n\}_r = F_{T^{-1}r}$.

$\mathbf{M3}_\mathfrak{f}$. We have

$$\begin{aligned}(cE)_{T^{-1}r} = \left\{\frac{aa_1}{tt_1}, \ldots, \frac{aa_m}{tt_m}\right\}_{T^{-1}r} &= T^{-1}\{aa_1, \ldots, aa_m\}_r \\ &\supset T^{-1}a\{a_1, \ldots, a_m\}_r = cT^{-1}\{a_1, \ldots, a_m\}_r = cE_{T^{-1}r}\,,\end{aligned}$$

and equality holds if r is a module system.

Since r is a weak D-module system, it follows that $\{c\}_{T^{-1}r} = T^{-1}\{a\}_r \supset T^{-1}aD \supset cT^{-1}D$.

It remains to prove the uniqueness of $T^{-1}r$. Thus let $\widetilde{r}$ be a finitary weak $T^{-1}D$-module system on $T^{-1}K$ satisfying $j_T(E)_{\widetilde{r}} = T^{-1}E_r$ for all finite subsets $E \subset K$. By Theorem 2.2.1.3 it suffices to prove that $F_{\widetilde{r}} = F_{T^{-1}r}$ for every finite subset $F \subset T^{-1}K$. Thus assume that

$$F = \left\{\frac{a_1}{t_1}, \ldots, \frac{a_m}{t_m}\right\}, \quad \text{where} \quad m \in \mathbb{N}\,, \quad a_1, \ldots, a_m \in K \quad \text{and} \quad t_1, \ldots, t_m \in T\,.$$

If $E = \{a_1, \ldots, a_m\}$, then $(T^{-1}D)F = (T^{-1}D)j_T(E)$, and

$$F_{\widetilde{r}} = ((T^{-1}D)F)_{\widetilde{r}} = ((T^{-1}D)j_T(E))_{\widetilde{r}} = j_T(E)_{\widetilde{r}} = T^{-1}E_r = F_{T^{-1}r}\,.$$

2. Observing that $\mathbb{P}_\mathfrak{f}(j_T(X)) = \{j_T(E) \mid E \in \mathbb{P}_\mathfrak{f}(X)\}$, we obtain

$$T^{-1}X_r = T^{-1}\bigcup_{E \in \mathbb{P}_\mathfrak{f}(X)} E_r = \bigcup_{E \in \mathbb{P}_\mathfrak{f}(X)} T^{-1}E_r = \bigcup_{E \in \mathbb{P}_\mathfrak{f}(X)} j_T(E)_{T^{-1}r} = \bigcup_{F \in \mathbb{P}_\mathfrak{f}(j_T(X))} F_{T^{-1}r} = j_T(X)_{T^{-1}r}\,.$$

Since $(T^{-1}D)(T^{-1}X) = (T^{-1}D)j_T(X)$ and $T^{-1}r$ is a weak $T^{-1}D$-module system, it follows that $(T^{-1}X)_{T^{-1}r} = (j_T(X))_{T^{-1}r}$.

3. Let $V \in \mathcal{M}_{T^{-1}r}(T^{-1}K)$ and $J = j_T^{-1}(V)$. We prove first that $V = T^{-1}J$.

If $\frac{x}{t} \in V$, where $x \in K$ and $t \in T$, then $\frac{x}{1} = \frac{t}{1}\frac{x}{t} \in (T^{-1}D)V = V$, hence $x \in J$ and $\frac{x}{t} \in T^{-1}J$. Conversely, if $x \in J$ and $t \in T$, then $\frac{x}{1} \in V$, $\frac{1}{t} \in T^{-1}D$ and $\frac{x}{t} = \frac{1}{t}\frac{x}{1} \in T^{-1}DV = V$.

It remains to prove that $J \in \mathcal{M}_r(K)$, and for this it suffices to show that $J_r \subset J$. If $a \in J_r$, then $\frac{a}{1} \in T^{-1}J_r = (T^{-1}J)_{T^{-1}r} = V_{T^{-1}r} = V$ and therefore $a \in J$.

4. If $J \in \mathcal{M}_r(K)$, then $(T^{-1}J)_{T^{-1}r} = T^{-1}J_r = T^{-1}J \in \mathcal{M}_{T^{-1}r}(T^{-1}K)$. If $J \in \mathcal{M}_{r,\mathsf{f}}(K)$, then $J = E_r$ for some $E \in \mathbb{P}_\mathsf{f}(K)$, and $T^{-1}J = j_T(E)_{T^{-1}r} \in \mathcal{M}_{r,\mathsf{f}}(T^{-1}K)$. Hence j_T^* is an inclusion-preserving map as asserted, and by 3. it is surjective.

If $V \in \mathcal{M}_{T^{-1}r,\mathsf{f}}(T^{-1}K)$, then $V = \{\frac{a_1}{t_1}, \ldots, \frac{a_m}{t_m}\}_{T^{-1}r}$ for some $m \in \mathbb{N}$, $a_1, \ldots, a_m \in K$ and $t_1, \ldots, t_m \in T$. If $E = \{a_1, \ldots, a_m\} \subset K$, then $V = j_T(E)_{T^{-1}r} \in j_T^*(\mathcal{M}_{r,\mathsf{f}}(K))$.

If $J_1,\, J_2 \in \mathcal{M}_r(K)$, then $T^{-1}(J_1 \cap J_2) = T^{-1}J_1 \cap T^{-1}J_2$, since $TJ_1 = J_1$ and $TJ_2 = J_2$. Moreover, $T^{-1}(J_1 \cdot_r J_2) = T^{-1}(J_1J_2)_r = (T^{-1}J_1J_2)_{T^{-1}r} = ((T^{-1}J_1)(T^{-1}J_2)_{T^{-1}r} = (T^{-1}J_1) \cdot_{T^{-1}r} (T^{-1}J_2)$, and therefore j_T^* is a homomorphism.

5. By 1., $T^{-1}r$ is a weak ideal system of $T^{-1}D$. If $V \in \mathcal{I}_{T^{-1}r}(T^{-1}D)$, then $j_T^{-1}(V) \in \mathcal{M}_r(K)$ by 3., and consequently $J = j_T^{-1}(V) \cap D \in \mathcal{I}_r(D)$. If $a \in J$ and $t \in T$, then $\frac{a}{t} = \frac{a}{1}\frac{1}{t} \in T^{-1}DV = V$, and therefore $T^{-1}J \subset V$. To prove the reverse inclusion, assume that $\frac{a}{t} \in V$, where $a \in D$ and $t \in T$. Then it follows that $\frac{a}{1} = \frac{t}{1}\frac{a}{t} \in V$, hence $a \in j_T(V) \cap D = J$ and $\frac{a}{t} \in T^{-1}J$. If $V \in \mathcal{I}_{T^{-1}r,\mathsf{f}}(T^{-1}D)$, then $V = \{\frac{a_1}{t_1}, \ldots, \frac{a_m}{t_m}\}_{T^{-1}r}$ for some $m \in \mathbb{N}$, $a_1, \ldots, a_m \in D$ and $t_1, \ldots, t_m \in T$. If $E = \{a_1, \ldots, a_m\} \subset D$, then $V = j_T(E)_{T^{-1}r} \in j_T^*(\mathcal{I}_{r,\mathsf{f}}(D))$.

Clearly, $j_T^*(\mathcal{I}_r(D)) \subset \mathcal{I}_{T^{-1}r}(T^{-1}D))$ and $j_T^*(\mathcal{I}_{r,\mathsf{f}}(D)) \subset \mathcal{I}_{T^{-1}r,\mathsf{f}}(T^{-1}D)$, and as we have just proved, equality holds. In particular, if D is r-noetherian, then $\mathcal{I}_r(D) = \mathcal{I}_{r,\mathsf{f}}(D)$, hence $\mathcal{I}_{T^{-r}}(T^{-1}D) = \mathcal{I}_{T^{-r},\mathsf{f}}(T^{-1}D)$, and therefore $T^{-1}D$ is $T^{-1}r$-noetherian. □

Theorem 2.4.2.

1. *Let K be a monoid, $D \subset K$ a submonoid, $s(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$ the semigroup system of D defined on K and $T \subset D$ a multiplicatively closed subset. Then*
$$T^{-1}s(D) = s(T^{-1}D)\colon T^{-1}K \to T^{-1}K$$
is the semigroup system of $T^{-1}D$ defined on $T^{-1}K$.
2. *Let K be a ring, $D \subset K$ a subring, $d(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$ the Dedekind system of D defined on K and $T \subset D$ a multiplicatively closed subset. Then*
$$T^{-1}d(D) = d(T^{-1}D)\colon T^{-1}K \to T^{-1}K$$
is the Dedekind system of $T^{-1}D$ defined on $T^{-1}K$.

PROOF. 1. We prove that $j_T(E)_{s(T^{-1}D)} = T^{-1}E_{s(D)}$ for all $E \in \mathbb{P}_\mathsf{f}(K)$. The the assertion follows from the uniqueness of $T^{-1}s(D)$ in Theorem 2.4.1.

If $E = \{a_1, \ldots, a_m\}$, where $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in K$, then
$$j_T(E)_{s(T^{-1}D)} = \bigcup_{j=1}^m T^{-1}D\,\frac{a_j}{1} = \bigcup_{j=1}^m T^{-1}(Da_j) = T^{-1}\bigcup_{j=1}^m Da_j = T^{-1}E_{s(D)}\,.$$

2. As in 1. it suffices to prove that $j_T(E)_{d(T^{-1}D)} = T^{-1}E_{d(D)}$ for all $E \in \mathbb{P}_\mathsf{f}(K)$.

If $E = \{a_1, \ldots, a_m\}$, where $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in K$, then
$$j_T(E)_{d(T^{-1}D)} = \sum_{j=1}^m T^{-1}D\,\frac{a_j}{1} = \sum_{j=1}^m T^{-1}(Da_j) = T^{-1}\sum_{j=1}^m Da_j = T^{-1}E_{s(D)}\,. \qquad \square$$

2.5. Extension and restriction of module systems

Definition 2.5.1. Let K be a monoid, $D \subset K$ a submonoid and r a weak module system on K. Then we define

$$r[D] \colon \mathbb{P}(K) \to \mathbb{P}(K) \quad \text{by} \quad X_{r[D]} = (XD)_r \quad \text{for all } X \subset K\,,$$

$$r_{(D)} \colon \mathbb{P}(D) \to \mathbb{P}(D) \;\text{ by }\; X_{r_{(D)}} = X_r \cap D \quad \text{for all } X \subset D\,, \quad \text{and we set } \; r_D = r[D]_{(D)} \colon \mathbb{P}(D) \to \mathbb{P}(D)\,.$$

By definition, we have $X_{r_D} = X_{r[D]} \cap D = (XD)_r \cap D$ for all $X \subset D$, and if $T \subset K$ is another submonoid, then $r[D][T] = r[DT]$.

We call $r[D]$ the *extension* of r by D and r_D the *weak ideal system induced by* r *on* D (see Theorem 2.5.2.4).

Theorem 2.5.2. *Let K be a monoid, $D \subset K$ a submonoid and r a weak module system on K.*

1. *$r_{(D)}$ is a weak module system on D. If r is finitary, then $r_{(D)}$ is also finitary, and if r is a weak D-module system, then $r_{(D)}$ is a weak ideal system of D.*
2. *If $D_r = D$, then $r_{(D)} = r \,|\, \mathbb{P}(D)$, and if r is a module system [an ideal system of D], then $r_{(D)}$ is also a module system [an ideal system of D].*
3. *$r[D]$ is a weak D-module system on K. If r is a module system, then $r[D]$ is also a module system, and if r is finitary, then $r[D]$ is also finitary.*
   Moreover, we have $r \le r[D]$, $\mathcal{M}_{r[D]}(K) = \{J \in \mathcal{M}_r(K) \mid DJ = J\}$, and $r = r[D]$ if and only if r is a weak D-module system.
4. *$r_D = r[D]_{(D)}$ is a weak ideal system on D and if $J \in \mathcal{M}_{r[D]}(K)$, then $J \cap D \in \mathcal{I}_{r_D}(D)$.*
   If r is finitary, then r_D is also finitary. If $D_r = D$, then $r[D]$ is an ideal system of D, and $r_D = r[D] \,|\, \mathbb{P}(D)$. In particular, if r is a weak ideal system of D, then $r_D = r \,|\, \mathbb{P}(D)$.

Proof. 1. We check the properties **M1**, **M2**, **M3** for $r_{(D)}$. Let $X, Y \in \mathbb{P}(D)$ and $c \in D$.

M1. $X_{r_{(D)}} = X_r \cap D \supset X \cup \{0\}$.

M2. If $X \subset Y_{r_{(D)}} = Y_r \cap D$, then $X_{r_{(D)}} = X_r \cap D \subset Y_r \cap D = Y_{r_{(D)}}$.

M3. $(cX)_{r_{(D)}} = (cX)_r \cap D \supset cX_r \cap D \supset c(X_r \cap D) = cX_{r_{(D)}}$.

Let r be finitary, $X \subset D$ and $a \in X_{r_{(D)}} = X_r \cap D$. Then there exists some $E \in \mathbb{P}_{\mathsf{f}}(X)$ such that $a \in E_r \cap D = E_{r_{(D)}}$. Hence $r_{(D)}$ is finitary.

If r is a weak D-module system and $X \subset D$, then $DX_{r_{(D)}} = D(X_r \cap D) \subset DX_r \cap D = X_r \cap D = X_{r_{(D)}}$. Hence $r_{(D)}$ is a weak D-module system, and since $D_{r_{(D)}} = D_r \cap D = D$, it is a weak ideal system of D.

2. If $X \subset D$, then $X_r \subset D$ and $X_{r_{(D)}} = X_r \cap D = X_r$. Hence $r_{(D)} = r \,|\, \mathbb{P}(D)$, and if r is a module system [an ideal system of D], then $r_{(D)}$ is also a module system [an ideal system of D].

3. We check the properties **M1**, **M2**, **M3** for $r[D]$. Let $X, Y \in \mathbb{P}(K)$ and $c \in K$.

M1. $X_{r[D]} = (DX)_r \supset DX \cup \{0\} \supset X \cup \{0\}$.

M2. If $X \subset Y_{r[D]} = (DY)_r$, then $DX \subset D(DY)_r \subset (DY)_r$, and $X_{r[D]} = (DX)_r \subset (DY)_r = Y_{r[D]}$.

M3. $(cX)_{r[D]} = (cDX)_r \supset c(DX)_r = cX_{r[D]}$, and equality holds if r is a module system.

Hence $r[D]$ is a weak module system on K, and it is a module system if r is a module system. If r is finitary, $X \subset K$ and $a \in X_{r[D]} = (DX)_r$, then there exists some $E \in \mathbb{P}_{\mathsf{f}}(X)$ such that $a \in (DE)_r = E_{r[D]}$, and therefore $r[D]$ is also finitary.

Next we prove that $\mathcal{M}_{r[D]}(K) = \{J \in \mathcal{M}_r(K) \mid DJ = J\}$. Once this is done, it follows that $r[D]$ is a D-module system, $r \le r[D]$, and $r = r[D]$ if and only if r is a weak D-module system.

If $J \in \mathcal{M}_r(K)$ and $DJ = J$, then $J_{r[D]} = (DJ)_r = J \in \mathcal{M}_{r[D]}(K)$. Conversely, if $J \in \mathcal{M}_{r[D]}(K)$, then $J = J_{r[D]} = (DJ)_r \in \mathcal{M}_r(K)$, and $DJ = (DJ)_r = J_{r[D]} = J$.

4. It suffices to prove that $J \in \mathcal{M}_{r[D]}(K)$ implies $J \cap D \in \mathcal{I}_{r_D}(D)$. The remaining assertions follow by 1., 2. and 3.

If $J \in \mathcal{M}_{r[D]}(K)$, then $DJ = J$ and therefore $(J \cap D)_{r_D} = ((J \cap D)D)_r \cap D \subset (JD)_r \cap D = J \cap D$. Hence $(J \cap D)_{r_D} = J \cap D$ is an r_D-ideal. □

Examples 2.5.3.

1. Let K be a monoid, and let $D \subset T \subset K$ be submonoids. If $s(D) \colon \mathbb{P}(K) \to \mathbb{P}(K)$ is the semigroup system of D defined on K, then $s(D)[T] \colon \mathbb{P}(K) \to \mathbb{P}(K)$ is the semigroup system of T defined on K, and $s(D)_T = s(D) \,|\, \mathbb{P}(T) \colon \mathbb{P}(T) \to \mathbb{P}(T)$ is the semigroup system of D defined on T.

2. Let K be a ring, and let $D \subset T \subset K$ be subrings. If $d(D) \colon \mathbb{P}(K) \to \mathbb{P}(K)$ is the Dedekind system of D defined on K, then $d(D)[T] \colon \mathbb{P}(K) \to \mathbb{P}(K)$ is the Dedekind system of T defined on K, and $d(D)_T = d(D) \,|\, \mathbb{P}(T) \colon \mathbb{P}(T) \to \mathbb{P}(T)$ is the Dedekind system of D defined on T.

Theorem 2.5.4. *Let K be a monoid, $D \subset K$ a submonoid, r a finitary D-module system on K and $T \subset K^\times \cap D$ a multiplicatively closed subset (then $T \subset K^*$ and $T^{-1}D \subset T^{-1}K = K$). Then*

$$T^{-1}r = r[T^{-1}D] \quad \textit{and} \quad r_{T^{-1}D} = T^{-1}r_D\,.$$

In particular:

1. *If $X \subset K$, then $X_{T^{-1}r} = T^{-1}X_r = (T^{-1}X)_r = X_{r[T^{-1}D]}$.*
2. *If $X \subset D$, then $X_{r_{T^{-1}D}} = T^{-1}X_{r_D}$.*

Proof. It suffices to prove 1. and 2. Indeed, 1. implies that $T^{-1}r = r[T^{-1}D]$, and from 2. and the uniqueness of $T^{-1}r_D$ in Theorem 2.4.1 it follows that $r_{T^{-1}D} = T^{-1}r_D$.

1. We start with a preliminary remark. If $Y \subset K$ and $TY = Y$, then

$$T^{-1}Y = \bigcup_{t \in T} t^{-1}Y \quad \text{and} \quad (T^{-1}Y)_r = \bigcup_{t \in T} t^{-1}Y_r\,,$$

since the family $(t^{-1}Y)_{t\in T}$ is directed [indeed, if $t_1,\, t_2 \in Y$, then $t_1^{-1}Y = (t_1t_2)^{-1}(t_2Y) \subset (t_1t_2)^{-1}Y$].

If $X \subset K$, then $TDX = DX$ and $TX_r = X_r$. By the preliminary remark we obtain

$$(T^{-1}X)_r = (T^{-1}DX)_r = \bigcup_{t\in T}(t^{-1}DX)_r = \bigcup_{t \in T} t^{-1}X_r = T^{-1}X_r = X_{T^{-1}r}\,.$$

2. If $X \subset D$, then

$$X_{r_{T^{-1}D}} = X_{r[T^{-1}D]} \cap T^{-1}D = (T^{-1}DX)_r \cap T^{-1}D = T^{-1}X_r \cap T^{-1}D = T^{-1}(X_r \cap D)T^{-1}X_{r_D}\,. \quad \square$$

Theorem 2.5.5. *Let K be a monoid, $D \subset K$ a submonoid, $K = \mathsf{q}(D)$, $r \colon \mathbb{P}(K) \to \mathbb{P}(K)$ a finitary ideal system of D and $T \subset D^*$ a multiplicatively closed subset (then $T \subset K^\times$ and $T^{-1}D \subset T^{-1}K = K$).*

1. $(T^{-1}D)_r = T^{-1}D$, $T^{-1}r \colon \mathbb{P}(K) \to \mathbb{P}(K)$ *is a finitary ideal system of $T^{-1}D$,*

$$\mathcal{F}_{T^{-1}r}(T^{-1}D) = \{a^{-1}T^{-1}I \mid I \in \mathcal{I}_r(D)\,,\ a \in D^*\} = \{T^{-1}J \mid J \in \mathcal{F}_r(D)\}\,,$$

$$\mathcal{F}_{T^{-1}r,\mathsf{f}}(T^{-1}D) = \{a^{-1}T^{-1}I \mid I \in \mathcal{I}_{r,\mathsf{f}}(D)\,,\ a \in D^*\} = \{T^{-1}J \mid J \in \mathcal{F}_{r,\mathsf{f}}(D)\}\,,$$

and the map $j_T^* \colon \mathcal{F}_r(D) \to \mathcal{F}_{T^{-1}r}(T^{-1}D)$, defined by $j_T^*(J) = T^{-1}J$, is a surjective monoid homomorphism satisfying $j_T^*(\mathcal{F}_{r,\mathsf{f}}(D)) = \mathcal{F}_{T^{-1}r,\mathsf{f}}(T^{-1}D)$ and $T^{-1}(J_1 \cap J_2) = T^{-1}J_1 \cap T^{-1}J_2$ for all $J_1,\, J_2 \in \mathcal{F}_r(D)$.

2. *Let D be r-noetherian. If $J \in \mathcal{F}_r(D)$ and $X \subset K$ is D-fractional, then*

$$T^{-1}(J\!:\!X) = (T^{-1}J\!:\!T^{-1}X) = (T^{-1}J\!:\!X)\,.$$

Proof. 1. By Theorem 2.5.4.1, $(T^{-1}D)_r = T^{-1}D_r = T^{-1}D$, and by Theorem 2.4.1 $T^{-1}r$ is a finitary ideal system of $T^{-1}D$. Next we prove that

$$\mathcal{F}_{T^{-1}r}(T^{-1}D) \subset \{a^{-1}T^{-1}I \mid I \in \mathcal{I}_r(D)\,,\ a \in D^*\} \subset \{T^{-1}J \mid J \in \mathcal{F}_r(D)\} \subset \mathcal{F}_{T^{-1}r}(T^{-1}D)$$

and

$$\mathcal{F}_{T^{-1}r,\mathsf{f}}(T^{-1}D) \subset \{a^{-1}T^{-1}I \mid I \in \mathcal{I}_{r,\mathsf{f}}(D)\,,\ a \in D^*\} \subset \{T^{-1}J \mid J \in \mathcal{F}_{r,\mathsf{f}}(D)\} \subset \mathcal{F}_{T^{-1}r,\mathsf{f}}(T^{-1}D)\,.$$

If $V \in \mathcal{F}_{T^{-1}r}(T^{-1}D)$, then Theorem 2.1.4 implies that $V = a_1^{-1}I_1$, where $a_1 \in (T^{-1}D)^*$ and $I_1 \in \mathcal{I}_{T^{-1}r}(T^{-1}D)$. By Theorem 1.2.6, $a_1 = t^{-1}a$ for some $t \in T$ and $a \in D^*$, and by Theorem 2.4.1 $I_1 = T^{-1}I$ for some $I \in \mathcal{I}_r(D)$. Hence we obtain $V = ta^{-1}T^{-1}I = a^{-1}T^{-1}I$. If V is $T^{-1}r$-finitely generated, then I_1 is also $T^{-1}r$-finitely generated and I is r-finitely generated.

If $I \in \mathcal{I}_r(D)$ and $a \in D^*$, then $J = a^{-1}I \in \mathcal{F}_r(D)$ and $a^{-1}T^{-1}I = T^{-1}J$. If I is r-finitely generated, then J is r-finitely generated, too.

If $J \in \mathcal{F}_r(D)$, then $T^{-1}J \in \mathcal{M}_{T^{-1}r}(K)$, and there is some $a \in D^*$ such that $aJ \in \mathcal{I}_r(D)$. Then $T^{-1}aJ = aT^{-1}J \in \mathcal{I}_{T^{-1}r}(T^{-1}D)$, and since $a \in (T^{-1}D)^*$, it follows that $T^{-1}J \in \mathcal{F}_{T^{-1}r}(T^{-1}J)$. If J is r-finitely generated, then $T^{-1}J$ is $T^{-1}r$-finitely generated.

By the above, $j_T^*\colon \mathcal{F}_r(D) \to \mathcal{F}_{T^{-1}r}(T^{-1}D)$ is surjective map and $j_T^*(\mathcal{F}_{r,\mathsf{f}}(D)) = \mathcal{F}_{T^{-1}r,\mathsf{f}}(T^{-1}D)$. The proof of the remaining assertions is literally the same as in Theorem 2.4.1.4.

2. Since X is D-fractional, it follows that $X_r \in \mathcal{F}_r(D) = \mathcal{F}_{r,\mathsf{f}}(D)$, and therefore $X_r = E_r$ for some $E \in \mathbb{P}_{\mathsf{f}}(X)$. Hence $X_{T^{-1}r} = T^{-1}X_r = T^{-1}E_r = E_{T^{-1}r} = (T^{-1}E)_{T^{-1}r}$ and, using Theorems 1.2.4.4 and 2.1.2.9,

$$T^{-1}(J\!:\!X) = T^{-1}(J\!:\!X_r) = T^{-1}(J\!:\!E) = (T^{-1}J\!:\!T^{-1}E) = (T^{-1}J\!:\!(T^{-1}E)_{T^{-1}r}) = (T^{-1}J\!:\!X)\,.$$

Finally, $X_{T^{-1}r} = (T^{-1}X)_{T^{-1}r}$ implies $(T^{-1}J\!:\!X) = (T^{-1}J\!:\!T^{-1}X)$. □

Theorem und Definition 2.5.6. *Let D be a cancellative monoid, $K = \mathsf{q}(D)$ and $r\colon \mathbb{P}(D) \to \mathbb{P}(D)$ a module system on D.*

1. *There exists a unique module system r_∞ on K such that, for all $X \subset K$,*

$$X_{r_\infty} = \begin{cases} K & \text{if } X \text{ is not } D\text{-fractional}\,,\\ a^{-1}(aX)_r & \text{if } a \in D^\bullet \text{ and } aX \subset D\,. \end{cases}$$

   $r_\infty \,|\, \mathbb{P}(D) = r$, and if r is an ideal system of D, then r_∞ is also an ideal system of D. If q is any module system on K such that $q \,|\, \mathbb{P}(D) = r$, then $q \le r_\infty$.

   r_∞ is called the *trivial extension* of r to K.

2. *$(r_\infty)_{\mathsf{f}}$ is the unique finitary module system on K satisfying $(r_\infty)_{\mathsf{f}} \,|\, \mathbb{P}(D) = r_{\mathsf{f}}$. If r_{f} is an ideal system of D, then $(r_\infty)_{\mathsf{f}}$ is also an ideal system of D.*

   $(r_\infty)_{\mathsf{f}}$ is called the *natural extension* of r_{f} to K.

   In particular, for every finitary module system r on D there exists a unique finitary module system $\overline{r}$ on K such that $\overline{r} \,|\, \mathbb{P}(D) = r$.

3. *If $q\colon \mathbb{P}(K) \to \mathbb{P}(K)$ is any finitary ideal system of D, then $q = ((q_D)_\infty)_{\mathsf{f}}$.*

Proof. 1. Uniqueness is obvious. We define r_∞ as in the assertion. Note that this definition does not depend on the choice of $a \in D^\bullet$ with $aX \subset D$. Indeed, if $X \subset K$ and $a_1, a_2 \in D^\bullet$ are such that $a_1X \subset D$ and $a_2X \subset D$, then $a_1a_2X \subset D$ and $(a_1a_2X)_r = a_1(a_2X)_r = a_2(a_1X)_r$ and therefore $a_2^{-1}(a_2X)_r = a_1^{-1}(a_1X)_r$. By definition, $r_\infty \,|\, \mathbb{P}(D) = r$.

We check the conditions **M1**, **M2**, **M3** for r_∞. Let $X, Y \subset K$ and $c = b^{-1}d \in K$, where $b \in D^\bullet$ and $d \in D$.

M1. If X is not D-fractional, then $X_{r_\infty} = K \supset X \cup \{0\}$. If $a \in D^\bullet$ is such that $aX \subset D$, then $X_{r_\infty} = a^{-1}(aX)_r \supset a^{-1}(aX) \cup \{0\} = X \cup \{0\}$.

M2. Suppose that $X \subset Y_{r_\infty}$. If Y is not D-fractional, then $Y_{r_\infty} = K \supset X_{r_\infty}$. Thus let $a \in D^\bullet$ be such that $aY \subset D$. Then $X \subset Y_{r_\infty} = a^{-1}(aY)_r$, hence $aX \subset (aY)_r \subset D$, and therefore it follows that $X_{r_\infty} = a^{-1}(aX)_r \subset a^{-1}(aY)_r = Y_{r_\infty}$.

M3. We may assume that $c \neq 0$, hence $c \in K^\times$ and $d \in D^\bullet$. If X is not D-fractional, then (by Lemma 1.4.2) also cX is not D-fractional, and $(cX)_{r_\infty} = K = cK = cX_{r_\infty}$.

Thus assume that $aX \subset D$ for some $a \in D^\bullet$. Then $ab(cX) = adX \subset dD \subset D$ and therefore $(cX)_{r_\infty} = (ab)^{-1}(abcX)_r = (ab)^{-1}bc(aX)_r = ca^{-1}(aX)_r = cX_{r_\infty}$.

Let q be any module system on K such that $q \,|\, \mathbb{P}(D) = r$ and $X \subset K$. If $a \in D^\bullet$ is such that $aX \subset D$, then $X_q = a^{-1}(aX)_q = a^{-1}(aX)_r = X_{r_\infty}$, and if X is not D-fractional, then $X_q \subset K = X_{r_\infty}$. Hence it follows that $q \le r_\infty$.

Let r be an ideal system of D. Since $D_{r_\infty} = D_r = D$ and $DX_{r_\infty} = X_{r_\infty}$ for all $X \in \mathbb{P}(K)$, it follows that r_∞ is also an ideal system of D.

2. By definition, $(r_\infty)_\mathfrak{f}$ is a finitary module system on K. If $X \subset D$, then

$$X_{(r_\infty)_\mathfrak{f}} = \bigcup_{E \in \mathbb{P}_\mathfrak{f}(X)} E_{r_\infty} = \bigcup_{E \in \mathbb{P}_\mathfrak{f}(X)} E_r = X_{r_\mathfrak{f}}\,, \quad \text{and therefore} \quad (r_\infty)_\mathfrak{f} \,|\, \mathbb{P}(D) = r_\mathfrak{f}\,.$$

Let now $r_\mathfrak{f}$ be an ideal system of D and $X \subset K$. For $E \in \mathbb{P}_\mathfrak{f}(X)$, let $a \in D^\bullet$ be such that $aE \subset D$. Then $DE_{r_\infty} = Da^{-1}(aE)_r = Da^{-1}(aE)_{r_\mathfrak{f}} = a^{-1}(aE)_{r_\mathfrak{f}} = a^{-1}(aE)_r = E_{r_\infty}$, and

$$DX_{(r_\infty)_\mathfrak{f}} = \bigcup_{E \in \mathbb{P}_\mathfrak{f}(X)} DE_{r_\infty} = \bigcup_{E \in \mathbb{P}_\mathfrak{f}(X)} E_{r_\infty} = X_{(r_\infty)_\mathfrak{f}}\,.$$

Hence $(r_\infty)_\mathfrak{f}$ is an ideal system of D.

To prove uniqueness of $r_\mathfrak{f}$, let $\widetilde{r}$ be a finitary module system on K such that $\widetilde{r} \,|\, \mathbb{P}_\mathfrak{f}(D) = r_\mathfrak{f}$. We must prove that $E_{\widetilde{r}} = E_{(r_\infty)_\mathfrak{f}}$ for all $E \in \mathbb{P}_\mathfrak{f}(K)$. If $E \in \mathbb{P}_\mathfrak{f}(K)$, let $c \in D^\bullet$ be such that $cE \subset D$. Then $E_{\widetilde{r}} = c^{-1}(cE)_{\widetilde{r}} = c^{-1}(cE)_{r_\mathfrak{f}} = c^{-1}(cE)_r = E_{r_\infty} = E_{(r_\infty)_\mathfrak{f}}$.

3. Let $q\colon \mathbb{P}(K) \to \mathbb{P}(K)$ be a finitary ideal system of D. Then $q_D = q \,|\, \mathbb{P}(D)$ by Theorem 2.5.2.4, and it suffices to prove that $E_{((q_D)_\infty)_\mathfrak{f}} = E_q$ for all $E \in \mathbb{P}_\mathfrak{f}(K)$. If $E \in \mathbb{P}_\mathfrak{f}(K)$ and $a \in D^\bullet$ is such that $aE \subset K$, then $E_{((q_D)_\infty)_\mathfrak{f}} = E_{(q_D)_\infty} = a^{-1}(aE)_{q_D} = a^{-1}(aE)_q = E_q$. □

Example 2.5.7. Let D be a domain, $K = \mathsf{q}(D)$ and $\overline{\mathcal{F}}(D) = \mathcal{M}_{d(D)}(K)^\bullet$ the set of all non-zero D-submodules of K.

A *semistar operation* of D is a map $*\colon \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$, $M \mapsto M^*$, such that, for all $c \in K$ and $M,\, N \in \overline{\mathcal{F}}(D)$, the following conditions are satisfied:

$$*\mathbf{1}.\quad M \subset M^*\,; \qquad *\mathbf{2}.\quad M \subset N^* \text{ implies } M^* \subset N^*\,; \qquad *\mathbf{3}.\quad cM^* = (cM)^*\,.$$

If moreover $D^* = D$, then $*$ is called a *(semi)star operation*, and the restriction $* \,|\, \mathcal{F}(D)$ is called a *star operation*.

Let $*$ be a semistar operation of D, and define $r_*\colon \mathbb{P}(K) \to \mathbb{P}(K)$, $X \mapsto X_{r_*}$, by

$$X_{r_*} = \begin{cases} \{0\} & \text{if } \; X \subset \{0\}\,, \\ {}_D(X)^* & \text{if } \; X \not\subset \{0\}\,. \end{cases}$$

Then r_* is a D-module system on K, $d(D) \le r_*$ and $D_{r_*} = D^*$. Hence r_* is an ideal system of D if and only if $*$ is a (semi)star operation.

Conversely, let $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$ be a D-module system on K and $d(D) \le r$. Then $\mathcal{M}_r(K)^\bullet \subset \overline{\mathcal{F}}(D)$, and we define $*_r\colon \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$ by $M^{*_r} = M_r$. Then $*_r$ is a semistar operation, $r_{*_r} = r$, and for every semistar operation $*$ of D we have $*_{r_*} = *$.

2.6. The ideal systems v and t

Throughout this section, let D be a cancellative monoid, $K = \mathsf{q}(D)$, and for $X \subset K$, let $X^{-1} = (D\colon X)$.

Definition 2.6.1. If $D \ne K$, we define $v = v(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$ by $X_v = (X^{-1})^{-1}$ for all $X \subset D$, and if $D = K$, we set $v(K) = s(K)\colon \mathbb{P}(K) \to \mathbb{P}(K)$. We shall see in Theorem 2.6.2 that $v(D)$ is an ideal system of D, and we define $t = t(D) = v(D)_{\mathsf{f}}\colon \mathbb{P}(K) \to \mathbb{P}(K)$.

$v(D)$ is called the *divisorial system* and $t(D)$ is called the *total system* of D defined on K.

Theorem 2.6.2. *Assume that $D \ne K$, and set $v = v(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$.*

1. *If $X \subset K$, then*
   - $X^{-1} = K$ *if and only if* $X^\bullet = \emptyset$,
   - $(X^{-1})^\bullet \ne \emptyset$ *if and only if X is D-fractional,*
   - $X_v = K$ *if and only if X is not D-fractional,*
   - $X_v = \{0\}$ *if and only if* $X \subset \{0\}$.

   In any case, we have
$$X_v = \bigcap_{\substack{z\in K\\ X\subset zD}} zD\,. \tag{$*$}$$
2. *If $X \subset K$, then $X \cup \{0\} \subset X_v$, $X_v^{-1} = X^{-1} = (X^{-1})_v$, and $(XX^{-1})^{-1} = (X^{-1}\colon X^{-1})$.*
3. *v is a ideal system of D, $\mathcal{M}_v(K) = \{X^{-1} \mid X \subset K\}$, and $(v_D)_\infty = v$. If q is any ideal system of D defined on K, then $q \le v$.*
4. *The system $t = t(D) = v(D)_{\mathsf{f}}\colon \mathbb{P}(K) \to \mathbb{P}(K)$ is a finitary ideal system of D. If q is any finitary ideal system of D defined on K, then $q \le t$.*
5. *Let D' be another cancellative monoid, $K' = \mathsf{q}(D')$, $v' = v(D')$ and $t' = t(D')$. Let $\varepsilon\colon K \to K'$ be a surjective monoid homomorsphism, $D' = \varepsilon(D)$, and let $G \subset D^\times$ be a subgroup such that $\varepsilon^{-1}(\varepsilon(x)) = xG$ for all $x \in K$. Then we have $\varepsilon(X)^{-1} = \varepsilon(X^{-1})$ for all subsets $X \subset K$, $v' = \varepsilon(v)$ and $t' = \varepsilon(t)$.*

Proof. 1. Let $X \subset K$.

If $X^\bullet = \emptyset$, then $KX = X$ and $X^{-1} = K$. If $z \in X^\bullet$, then $zK = K \ne D$, and therefore $X^{-1} \ne K$. By definition, $(X^{-1})^\bullet \ne \emptyset$ if and only if X is D-fractional. Therefore we obtain $X_v = (X^{-1})^{-1} = K$ if and only if $(X^{-1})^\bullet = \emptyset$, that is, if and only if X is not D-fractional. Similarly, $X_v = (X^{-1})^{-1} \subset \{0\}$ if and only if X^{-1} is not D-fractional which holds if and only if $X^\bullet = \emptyset$.

It remains to prove $(*)$. If $X \subset \{0\}$, $(*)$ holds by Theorem 1.2.8. Thus assume that $X \ne \{0\}$. Since $(X^{-1})^\bullet = \{y \in K^\times \mid yX \subset D\} = \{z^{-1} \mid z \in K^\times,\ X \subset zD\}$, we obtain
$$X_v = (D\colon X^{-1}) = (D\colon (X^{-1})^\bullet) = \bigcap_{y\in (X^{-1})^\bullet} y^{-1}D = \bigcap_{\substack{z\in K^\times\\ X\subset zD}} zD = \bigcap_{\substack{z\in K\\ X\subset zD}} zD\,.$$

2. If $X \subset K$, then $(X\cup\{0\})X^{-1} \subset D$ implies that $X \cup \{0\} \subset (X^{-1})^{-1} = X_v$. Hence we obtain $X_v^{-1} \subset X^{-1} \subset (X^{-1})_v = [(X^{-1})^{-1}]^{-1} = X_v^{-1}$, and thus $X_v^{-1} = X^{-1} = (X^{-1})_v$. Finally,
$$(X^{-1}\colon X^{-1}) = ((D\colon X)\colon X^{-1}) = (D\colon XX^{-1}) = (XX^{-1})^{-1}\,.$$

3. We verify the conditions **M1**, **M2** and **M3**. Let $X, Y \subset K$ and $c \in K$.

M1. By 1.

M2. If $X \subset Y_v$, then $Y^{-1} = Y_v^{-1} \subset X^{-1}$, and therefore $X_v = (X^{-1})^{-1} \subset (Y^{-1})^{-1} = Y_v$.

M3. We may assume that $c \ne 0$. Then $cX_v = c(X^{-1})^{-1} = (c^{-1}X^{-1})^{-1} = ((cX)^{-1})^{-1} = (cX)_v$.

If $c \in D$ and $X \subset K$, then $cX_v \subset X_v$ by $(*)$. Hence v is a D-module system, and since $D_v = D$ it is even an ideal system of D. In particular, $v_D = v \,|\, \mathbb{P}(D)$, and if $X \subset K$ is not D-fractional, then $X_v = K = X_{(v_D)_\infty}$. Hence it follows that $v = (v_D)_\infty$.

If $X \in \mathcal{M}_v(K)$, then $X_v = (X^{-1})^{-1}$, and if $X \subset K$, then $(X^{-1})_v = X^{-1}$. Hence we obtain $\mathcal{M}_v(K) = \{X^{-1} \mid X \subset K\}$.

Let q be any ideal system of D defined on K and $X \subset K$. If $z \in K$ is such that $X \subset zD$, then $X_q \subset zD$, and therefore $X_q \subset X_v$ by $(*)$. Hence $q \le v$.

4. By Theorem 2.2.2, t is a finitary ideal system of D. If q is any finitary ideal system of D defined on K, then $q \le v$ by 3., and therefore $q = q_{\mathsf{f}} \le v_{\mathsf{f}} = t$.

5. If $X \subset K$ and $x' = \varepsilon(x) \in K'$, then $x'\varepsilon(X) = \varepsilon(xX) \subset D' = \varepsilon(D)$ if and only if $xX \subset D$. Hence we obtain $\varepsilon(X)^{-1} = \varepsilon(X^{-1})$, and $\varepsilon(X)_{v'} = (\varepsilon(X)^{-1})^{-1} = \varepsilon((X^{-1})^{-1}) = \varepsilon(X_v) = \varepsilon(X)_{\varepsilon(v)}$. Consequently, $v' = \varepsilon(v)$, and by Theorem 2.3.7 it follows that $\varepsilon(t) = \varepsilon(v_{\mathsf{f}}) = \varepsilon(v)_{\mathsf{f}} = v'_{\mathsf{f}} = t'$. □

Theorem 2.6.3. *Let $v = v(D) \colon \mathbb{P}(K) \to \mathbb{P}(K)$, $X \subset D$ and $a,\, d \in D$.*

1. *If $X_v = dD$, then $\mathrm{GCD}(X) = dD^\times$.*
2. *If $\mathrm{GCD}(X) = dD^\times$ and $\mathrm{GCD}(bX) \ne \emptyset$ for all $b \in D$, then $X_v = dD$.*
3. *The following assertions are equivalent:*
   (a) $\mathrm{GCD}(X) \ne \emptyset$ *for all* $X \in \mathbb{P}(D)$.
   (b) *Every (fractional) v-ideal of D is principal.*
4. *If D is a GCD-monoid, $X \subset D$ and $d \in D$, then*

$$X_v = \bigcap_{\substack{a \in D \\ X \subset aD}} aD\,,$$

and $X_v = dD$ if and only if $d \in \mathrm{GCD}(X)$.

Proof. We may assume that $D \ne K$.

1. If $X_v = dD$, then $X \subset dD$, and if $b \in D$ is such that $X \subset bD$, then $dD = X_v \subset bD$. Hence dD is the smallest principal ideal containing X, and $dD^\times = \mathrm{GCD}(X)$.

2. If $\mathrm{GCD}(X) = dD^\times$, then $X \subset dD$, and therefore

$$X_v = \bigcap_{\substack{z \in K \\ X \subset zD}} zD \subset dD\,.$$

Hence it suffices to prove that, for all $z \in K$, $X \subset zD$ implies $dD \subset zD$. Thus suppose that $z = b^{-1}c \in K$, where $b \in D^\bullet$ and $c \in D$, and $X \subset zD$. Then $bX \subset cD$, and since $\mathrm{GCD}(bX) \ne \emptyset$, it follows that $\mathrm{GCD}(bX) = bdD^\times$. Therefore we obtain $bdD \subset cD$, and $dD \subset b^{-1}cD = zD$.

3. Obvious by 1. and 2.

4. Clearly,

$$\overline{X} = \bigcap_{\substack{a \in D \\ X \subset aD}} aD \supset \bigcap_{\substack{z \in K \\ X \subset zD}} zD = X_v\,.$$

To prove the converse, suppose that $x \in \overline{X} \subset D$, and let $z \in K$ be such that $X \subset zD$. Then $z = a^{-1}b$, where $a \in D^\bullet$, $b \in D$, $\mathrm{GCD}(a,b) = D^\times$, and it suffices to prove that $X \subset bD$. If $x \in X$, then $x = zc$ for some $c \in D$, hence $ax = bc$, and since a is coprime to b, it follows that $a \,|\, c$, say $c = ad$ for some $d \in D$. But then $x = bd \in bD$.

By 1. and 2. it follows that $X_v = dD$ if and only if $d \in \mathrm{GCD}(X)$. □

Theorem 2.6.4. *Let $v = v(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$ and $t = t(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$.*

1. *D is a* GCD*-monoid if and only if every v-finitely generated v-ideal is principal [equivalently, every t-finitely generated t-ideal is principal].*

   In this case, $\mathcal{F}_{t,\mathsf{f}}(D)^\bullet = \mathcal{F}_{v,\mathsf{f}}(D)^\bullet = \{aD \mid a \in K^\times\} \cong K^\times/D^\times$ is a group.
2. *D is factorial if and only if every t-ideal of D is principal.*

   In this case, $\mathcal{F}_t(D)^\bullet = \{aD \mid a \in K^\times\} \cong K^\times/D^\times$ is a group.
3. *Let D' be another cancellative monoid, $\varepsilon\colon D \to D'$ a surjective monoid homomorphism and $G \subset D^\times$ a subgroup such that $\varepsilon^{-1}(\varepsilon(x) = xG$ for all $x \in D$. Then D' is factorial [a* GCD*-monoid] if and only if D is factorial [a* GCD*-monoid].*

Proof. 1. Let D be a GCD-monoid and $J \in \mathcal{I}_{v,\mathsf{f}}(D)$. Then $J = E_v$ for some $E \in \mathbb{P}_\mathsf{f}(D)$, and if $d \in \mathrm{GCD}(E)$, then $J = E_v = dD$ by Theorem 2.6.3.2.

Conversely, if every v-finitely generated v-ideal is principal and $E \in \mathbb{P}_\mathsf{f}(D)$, then $E_v = dD$ for some $d \in D$, and then $d \in \mathrm{GCD}(E)$ by Theorem 1.5.2.1.

2. Let D be factorial. Then Theorem 1.5.6.3 implies that $\mathrm{GCD}(X) \neq \emptyset$ for every subset $X \subset D$. If $J \in \mathcal{I}_t(D)^\bullet$ and $d \in \mathrm{GCD}(J)$, then $J = dD$ by Theorem 2.6.3.2.

Conversely, assume that every t-ideal is principal. Then D is t-noetherian, and as every principal ideal is a t-ideal, it satisfies the ACCP. By 1., D is a GCD-monoid, and by Theorem 1.5.5, it is an atomic GCD-monoid and thus it is factorial by Theorem 1.5.6.4.

3. Let $\varepsilon\colon K \to K'$ be the extension of ε to the quotient monoids and $t' = t(D')$. By the Theorems 2.6.2 and 2.3.7 we have $\varepsilon(t) = t'$, $\varepsilon(X)_{t'} = \varepsilon(X_t)$ for all subsets $X \subset D$, and $J \mapsto \varepsilon(J)$ defines a bijective map $\mathcal{I}_t(D) \to \mathcal{I}_{t'}(D')$. Hence every [t-finitely generated] t-ideal of D is principal if and only if every [t'-finitely generated] t'-ideal of D' is principal, and the assertion follows by 1. and 2. □

Theorem 2.6.5. *For $i \in \{1,2\}$, let D_i be a* GCD*-monoid, $K_i = \mathsf{q}(D_i)$, $t_i = t(D_i)\colon \mathbb{P}(K_i) \to \mathbb{P}(K_i)$, and let $\varphi\colon K_1 \to K_2$ be a monoid homomorphism. Then φ is a (t_1,t_2)-homomorphism if and only if $\varphi(D_1) \subset D_2$ and $\varphi \,|\, D_1\colon D_1 \to D_2$ is a* GCD*-homomorphism. In particular, there is a bijective map*

$$\mathrm{Hom}_{(t_1,t_2)}(K_1,K_2) \;\to\; \mathrm{Hom}_{\mathrm{GCD}}(D_1,D_2)\,, \quad \textit{given by} \quad \varphi \;\mapsto\; \varphi \,|\, D_1\,.$$

Proof. Let first φ be a (t_1,t_2)-homomorphism. Then

$$\varphi(D_1) = \varphi(\{1_{D_1}\}_{t_1}) \subset \{\varphi(1_{D_1})\}_{t_2} = \{1_{D_2}\}_{t_2} = D_2\,.$$

Let $E \subset D_1$ be finite and $d \in \mathrm{GCD}(E)$. Then $E_{t_1} = dD_1$ and $\varphi(d) \in \varphi(E_{t_1}) \subset \varphi(E)_{t_2} = d'D_2$, where $d' \in \mathrm{GCD}(\varphi(E))$. Since $E \subset dD_1$, it follows that $\varphi(E) \subset \varphi(d)D_2$, hence $d'D_2 \subset \varphi(d)D_2$, and since $\varphi(d) \in d'D_2$, we obtain $\varphi(d) \in d'D_2^\times = \mathrm{GCD}(\varphi(E))$.

Assume now that $\varphi(D_1) \subset D_2$, and let $\varphi \,|\, D_1\colon D_1 \to D_2$ be a GCD-homomorphism. We must prove that $\varphi(E_{t_1}) \subset \varphi(E)_{t_2}$ for all $E \in \mathbb{P}_\mathsf{f}(K_1)$. If $E \in \mathbb{P}_\mathsf{f}(K_1)$ and $c \in D_1^\bullet$ such that $cE \subset D_1$. If $d \in \mathrm{GCD}(cE)$, then $\varphi(d) \in \mathrm{GCD}(\varphi(c)\varphi(E))$ and therefore

$$\varphi(E_{t_1}) = \varphi(c^{-1}(cE)_{t_1}) = \varphi(c)^{-1}\varphi(dD_1) \subset \varphi(c)^{-1}\varphi(d)D_2 = \varphi(c)^{-1}(\varphi(c)\varphi(E))_{t_2} = \varphi(E)_{t_2}\,. \qquad \square$$

Theorem und Definition 2.6.6. *Let $v = v(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$ and $t = t(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$.*

1. *The following assertions are equivalent:*
   (a) *D is v-noetherian.*
   (b) *D is t-noetherian.*

(c) *For every sequence* $(J_n)_{n\ge0}$ *in* $\mathcal{F}_v(D)$ *such that*

$$J_n \supset J_{n+1} \quad \text{for all } n \ge 0, \quad \text{and} \quad \Big(\bigcap_{n\ge0} J_n\Big)^\bullet \ne \emptyset\,,$$

there exists some $m \ge 0$ *such that* $J_n = J_m$ *for all* $n \ge m$.

(d) *Every non-empty subset* $\Omega \subset \mathcal{F}_v(D)$ *satisfying*

$$\Big(\bigcap_{J\in\Omega} J\Big)^\bullet \ne \emptyset$$

possesses a minimal element (with respect to inclusion).

(e) *For every subset* $X \subset D$ *there exists some* $E \in \mathbb{P}_{\mathsf{f}}(X)$ *such that* $X^{-1} = E^{-1} \subset K$.

If these conditions are satisfied, then D is called a *Mori monoid.*

In particular, if D *is a Mori monoid, then* $X_v = X_t$ *for every* D*-fractional subset* $X \subset K$, $\mathcal{F}_v(D) = \mathcal{F}_t(D)$ *and* $\mathcal{I}_v(D) = \mathcal{I}_t(D)$.

2. *Let* D *be a Mori monoid and* $T \subset D$ *be a multiplicatively closed subset.*
   (a) $T^{-1}D$ *is a Mori monoid, and* $t(T^{-1}D) = T^{-1}t \colon \mathbb{P}(K) \to \mathbb{P}(K)$.
   (b) *If* $X \subset K$ *is* D*-fractional, then* $T^{-1}(D:X) = (T^{-1}D : T^{-1}X) = (T^{-1}D : X)$*, and* $T^{-1}X_v = (T^{-1}X)_{v(T^{-1}D)} = X_{v(T^{-1}D)}$.
   (c) *Let* $P \subset D$ *be a prime ideal such that* $P \cap T = \emptyset$. *Then* $P \in v\text{-spec}(D)$ *if and only if* $T^{-1}P \in v(T^{-1}D)\text{-spec}(T^{-1}D)$.
3. *Let* $C \in \mathcal{F}_t(D)$ *be an overmonoid of* D. *Then* $\mathcal{F}_{t(C)}(C) \subset \mathcal{F}_t(D)$. *In particular, if* D *is a Mori monoid, then* C *is also a Mori monoid.*

Proof. We may assume that $D \ne K$.

1. (a) $\Leftrightarrow$ (b) By Theorem 2.2.5.3, since $t = v_{\mathsf{f}}$. In particular, it follows that $v\,|\,\mathbb{P}(D) = t\,|\,\mathbb{P}(D)$, and therefore $X_v = X_t$ for every D-fractional subset $X \subset K$, $\mathcal{F}_v(D) = \mathcal{F}_t(D)$ and $\mathcal{I}_v(D) = \mathcal{I}_t(D)$.

(b) $\Rightarrow$ (c) Let $(J_n)_{n\ge0}$ be a sequence in $\mathcal{F}_v(D)$ such that $J_n \supset J_{n+1}$ for all $n \ge 0$, and let $c \in K^\times$ be such that $c \in J_n$ for all $n \ge 0$. Then $(cJ_n^{-1})_{n\ge0}$ is an ascending sequence in $\mathcal{I}_v(D)$. Hence it becomes stationary, and therefore the sequence $(J_n)_{n\ge0}$ becomes stationary, too.

(c) $\Rightarrow$ (d) Assume to the contrary that there exists a subset $\emptyset \ne \Omega \subset \mathcal{F}_v(D)$ without a smallest element, and that there is some $c \in K^\bullet$ such that $c \in J$ for all $J \in \Omega$. Consequently, for every $J \in \Omega$ there exists some $J' \in \Omega$ such that $J' \subsetneq J$. If $J_0 \in \Omega$ is arbitrary and $(J_n)_{n\ge0}$ is recursively defined by $J_{n+1} = J_n'$ for all $n \ge 0$, then the sequence $(J_n)_{n\ge0}$ contradicts (c).

(d) $\Rightarrow$ (e) If $X \subset D$ and $X^\bullet = \emptyset$, we set $E = X$. Thus assume that $X \subset D$, $X^\bullet \ne \emptyset$, and set $\Omega = \{F^{-1} \mid F \in \mathbb{P}_{\mathsf{f}}(X),\ F^\bullet \ne \emptyset\}$. Then $\Omega \ne \emptyset$, and if $F \in \mathbb{P}_{\mathsf{f}}(X)$ and $F^\bullet \ne \emptyset$, then $F^{-1} \in \mathcal{F}_v(D)$ and $1 \in F^{-1}$. Thus by (d) there exists some $E \in \mathbb{P}_{\mathsf{f}}(X)$ such that $E^\bullet \ne \emptyset$ and E^{-1} is minimal in Ω. Clearly, $X^{-1} \subset E^{-1}$, and we assert that equality holds. Indeed, suppose to the contrary that there is some $c \in E^{-1} \setminus X^{-1}$, and let $a \in X$ be such that $ca \notin D$. Then $(E \cup \{a\})^{-1} \in \Omega$, $c \notin (E \cup \{a\})^{-1}$ and therefore $(E \cup \{a\})^{-1} \subsetneq E^{-1}$, a contradiction.

(e) $\Rightarrow$ (a) If $X \subset D$, there exists some $E \in \mathbb{P}_{\mathsf{f}}(X)$ such that $E^{-1} = X^{-1}$ and thus $E_v = X_v$. Hence D is v-noetherian.

2. (a), (b) By Theorem 2.4.1.5 $T^{-1}D$ is $T^{-1}t$-noetherian, and thus it is a Mori monoid. If $X \subset K$ is D-fractional, then $T^{-1}(D{:}X) = (T^{-1}D{:}T^{-1}X) = (T^{-1}D{:}X)$ by Theorem 2.5.5.2, and therefore

$$\begin{aligned} T^{-1}X_v = T^{-1}(D{:}(D{:}X)) &= (T^{-1}D{:}(T^{-1}D{:}T^{-1}X)) = (T^{-1}X)_{v(T^{-1}D)} \\ &= (T^{-1}D{:}(T^{-1}D{:}X)) = X_{v(T^{-1}D)}\,. \end{aligned}$$

In particular, if $E \in \mathbb{P}_{\mathsf{f}}(K)$, then $E_{T^{-1}t} = T^{-1}E_t = T^{-1}E_v = E_{v(T^{-1}D)} = E_{t(T^{-1}D)}$, and therefore $T^{-1}t = t(T^{-1}D)$.

(c) If $P \in v$-spec(D), then $(T^{-1}P)_{v(T^{-1}D)} = T^{-1}P_v = T^{-1}P \in v(T^{-1}D)$-spec$(T^{-1}D)$. Conversely, if $T^{-1}P \in v(T^{-1}D)$-spec$(T^{-1}D) = t(T^{-1}D)$-spec$(T^{-1}D)$, then $t \le t(T^{-1}D)$ implies $(T^{-1}P)_t = T^{-1}P$, hence $P_t = (T^{-1}P \cap D)_t = T^{-1}P \cap D = P$, and consequently $P \in t$-spec$(D) = v$-spec(D).

3. Since $t[C]$ is an ideal system of C, it follows that $t \le t[C] \le t(C)$, and therefore we obtain $\mathcal{F}_{t(C)}(C) \subset \mathcal{M}_{t(C)}(K) \subset \mathcal{M}_t(K)$. By Theorem 1.4.2.6 every C-fractional subset of K is D-fractional, and therefore it follows that $\mathcal{F}_{t(C)}(C) \subset \mathcal{F}_t(D)$. □

CHAPTER 3

Prime Ideals and Valuation Monoids

Throughout this chapter, let D be a monoid, $K = \mathsf{q}(D)$, $s = s(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$, and if D is cancellative, then $v = v(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$ and $t = t(D)\colon \mathbb{P}(K) \to \mathbb{P}(K)$.

3.1. Prime ideals and Krull's Theorem

Definition 3.1.1. Let r be a weak ideal system of D.

1. We denote by
   - r-spec$(D) \subset \mathcal{I}_r(D)$ the set of all prime r-ideals (in particular, s-spec(D) is the set of all prime ideals of D);
   - $\mathfrak{X}(D)$ the set of all minimal non-zero prime ideals of D;
   - r-max(D) the set of all maximal elements of $\mathcal{I}_r(D) \setminus \{D\}$ (they are called r-maximal r-ideals).
2. An r-ideal $Q \in \mathcal{I}_r(D)$ is called *r-irreducible* if $Q \ne D$ and, for all $I, J \in \mathcal{I}_r(D)$, $Q = I \cap J$ implies $Q = I$ or $Q = J$.
3. D is called r-local if $|r\text{-max}(D)| = 1$.

   If $D \setminus D^\times \in \mathcal{I}_r(D)$, then r-max$(D) = \{D \setminus D^\times\}$, and D is r-local. In particular, $D \setminus D^\times \in \mathcal{I}_s(D)$, and D is s-local.

Theorem 3.1.2 (Krull)**.** *Let r be a weak ideal system of D. Let $\emptyset \ne \mathfrak{L} \subset \mathbb{P}(D)$ be such that, for all $M, N \in \mathfrak{L}$ it follows that $MN \in \mathfrak{L}$, and set $\Omega = \{C \in \mathcal{I}_r(D) \mid M \not\subset C$ for all $M \in \mathfrak{L}\}$.*

1. *Every (with respect to the inclusion) maximal element of Ω is a prime ideal.*
2. *Suppose that r is finitary and $M_r \in \mathcal{I}_{r,\mathsf{f}}(D)$ for all $M \in \mathfrak{L}$. For every $C_0 \in \Omega$, there exists a maximal element $P \in \Omega$ such that $C_0 \subset P$.*

   In particular, there exists some $P \in \Omega \cap r$-spec(D) such that $C_0 \subset P$.

Proof. 1. Assume to the contrary that there is a maximal element $P \in \Omega$ which is not a prime ideal. As $\mathfrak{L} \ne \emptyset$, it follows that $P \ne D$. Let $a, b \in D \setminus P$ be such that $ab \in P$. Then it follows by the maximality of P that $(P \cup \{a\})_r, (P \cup \{b\})_r \notin \Omega$, and there exist $M, N \in \mathfrak{L}$ such that $M \subset (P \cup \{a\})_r$ and $N \subset (P \cup \{b\})_r$. Hence we obtain $MN \subset (P \cup \{a\})_r (P \cup \{b\})_r \subset (P^2 \cup Pa \cup Pb \cup \{ab\})_r \subset P$, a contradiction, since $MN \in \mathfrak{L}$.

2. By assumption, $\Omega_1 = \{C \in \Omega \mid C_0 \subset C\} \ne \emptyset$, and we prove that every chain in $(\Omega_1, \subset)$ has an upper bound in Ω_1. Then the assertion follows by 1. and Zorn's Lemma. Let $\Sigma \subset \Omega_1$ be a chain, and

$$P = \bigcup_{C \in \Sigma} C\,.$$

Then $P \in \mathcal{I}_r(D)$, and we assert that $P \in \Omega_1$. Clearly, $C_0 \subset P$, and we assume to the contrary that $M \subset P$ for some $M \in \mathfrak{L}$. Then there is some $E \in \mathbb{P}_{\mathsf{f}}(D)$ such that $M_r = E_r$, hence $E \subset P$, and as Σ is a chain, we obtain $E \subset C$ for some $C \in \Sigma$. But then it follows that $M \subset M_r = E_r \subset C$, a contradiction. □

Corollary 3.1.3. *Let r be a weak ideal system of D, $T \subset D^\bullet$ a multiplicatively closed subset and $\Omega = \{C \in \mathcal{I}_r(D) \mid C \cap T = \emptyset\}$.*

1. *Every (with respect to the inclusion) maximal element of Ω is a prime ideal.*
2. *Suppose that r is finitary and $C_0 \in \Omega$. Then there exists a maximal element $P \in \Omega$ such that $C_0 \subset P$. In particular, there exists some $P \in \Omega \cap r\text{-spec}(D)$ such that $C_0 \subset P$.*

Proof. By Theorem 3.1.2, applied with $\mathfrak{L} = \big\{\{a\} \mid a \in T\big\}$. □

Corollary 3.1.4. *Let r be a weak ideal system of D.*

1. *$r\text{-max}(D) \subset r\text{-spec}(D)$.*
2. *If r is finitary and $J \in \mathcal{I}_r(D) \setminus \{D\}$, then there exists some $M \in r\text{-max}(D)$ such that $J \subset M$. In particular, if $\emptyset_r \ne D$, then $r\text{-max}(D) \ne \emptyset$.*

Proof. We apply Corollary 3.1.3 with $T = D^\times$.

1. If $M \in r\text{-max}(D)$, then M is maximal in $\{C \in \mathcal{I}_r(D) \mid C \cap D^\times = \emptyset\}$.
2. If $J \in \mathcal{I}_r(D)$, and M is maximal in $\{C \in \mathcal{I}_r(D) \mid J \subset C\,,\ J \cap D^\times = \emptyset\}$, then $M \in r\text{-max}(D)$. □

Corollary 3.1.5. *Let r be a finitary ideal system of D. If D is r-local, then $r\text{-max}(D) = \{D \setminus D^\times\}$.*

Proof. Let D be r-local and $r\text{-max}(D) = \{M\}$. If $a \in D \setminus D^\times$, then $aD \in \mathcal{I}_r(D)$ and $aD \ne D$. By Corollary 3.1.4 there exists some $P \in r\text{-max}(D)$ such that $aD \subset P$, and by assumption we have $P = M$. □

Theorem 3.1.6. *Let r be a finitary weak ideal system of D and $J \in \mathcal{I}_r(D) \setminus \{D\}$.*

1. *$\mathcal{P}(J) \subset r\text{-spec}(D)$.*
2. *If $\mathcal{P}(J) \cap r\text{-spec}(D) \subset \mathcal{I}_{r,\mathsf{f}}(D)$, then $\mathcal{P}(J)$ is finite.*
3. *Suppose that every principal ideal of D is an r-ideal. Then $\mathfrak{X}(D) \subset r\text{-spec}(D)$. In particular, if D is cancellative, then $\mathfrak{X}(D) \subset t\text{-spec}(D)$.*
4. *If r is finitary, then $\sqrt{J} \in \mathcal{I}_r(D)$. If $I \in \mathcal{I}_{r,\mathsf{f}}(D)$ and $I \subset \sqrt{J}$, then there is some $n \in \mathbb{N}$ such that $I^n \subset J$.*
5. *If r is finitary, then $\sqrt{r}\colon \mathbb{P}(D) \to \mathbb{P}(D)$, defined by $X_{\sqrt{r}} = \sqrt{X_r}$, is a finitary weak ideal system of D, and $\sqrt{r} \le r$.*

Proof. 1. If $P \in \mathcal{P}(J)$, then $D \setminus P$ is multiplicatively closed, and by Corollary 3.1.3 there exists some $P_0 \in r\text{-spec}(D)$ such that $J \subset P_0 \subset P$. Hence $P_0 \in \Sigma(J)$ and therefore $P_0 = P \in r\text{-spec}(D)$.

2. Let $\mathfrak{L} = \{P_1 \cdot \ldots \cdot P_m \mid m \in \mathbb{N}\,,\ P_1, \ldots, P_m \in \Sigma(J)\}$, $\Omega = \{C \in \mathcal{I}_r(D) \mid L \not\subset C \text{ for all } L \in \mathfrak{L}\}$, and assume that $J \in \Omega$. For every $L \in \mathcal{L}$, we have $L_r \in \mathcal{I}_{r,\mathsf{f}}(D)$, and if $L_1, L_2 \in \mathfrak{L}$, then $L_1L_2 \in \mathfrak{L}$. By Theorem 3.1.2 there exists some $P \in r\text{-spec}(D) \cap \Omega$ such that $J \subset P$, and by Theorem 1.3.2 there exists some $P_0 \in \mathcal{P}(J)$ such that $P_0 \subset P$, which implies $P_0 \in \Omega \cap \mathfrak{L}$, a contradiction. Hence there exists some $L \in \mathfrak{L}$ such that $L \subset J$, say $L = P_1 \cdot \ldots \cdot P_m$, where $m \in \mathbb{N}$ and $P_1, \ldots, P_m \in \mathcal{P}(J)$. We assert that $\mathcal{P}(J) \subset \{P_1, \ldots, P_m\}$. Indeed, if $P \in \mathcal{P}(J)$, then $P_1 \cdot \ldots \cdot P_m \subset J \subset P$ implies $P_j \subset P$ for some $j \in [1, m]$ and hence $P = P_j$ by the minimality of P.

3. If $P \in \mathfrak{X}(D)$ and $a \in P^\bullet$, then $aD \in \mathcal{I}_r(D)$ and $P \in \mathcal{P}(aD) \subset r\text{-spec}(D)$.

4. By Theorem 1.3.2,

$$\sqrt{J} = \bigcap_{P \in \mathcal{P}(J)} P\,,$$

and as $\mathcal{P}(J) \subset r\text{-spec}(D)$, we obtain $\sqrt{J} \in \mathcal{I}_r(D)$.

Assume now that $I \in \mathcal{I}_{r,\mathsf{f}}(D)$ and $I \subset \sqrt{J}$, say $I = E_r$, where $E = \{a_1, \ldots, a_m\} \in \mathbb{P}_{\mathsf{f}}(D)$. For $j \in [1,m]$, let $n_j \in \mathbb{N}$ be such that $a_j^{n_j} \in J$, and set $n = n_1 + \ldots + n_m$. We assert that $E^n \subset J$. Indeed, if $a \in E^n$, then $a = a_1^{\nu_1} \cdot \ldots \cdot a_m^{\nu_m}$, where $\nu_1, \ldots, \nu_m \in \mathbb{N}_0$, $\nu_1 + \ldots + \nu_m = n$, and there is some $j \in [1,m]$ such that $\nu_j \ge n_j$, which implies $a \in J$. Now it follows that $I^n = E_r^n \subset (E^n)_r \subset J$.

5. We verify the properties **M1**, **M2** and **M3**. Let $X, Y \subset D$ and $c \in D$.

M1. $X_{\sqrt{r}} = \sqrt{X_r} \supset X_r \supset X \cup \{0\}$.

M2. If $X \subset Y_{\sqrt{r}} = \sqrt{Y_r}$, then $X_r \subset \sqrt{Y_r}$ (since $\sqrt{Y_r} \in \mathcal{I}_r(D)$), and consequently $\sqrt{X_r} \subset \sqrt{Y_r}$.

M3. If $x \in X_{\sqrt{r}} = \sqrt{X_r}$ and $n \in \mathbb{N}$ is such that $x^n \in X_r$, then $(cx)^n \in c^n X_r \subset cX_r \subset (cX)_r$ and therefore $cx \in \sqrt{(cX)_r} = (cX)_{\sqrt{r}}$. Hence $cX_{\sqrt{r}} \subset (cX)_{\sqrt{r}}$.

Clearly, $X_r \subset X_{\sqrt{r}}$ implies $\sqrt{r} \le r$. If $X \subset D$ and $x \in X_{\sqrt{r}}$, let $n \in \mathbb{N}$ be such that $x^n \in X_r$. As r is finitary, there exists some $E \in \mathbb{P}_{\mathsf{f}}(X)$ such that $x^n \in E_r$ and consequently $x \in E_{\sqrt{r}}$. Hence $\sqrt{r}$ is finitary. If $X \subset D$, then $\sqrt{X_r} \subset D$ is an ideal, and therefore $\sqrt{r}$ is a weak ideal system of D. □

Theorem 3.1.7. *Let r be a finitary weak ideal system of D. Then D is $\sqrt{r}$-noetherian if and only if r-spec(D) satisfies the* ACC *and for every $J \in \mathcal{I}_r(D)$ the set $\mathcal{P}(J)$ is finite.*

PROOF. Assume first that D is $\sqrt{r}$-noetherian. As $r\text{-spec}(D) \subset \{J \in \mathcal{I}_r(D) \mid \sqrt{J} = J\} = \mathcal{I}_{\sqrt{r}}(D)$, it satisfies the ACC. If $J \in \mathcal{I}_r(D)$, then $\sqrt{J} \in \mathcal{I}_{\sqrt{r}}(D)$, and $\mathcal{P}(J) = \mathcal{P}(\sqrt{J}\,) \subset \mathcal{I}_{\sqrt{r}}(D) = \mathcal{I}_{\sqrt{r},\mathsf{f}}(D)$. Hence $\mathcal{P}(J)$ is finite by Theorem 3.1.6.2.

Assume now that r-spec(D) satisfies the ACC , $\mathcal{P}(J)$ is finite for all $J \in \mathcal{I}_r(D)$, and yet there exists a properly ascending sequence $(J_n)_{n \ge 0}$ in $\mathcal{I}_{\sqrt{r}}(D)$. As $\sqrt{r}$ is finitary, we obtain

$$J = \bigcup_{n \ge 0} J_n \in \mathcal{I}_{\sqrt{r}}(D)\,.$$

Let $\mathcal{P}(D) = \{J^{(1)}, \ldots, J^{(N)}\}$. For $n \ge 0$, let $\{P \in \mathcal{P}(J_n) \mid J \not\subset P\} = \{P_n^{(1)}, \ldots, P_n^{(N_n)}\}$. By Theorem 1.3.2.3 it follows that $J = P^{(1)} \cap \ldots \cap P^{(N)}$ and $J_n = J \cap P_n^{(1)} \cap \ldots \cap P_n^{(N_n)}$. We denote by L_n the (finite) set of all sequences $(\nu_0, \ldots, \nu_n) \in [1, N_0] \times \ldots \times [1, N_n]$ such that $P_0^{(\nu_0)} \subset P_1^{(\nu_1)} \subset \ldots \subset P_n^{(\nu_n)}$, and we assert that $L_n \ne \emptyset$.

We proceed by induction on n. For $n = 0$, there is nothing to do. Thus suppose that $n \ge 1$ and $\nu_n \in [1, N_n]$. Since $J_{n-1} = J \cap P_{n-1}^{(1)} \cap \ldots \cap P_{n-1}^{(N_{n-1})} \subset J_n \subset P_n^{(\nu_n)}$ and $J \not\subset P_n^{(\nu_n)}$, it follows that $P_{n-1}^{(\nu_{n-1})} \subset P_n^{(\nu_n)}$ for some $\nu_{n-1} \in [1, N_{n-1}]$, and the induction hypothesis yields the complementary sequence $(\nu_0, \ldots, \nu_{n-1})$.

Now the assignment $(\nu_0, \ldots, \nu_n) \mapsto (\nu_0, \ldots, \nu_{n-1})$ defines a map $L_n \to L_{n-1}$, and as the projective limit of a system of non-empty finite sets is not empty, there exists a sequence

$$(\nu_n)_{n \ge 0} \in \varprojlim_{n \ge 0} L_n\,.$$

By construction, $(P_n^{(\nu_n)})_{n \ge 0}$ is an ascending sequence in r-spec(D). Hence there exists some $m \ge 0$ such that $P_n^{(\nu_n)} = P_m^{(\nu_m)}$ for all $m \ge n$, and consequently

$$J = \bigcup_{n \ge 0} J_n \subset \bigcup_{n \ge 0} P_n^{(\nu_n)} = P_m^{(\nu_m)} \not\supset J\,, \quad \text{a contradiction.}$$

□

3.2. Associated primes, localizations and primary decompositions

Throughout this section, we set $(X\,{:}\,Y) = (X :_D Y)$ *for all subsets* $X,\, Y \subset D$.

Definition 3.2.1. Let $B \supset D$ be an overmonoid and $P \subset D$ be a prime ideal. Recall from Definition 1.3.7 that the *localization* B_P of B at P is defined by $B_P = (D \setminus P)^{-1}B$, that $j_P \colon B \to B_P$ denotes the natural embedding, and for every subset $X \subset B$, $X_P = (D \setminus P)^{-1}X$.

For a finitary weak module system r on B, we define $r_P = (D \setminus P)^{-1}r \colon \mathbb{P}(B_P) \to \mathbb{P}(B_P)$.

If r is a finitary weak module system of B, then r_D is a finitary weak ideal system on D by Theorem 2.5.2, r_P is a finitary weak module system on B_P and if $X \subset B$, then $(X_r)_P = j_P(X)_{r_P} = (X_P)_{r_P}$ by Theorem 2.4.1.

Theorem 3.2.2. *Let $B \supset D$ be an overmonoid, r a finitary weak module system on B, and for $P \in r_D$-spec(D), let $j_P \colon B \to B_P$ be the natural embedding. If $A \in \mathcal{M}_r(B)$ is a D-module, then*

$$A = \bigcap_{P \in r_D\text{-}\max(D)} j_P^{-1}(A_P)\,.$$

In particular:

- *If $A,\, A' \in \mathcal{M}_r(B)$ are D-modules and $A_P = A'_P$ for all $P \in r_D$-$\max(D)$, then $A = A'$.*
- *Assume that $D^\bullet \subset B^\times$. Then $B = B_P \supset A_P \supset A$, $j_P = \mathrm{id}_B$ for all $P \in r_D$-spec(D), and*

$$A = \bigcap_{P \in r_D\text{-}\max(D)} A_P\,.$$

Proof. By Theorem 2.5.2.4, r_D is a finitary weak ideal system on D. Obviously, $A \subset j_P^{-1}(A_P)$ for all $P \in r_D$-max(D). Thus assume that $z \in B$, $j_P(z) \in A_P$ for all $P \in r_D$-max(D), and set $J = (A\,{:}\,z) \cap D$. Then $J \subset D$ is an r_D-ideal, and therefore it suffices to prove that $J \not\subset P$ for all $P \in r_D$-max(D), for then $J = D$ by Corollary 3.1.4.2, hence $1 \in J$ and therefore $z \in A$.

If $P \in r_D$-max(D), then

$$\frac{z}{1} = \frac{a}{t} \quad \text{for some} \quad a \in A \ \text{ and } \ t \in D \setminus P\,,$$

and there exists some $s \in D \setminus P$ such that $stz = sa \in A$ and therefore $st \in J \setminus P$. □

Theorem 3.2.3. *Let r be a finitary weak ideal system of D and $P \in r$-spec(D).*

1. *D_P is r_P-local with r_P-maximal ideal $P_P = D_P \setminus D_P^\times$.*
2. *If $J \in \mathcal{I}_r(D)$ and $\sqrt{J} \in r$-max(D), then J is primary.*
3. *Let $J \in \mathcal{I}_r(D)$ and $P \in \mathcal{P}(J)$.*
   (a) *P_P is the only prime r_P-ideal of D_P containing J_P, J_P is P_P-primary, and $j_P^{-1}(J_P)$ is the smallest P-primary r-ideal of D which contains J.*
   (b) *Assume that $P = \sqrt{J}$ and J_M is P_M-primary for all $M \in r$-max(D) such that $M \supset J$. Then $J = j_P^{-1}(J_P)$ is P-primary.*

Proof. 1. By Theorem 1.2.4, $D_P^\times = (D \setminus P)^{-1}(D \setminus P) = D_P \setminus P_P$, and therefore P_P is the greatest ideal of D_P.

2. Let $a,\, b \in D$ be such that $ab \in J$ and $a \notin J$. Then $(J\,{:}\,a) \in \mathcal{I}_r(D)$, and $J \cup \{b\} \subset (J\,{:}\,a) \subsetneq D$. By Corollary 3.1.4.2 there exists some $M \in r$-max(D) such that $(J\,{:}\,a) \subset M$. Now $J \subset M$ implies $\sqrt{J} \subset M$, hence $\sqrt{J} = M$ and $b \in \sqrt{J}$.

3. (a) Let $\overline{Q} \in r_P$-spec(D_P) be such that $J_P \subset \overline{Q}$. By Theorem 1.3.6.2 we have $\overline{Q} = Q_P$ for some $Q \in r$-spec(D) such that $Q \subset P$. Now $J_P \subset Q_P \subset P_P$ implies $J \subset j_P^{-1}(J_P) \subset Q \subset P$, hence $Q = P$ and

therefore $\overline{Q} = P_P$. Hence P_P is the only prime r_P-ideal containing J_P, $P_P = \sqrt{J_P}$, and J_P is P_P-primary by 1. By Theorem 1.3.6 $j_P^{-1}(J_P)$ is primary, and $\sqrt{j_P^{-1}(J_P)} = j_P^{-1}(\sqrt{J_P}) = P$. If Q is any P-primary r-ideal containing J, then $J_P \subset Q_P \subset P_P$, and $j_P^{-1}(J_P) \subset j_P^{-1}(Q_P) = Q$.

(b) Let J_M be P_M-primary for all $M \in r\text{-max}(D)$ satisfying $M \supset J$. It suffices to prove that $j_P^{-1}(J) \subset J$. If $a \in j_P^{-1}(J)$, then $\frac{a}{1} = \frac{c}{t}$ for some $c \in J$ and $t \in D \setminus P$, and therefore there exists some $s \in D \setminus P$ such that $sta = sc \in J$. By Theorem 3.2.2 it follows that

$$J = \bigcap_{M \in r\text{-}\max(D)} j_M^{-1}(J_M)\,,$$

and therefore it suffices to prove that $a \in j_M^{-1}(J_M)$ for all $M \in r\text{-max}(D)$. If $M \in r\text{-max}(D)$ and $J \not\subset M$, then $j_M^{-1}(J_M) = D$ and there is nothing to do. If $M \in r\text{-max}(D)$ and $J \subset M$, then $\frac{sta}{1} \in J_M$, and we assert that $\frac{a}{1} \in J_M$ (which implies $a \in j_M^{-1}(J_M)$). Indeed, if $\frac{a}{1} \notin J_M$, then $\frac{st}{1} \in P_M$ and $st \in j_M^{-1}(P_M) = P$, a contradiction. □

Definition 3.2.4. Let r be a weak ideal system of D and $J \in \mathcal{I}_r(D)$.

1. A prime ideal $P \subset D$ is called an *associated prime* of J if $P = (J \colon z)$ for some $z \in D \setminus J$. Let $\mathrm{Ass}_D(J) = \mathrm{Ass}(J) \subset r\text{-spec}(D)$ the set of all associated primes of J.
   If D is cancellative, $K = \mathsf{q}(D)$ and $z \in K^\times$, then $(J \colon z) = z^{-1}J \cap D$.
2. A primary decomposition $\mathfrak{Q}$ of J is called an *r-primary decomposition* if $\mathfrak{Q} \subset \mathcal{I}_r(D)$.
   By definition, a primary decomposition is just an s-primary decomposition. If J possesses an r-primary decomposition, then it also possesses a reduced one (this is proved as in Theorem 1.3.5). If $\mathfrak{Q}$ is a reduced r-primary decomposition of J, then $\{\sqrt{Q} \mid Q \in \mathfrak{Q}\} \subset \mathrm{Ass}(J)$ by Theorem 1.3.5.2.
3. D is called *r-laskerian* if every r-ideal of D possesses an r-primary decomposition.

Theorem 3.2.5. *Let r be a weak ideal system of D and $J \in \mathcal{I}_r(D)$.*

1. *Every maximal element in the set $\{(J \colon z) \mid z \in D \setminus J\}$ belongs to $\mathrm{Ass}(J)$.*
2. *Let r be finitary, $T \subset D$ a multiplicatively closed subset, $P \in r\text{-spec}(D)$ and $P \cap T = \emptyset$.*
   (a) *If $P \in \mathrm{Ass}(J)$, then $T^{-1}P \in \mathrm{Ass}(T^{-1}J)$.*
   (b) *If $P \in \mathcal{I}_{r,\mathsf{f}}(D)$ and $T^{-1}P \in \mathrm{Ass}(T^{-1}J)$, then $P \in \mathrm{Ass}(J)$.*

PROOF. 1. Let $c \in D \setminus J$ be such that $(J \colon c)$ is maximal in the set $\{(J \colon z) \mid z \in D \setminus J\}$. Let $a, b \in D$ be such that $ab \in (J \colon c)$ and $a \notin (J \colon c)$. Then it follows that $ac \notin J$, $b \in (J \colon ac)$. Since obviously $(J \colon c) \subset (J \colon ac)$, equality holds by the maximal choice of $(J \colon c)$, and thus $b \in (J \colon c)$. Therefore $(J \colon c)$ is a prime ideal and belongs to $\mathrm{Ass}(J)$.

2. (a) If $P = (J \colon z) \in \mathrm{Ass}(J)$, then $T^{-1}P = (T^{-1}J \colon_{T^{-1}D} j_P(z))$ is a prime ideal of $T^{-1}D$ and thus it belongs to $\mathrm{Ass}(T^{-1}J)$.

(b) Suppose that $P = \{a_1, \ldots, a_n\}_r$, where $n \in \mathbb{N}_0$ and $a_1, \ldots, a_n \in P$, and $T^{-1}P = \big(T^{-1}J \colon_{T^{-1}D} \frac{z}{t}\big)$, where $z \in D$ and $t \in T$. For $i \in [1, n]$, we obtain

$$\frac{a_i}{1}\,\frac{z}{t} = \frac{c_i}{s_i}\,, \quad \text{where} \quad c_i \in J \text{ and } \ s_i \in T\,, \quad \text{and therefore} \quad w_i s_i a_i z = w_i t c_i \in J \quad \text{for some } \ w_i \in T\,.$$

If $v = (w_1 s_1) \cdot \ldots \cdot (w_n s_n)$, then $v \in T$ and $vza_i \in J$ for all $i \in [1, n]$. Hence it follows that $vzP \subset J$ and $P \subset (J \colon vz)$. We assert that equality holds (which implies $P \in \mathrm{Ass}(J)$). Thus let $x \in (J \colon vz)$. Then $xvz \in J$, and

$$\frac{xv}{1}\,\frac{z}{t} \in T^{-1}J\,, \quad \text{which implies} \quad \frac{xv}{1} \in T^{-1}P\,.$$

Hence $xv \in P$ and finally $x \in P$, since $v \in T \subset D \setminus P$. □

Theorem 3.2.6. *Let r be a weak ideal system of D such that D is r-noetherian and $J \in \mathcal{I}_r(D)$.*

1. *$\mathcal{P}(J)$ is finite, and $\mathcal{P}(J) \subset \mathrm{Ass}(J)$.*
2. *If $\mathfrak{Q}$ is a reduced r-primary decomposition of J, then $\mathrm{Ass}(J) = \{\sqrt{Q} \mid Q \in \mathfrak{Q}\}$.*
3. *J possesses a representation $J = Q_1 \cap \ldots \cap Q_n$, where $n \in \mathbb{N}_0$ and $Q_1, \ldots, Q_n \in \mathcal{I}_r(D)$ are r-irreducible.*
4. *D is r-laskerian if and only if every r-irreducible r-ideal is primary.*

Proof. 1. By Theorem 3.1.6.2 the set $\mathcal{P}(J)$ is finite. Thus let $P \in \mathcal{P}(J)$. By Theorem 3.2.5.2 (b) it suffices to prove that $P_P \in \mathrm{Ass}(J_P)$. Since D_P is r_P-noetherian, the set $\{(J_P : z) \mid z \in D_P \setminus J_P\}$ has maximal elements, and thus $\mathrm{Ass}(J_P) \neq \emptyset$ by Theorem 3.2.5.1. If $\overline{Q} \in \mathrm{Ass}(J_P)$, then $J_P \subset \overline{Q} \subset P_P$, and P_P is the only prime r_P-ideal of D_P containing J_P by Theorem 3.2.3.3 (a). Hence $P_P = \overline{Q} \in \mathrm{Ass}(J_P)$.

2. If $P = (J : z) \in \mathrm{Ass}(J)$, where $z \in D \setminus P$, then $P = \sqrt{Q}$ for some $Q \in \mathfrak{Q}$ by Theorem 1.3.5.2. To prove the converse, let $P = \sqrt{Q}$ for some $Q \in \mathfrak{Q}$. Then $\mathfrak{Q}_P = \{Q_P \mid Q \in \mathfrak{Q},\ Q \subset P\}$ is the reduced primary decomposition of J_P, and $P_P = \sqrt{Q_P} = \sqrt{(J_P : z)}$ for some $z \in D_P \setminus J_P$. As D_P is r_P-noetherian, it follows that P_P is r_P-finitely generated, and by Theorem 3.1.6.4 there is some $k \in \mathbb{N}$ such that $P_P^k \subset (J_P : z)$. If k is minimal with this property, then there exists some $y \in P_P^{k-1}$ such that $yz \notin J_P$. It follows that $P_P yz \subset P_P^k z \subset J_P$, hence $P_P \subset (J_P : yz) \subsetneq D_P$, and therefore $P_P = (J_P : yz) \in \mathrm{Ass}(J_P)$. Hence we obtain $P \in \mathrm{Ass}(J)$ by Theorem 3.2.5.2 (b).

3. We assume that the set Ω of all $I \in \mathcal{I}_r(D)$, which are not intersections of finitely many r-irreducible r-ideals, is not empty. Then Ω possesses a maximal element I. Since I is not r-irreducible, there exist $I_1, I_2 \in \mathcal{I}_r(D)$ such that $I = I_1 \cap I_2$, $I_1 \neq I$ and $I_2 \neq I$. Since $I \subsetneq I_1$ and $I \subsetneq I_2$, it follows that $I_1, I_2 \notin \Omega$. Since both I_1 and I_2 are intersections of finitely many r-irreducible r-ideals, the same is true for I, a contradiction.

4. If every r-irreducible r-ideal is primary, then D is r-laskerian by 3. If D is r-laskerian and $Q \in \mathcal{I}_r(D)$ is irreducible and $\mathfrak{Q}$ is a reduced r-primary decomposition of Q, then $\mathfrak{Q} = \{Q\}$ and thus Q is primary. □

Theorem 3.2.7. *Let D be a Mori monoid.*

1. *If $I \in \mathcal{I}_v(D)^\bullet$ is v-irreducible, then $I = zD \cap D$ for some $z \in K^\times$.*
2. *If $a \in D^\bullet$, then $\mathrm{Ass}(aD) = \{P \in v\text{-spec}(D) \mid a \in P\}$ is a finite set.*
   In particular, if $X \subset D$ and $X^\bullet \neq \emptyset$, then the set $\{P \in v\text{-spec}(D) \mid X \subset P\}$ is finite.

Proof. 1. Let $I \in \mathcal{I}_v(D)^\bullet$ be v-irreducible. By Theorem 2.6.6, the set $\Omega = \{J \in \mathcal{I}_v(D) \mid J \supsetneq I\}$ has minimal elements, and we assert that it even has a smallest element. Indeed, if $J_1, J_2 \in \Omega$ are minimal elements, then $J_1 \cap J_2 \supsetneq I$, since I is v-irreducible, hence $J_1 \cap J_2 \in \Omega$ and therefore $J_1 = J_2$.

Let I^* be the smallest element of Ω. Since

$$I = I_v = \bigcap_{\substack{z \in K^\times \\ I \subset zD}} zD \subsetneq I^*\,,$$

there is some $z \in K^\times$ such that $I \subset zD$ and $I^* \not\subset zD$. Since $zD \cap D \in \mathcal{I}_t(D)$, $I \subset zD \cap D$ and $I^* \not\subset zD \cap D$, we obtain $I = zD \cap D$.

2. Let $a \in D^\bullet$. If $P \in \mathrm{Ass}(aD)$, then clearly $P \in v\text{-spec}(D)$ and $a \in P$. Conversely, suppose that $P \in v\text{-spec}(D)$ and $a \in P$. As P is v-irreducible, we obtain $P = zD \cap D$ for some $z \in K^\times$ by 1. Hence $z^{-1}a \in D$, and $P = zD \cap D = (z^{-1}a)^{-1}aD \cap D = (aD :_D z^{-1}a) \in \mathrm{Ass}(aD)$.

It remains to prove finiteness. Assume to the contrary that the set $\Omega = \{P \in v\text{-spec}(D) \mid a \in P\}$ is infinite. Since D is v-noetherian, there exists a sequence $(P)_{n \geq 0}$ in Ω such that, for every $n \geq 0$, P_n is maximal in $\Omega \setminus \{P_0, \ldots, P_{n-1}\}$. By Theorem 2.6.6, there exists some $m \geq 0$ such that

$$P_0 \cap \ldots \cap P_m = P_0 \cap \ldots \cap P_{m+1} \subset P_{m+1}$$

and therefore $P_j \subset P_{m+1}$ for some $j \in [1, m]$. However, P_j is maximal in $\Omega \setminus \{P_0, \ldots, P_{j-1}\}$, and since $P_{m+1} \in \Omega \setminus \{P_0, \ldots, P_m\} \subset \Omega \setminus \{P_0, \ldots, P_{j-1}\}$, it follows that $P_{m+1} = P_j$, a contradiction. □

Theorem 3.2.8. *Let D be a Mori monoid and $I \in \mathcal{I}_v(D)^\bullet$.*

1. *If $P \in \mathrm{Ass}(I)$ and $I = I_P \cap D$, then P is the greatest element of $\mathrm{Ass}(I)$.*
2. *If I is v-irreducible, then $\mathrm{Ass}(I)$ has a greatest element P, and $I = I_P \cap D$.*
3. *If $P \in v\text{-spec}(D)$, $a \in P^\bullet$ and $I = aD_P \cap D$, then I is v-irreducible, and P is the greatest element of $\mathrm{Ass}(I)$.*

PROOF. 1. Assume to the contrary that there is some $Q \in \mathrm{Ass}(I)$ such that $Q \not\subset P$, and fix an element $s \in Q \setminus P$. Let $b \in D \setminus I$ be such that $Q = (I\!:\!b)$. Then $sb \in I$ and therefore $b \in I_P \cap D = I$, a contradiction.

2. Let Ω be the (finite non-empty) set of all maximal elements of $\mathrm{Ass}(I)$. We assert that

$$I = \bigcap_{P \in \Omega} I_P \cap D\,.$$

Once this is proved, it follows that $|\Omega| = 1$ since I is v-irreducible, hence $\mathrm{Ass}(I)$ has a greatest element P, and $I = I_P \cap D$.

Clearly, $I \subset I_P \cap D$ for all $P \in \Omega$. Thus suppose that $x \in D \setminus I$. By Theorem 3.2.5.1, every maximal element in the set $\{(I : y) \mid y \in D \setminus I\}$ belongs to $\mathrm{Ass}(I)$. Hence there is some $Q \in \Omega$ such that $(I\!:\!x) \subset Q$, and we assert that $x \notin I_Q$. Indeed, if $x \in I_Q$, then there is some $s \in D \setminus Q$ such that $xs \in I$ and therefore $s \in (I\!:\!x) \subset Q$, a contradiction.

3. If $P \in v\text{-spec}(D)$, $a \in P^\bullet$ and $I = aD_P \cap D$, then $I \in \mathcal{I}_v(D)$, $I_P \cap D = I$, $P \in \mathrm{Ass}(aD)$ by Theorem 3.2.7.2, and therefore there exists some $b \in D$ such that $P = (aD\!:\!b) = b^{-1}aD \cap D \subset b^{-1}I \cap D$, and we assert that equality holds. Indeed, if $x \in b^{-1}I \cap D$, then $xb \in I = aD_P \cap D$, hence $xbs \in aD$ for some $s \in D \setminus P$ and therefore $xs \in ab^{-1}D \cap D = P$, which implies $x \in P$.

Hence it follows that $P = (I\!:\!b) \in \mathrm{Ass}(I)$, and by 1. P is the greatest element of $\mathrm{Ass}(I)$. It remains to show that I is t-irreducible, and for this we prove:

A. If $J \in \mathcal{I}_v(D)$ and $J \supsetneq I$, then $aJ^{-1} \subset P$ and $b \in J$.

Assume that **A** holds. If $I = J_1 \cap J_2$ for some $J_1, J_2 \in \mathcal{I}_t(D)$ such that $J_1 \supsetneq I$ and $J_2 \supsetneq I$, then $b \in J_1 \cap J_2 = I$ and therefore $P = (I\!:\!b) = D$, a contradiction.

Proof of **A.** Let $J \in \mathcal{I}_t(D)$ be such that $J \supsetneq I$. If $aJ^{-1} \not\subset P$, then $D_P = (aJ^{-1})_P = aJ_P^{-1}$, and as $J_P \in \mathcal{I}_v(D_P)$, it follows that $J_P = (J_P^{-1})^{-1} = aD_P$ and $J \subset aD_P \cap D = I$, a contradiction. Hence $aJ^{-1} \subset P$ and $aJ^{-1}b \subset Pb \subset aD$, which implies $J^{-1} \subset b^{-1}D$ and therefore $b \in bD = (J^{-1})^{-1} = J$. □

Theorem 3.2.9.

1. *A Mori monoid D is v-laskerian if and only if $\mathfrak{X}(D) = \{P \in v\text{-spec}(D) \mid P^\bullet \neq \emptyset\}$.*
2. *Every s-noetherian monoid is s-laskerian.*

PROOF. 1. Let D be a Mori monoid.

Let first D be v-laskerian and $P, Q \in v\text{-spec}(D)$ such that $Q^\bullet \neq \emptyset$ and $Q \subset P$. We must prove that $Q = P$. If $a \in Q^\bullet$, then $P, Q \in \mathrm{Ass}(aD)$ by Theorem 3.2.7.2, and $I = aD_P \cap D$ is v-irreducible by Theorem 3.2.8.3. By Theorem 3.2.6.4 I is primary, and since $I = aD_P \cap D \subset Q_P \cap D = Q$, it follows that $I = I_Q \cap D$. By Theorem 3.2.8.1 Q is the greatest element of $\mathrm{Ass}(I)$, and therefore $Q = P$.

Assume now that $\mathfrak{X}(D) = \{P \in v\text{-spec}(D) \mid P^\bullet \neq \emptyset\}$. By Theorem 3.2.6.4 we must prove that every v-irreducible v-ideal of D is primary. Let $Q \in \mathcal{I}_v(D)^\bullet$ be v-irreducible. By Theorem 3.2.8.2 $\mathrm{Ass}(Q)$ has a greatest element P, and as $P \in \mathfrak{X}(D)$, it follows that $P \in v\text{-max}(D)$, and $\mathrm{Ass}(Q) = \mathcal{P}(Q) = \{P\}$. In particular, $P = \sqrt{Q}$, and Theorem 3.2.3.2 implies that Q is primary.

2. Let D be an s-noetherian monoid. By Theorem 3.2.6.4 we must prove that every s-irreducible ideal of D is primary. Let $Q \subsetneq D$ be an ideal which is not primary. Then there exist $a, b \in D$ such that $ab \in Q$, $a \notin Q$ and $b \notin \sqrt{Q}$. For all $n \in \mathbb{N}$, we have $Q \subsetneq (Q : b) \subset (Q : b^n) \subset (Q : b^{n+1})$, and as D is s-noetherian, there exists some $n \in \mathbb{N}$ such that $(Q : b^n) = (Q : b^{2n})$. We assert that $Q = (Q : b^n) \cap (Q \cup b^n D)$, which shows that Q is not s-irreducible. Clearly, $Q \subset (Q : b^n) \cap (Q \cup b^n D)$, and we assume that there is some $x \in (Q : b^n) \cap (Q \cup b^n D) \setminus Q$. Then $x = b^n u$ for some $u \in D$ and $b^n x = b^{2n} u \in Q$. Since $(Q : b^n) = (Q : b^{2n})$, it follows that $b^n u = x \in Q$, a contradiction. □

3.3. Laskerian rings

In this Section, we use the common terminology of commutative ring theory.

Theorem 3.3.1. *Every noetherian ring is laskerian.*

Proof. Let D be a noetherian ring. By Theorem 3.2.6.4 it suffices to prove that every (d-)irreducible ideal of D is primary. Let $Q \subsetneq D$ be an ideal which is not primary. Then there exist $a, b \in D$ such that $ab \in Q$, $a \notin Q$ and $b \notin \sqrt{Q}$. For all $n \in \mathbb{N}$, we have $Q \subsetneq (Q : b) \subset (Q : b^n) \subset (Q : b^{n+1})$, and as D is noetherian, there exists some $n \in \mathbb{N}$ such that $(Q : b^n) = (Q : b^{2n})$. We assert that $Q = (Q : b^n) \cap (Q + b^n D)$, which shows that Q is not irreducible. Clearly, $Q \subset (Q : b^n) \cap (Q + b^n D)$, and we assume that there is some $x \in (Q : b^n) \cap (Q + b^n D) \setminus Q$. Then $x = q + b^n u$ for some $q \in Q$ and $u \in D$, and $b^n x = b^n q + b^{2n} u \in Q$. Hence $b^{2n} u \in Q$, and since $(Q : b^n) = (Q : b^{2n})$, it follows that $b^n u \in Q$ and therefore also $x \in Q$, a contradiction. □

Theorem 3.3.2. *Every laskerian ring satisfies the* ACC *for radical ideals.*

Proof. Let D be a laskerian ring. Then D satisfies the ACC for radical ideals if and only if D is $\sqrt{d(D)}$-noetherian. By Theorem 3.1.7 we must prove:

1. For every ideal $J \subset D$ the set $\mathcal{P}(J)$ is finite.
2. D satisfies the ACC on prime ideals.

1. Let $J \subset D$ be an ideal and $\mathfrak{Q} = \{Q_1, \ldots, Q_m\}$ a primary decomposition of J. If $P \in \mathcal{P}(J)$, then $P \supset J = Q_1 \cap \ldots \cap Q_m$, and there exists some $j \in [1, m]$ such that $Q_j \subset P$. Since $J \subset \sqrt{Q_j} \subset P$, it follows that $P = \sqrt{Q_j}$, and thus $\mathcal{P}(J) \subset \{\sqrt{Q_1}, \ldots, \sqrt{Q_m}\}$.

2. Assume to the contrary that there exists a sequence $(P_n)_{n \ge 0}$ of prime ideals such that $P_n \subsetneq P_{n+1}$ for all $n \ge 0$. For every $n \ge 1$, we fix an element $p_n \in P_n \setminus P_{n-1}$, and we consider the ideals

$$J = \sum_{i \ge 0} p_1 \cdot \ldots \cdot p_i P_i \quad \text{and} \quad J_n = (J : p_1 \cdot \ldots \cdot p_n) \supset P_n\,.$$

Let $\mathfrak{Q} = \{Q_1, \ldots, Q_m\}$ be a primary decomposition of J. For $n \ge 1$, we obtain

$$J_n = \Big(\bigcap_{j=1}^{m} Q_j : p_1 \cdot \ldots \cdot p_n \Big) = \bigcap_{\substack{j=1 \\ p_1 \cdot \ldots \cdot p_n \notin Q_j}}^{m} (Q_j : p_1 \cdot \ldots \cdot p_n)\,,$$

and we set $\mathfrak{Q}_n = \{(Q_j : p_1 \cdot \ldots \cdot p_n) \mid j \in [1, m]\,,\ p_1 \cdot \ldots \cdot p_n \notin Q_j\}$. If $j \in [1, m]$ and $p_1 \cdot \ldots \cdot p_n \notin Q_j$, then $(Q_j : p_1 \cdot \ldots \cdot p_n)$ is primary, and $\sqrt{(Q_j : p_1 \cdot \ldots \cdot p_n)} = \sqrt{Q_j}$ by Theorem 1.3.3.3 (b). In particular, it follows that $\{\sqrt{Q} \mid Q \in \mathfrak{Q}_n\} \subset \{\sqrt{Q_1}, \ldots, \sqrt{Q_m}\}$ for all $n \ge 1$. Now we prove the following assertion:

A. For all $n \ge 1$ and all $j \in [1, n+1]$, we have

$$p_j p_{j+1} \cdot \ldots \cdot p_n J_n \subset P_{j-1} + \sum_{i \ge j} p_j p_{j+1} \cdot \ldots \cdot p_i P_i\,.$$

Suppose that **A** holds. If $n \ge 1$ and $j = n+1$, we obtain

$$J_n \subset P_n + \sum_{i \ge n+1} p_{n+1}p_{n+2} \cdot \ldots \cdot p_i P_i \subset P_{n+1} \quad \text{and therefore} \quad P_n \subset J_n = \bigcap_{Q \in \mathfrak{Q}_n} Q \subset P_{n+1}\,.$$

In particular, for every $n \ge 1$, there exists some $Q \in \mathfrak{Q}_n$ such that $P_n \subset \sqrt{Q} \subset P_{n+1}$. This is impossible since the set $\{\sqrt{Q} \mid Q \in \mathfrak{Q}_n \text{ for some } n \ge 1\}$ is finite. Hence it suffices to prove **A**.

Proof of **A.** Let $n \ge 1$ and proceed by induction on j.

$j = 1$: By definition,

$$p_1 \cdot \ldots \cdot p_n J_n \subset J = P_0 + \sum_{i \ge 1} p_1 \cdot \ldots \cdot p_i P_i\,.$$

$j \in [1, n]$, $j \to j+1$: Let $a \in J_n$. By the induction hypothesis, we have

$$p_j p_{j+1} \cdot \ldots \cdot p_n a = q_{j-1} + \sum_{i \ge j} p_j p_{j+1} \cdot \ldots \cdot p_i q_i\,, \quad \text{where} \quad q_\nu \in P_\nu \ \text{ for all } \ \nu \ge j-1\,.$$

Hence

$$p_j \Big(p_{j+1} \cdot \ldots \cdot p_n a - \sum_{i \ge j} p_{j+1} \cdot \ldots \cdot p_i q_i \Big) = q_{j-1} \in P_{j-1}\,,$$

and as $p_j \notin P_{j-1}$, it follows that

$$p_{j+1} \cdot \ldots \cdot p_n a \in P_{j-1} + q_j + \sum_{i \ge j+1} p_{j+1} \cdot \ldots \cdot p_i P_i \subset P_j + \sum_{i \ge j+1} p_{j+1} \cdot \ldots \cdot p_i P_i\,. \qquad \square$$

3.4. Valuation monoids and primary monoids

Remarks and Definition 3.4.1.

1. Let Γ be a (multiplicative) abelian group.
   (a) Let $\le$ a partial ordering on Γ. Then $(\Gamma, \le)$ is called a *partially ordered* abelian group if, for all $a, b, c \in \Gamma$, $a \le b$ implies $ac \le bc$. The set $\Gamma_+ = \{x \in \Gamma \mid x \ge 1\}$ is called the *positive cone* of Γ. If $\Gamma_+^{-1} = \{x^{-1} \mid x \in \Gamma_+\}$, then $\Gamma_+ \cap \Gamma_+^{-1} = \{1\}$ (that means, Γ_+ is a reduced submonoid of Γ), and $\le$ is a total order (and thus $(\Gamma, \le)$ a totally ordered abelian group) if and only if $\Gamma = \Gamma_+ \cup \Gamma_+^{-1}$.
2. Let $\Delta \subset \Gamma$ be a reduced submonoid. Then there exists a unique partial ordering $\le$ on Γ such that $(\Gamma, \le)$ is a partially ordered abelian group and $\Gamma_+ = \Delta$ [indeed, define $\le$ by $a \le b$ if and only if $a^{-1}b \in \Delta$].
3. Let Γ be an additive abelian group and $\le$ a total ordering on Γ such that, for all $a, b, c \in \Gamma$, $a \le b$ implies $a + c \le b + c$. Then we call $\Gamma = (\Gamma, \le)$ a *totally ordered additive abelian group*, and we set $\Gamma_+ = \{x \in \Gamma \mid x \ge 0\}$. Then $\Gamma = \Gamma_+ \cup -\Gamma_+$ and $\Gamma_+ \cap -\Gamma_+ = \{0\}$.
4. Let D be a cancellative monoid and $K = \mathsf{q}(D)$. On $K^\times/D^\times$, we define a partial ordering $\le$ by $aD^\times \le bD^\times$ if $aD \supset bD$ (equivalently, if $a^{-1}b \in D$). Obviously, this definition is independent of the choice of representatives, and it makes $K^\times/D^\times$ into a partially ordered abelian group. $\mathcal{G}(D) = (K^\times/D^\times, \le)$ is called the *group of divisibility* of D. By definition, $\mathcal{G}(D)_+ = D^\bullet/D^\times$.

Theorem und Definition 3.4.2. *Let D be cancellative.*

1. *The following assertions are equivalent:*
   (a) *For all $a, b \in D$, if $a \notin bD$, then $b \in aD$.*
   (b) *Every s-finitely generated s-ideal $J \in \mathcal{I}_{s,\mathsf{f}}(D)^\bullet$ is principal.*
   (c) *For all $z \in K^\times$, if $z \notin D$, then $z^{-1} \in D$.*

(d) *The group of divisibility* $\mathcal{G}(D)$ *is totally ordered.*

(e) *There exists a surjective group homomorphism* $w\colon K^\times \to \Gamma$ *onto a totally ordered additive abelian group* Γ *such that* $D^\bullet = w^{-1}(\Gamma_+) = \{x \in K^\times \mid w(x) \geq 0\}$.

(f) *The set* $\mathcal{M}_{s(D)}(K)$ *of all* D*-submodules of* K *is a chain.*

(g) *The set* $\mathcal{I}_s(D)$ *of all ideals of* D *is a chain.*

If these conditions are fulfilled, then D is called a *valuation monoid* (of K), and a group epimorphism $w\colon K^\times \to \Gamma$ onto a totally ordered abelian group Γ such that $D^\bullet = w^{-1}(\Gamma_+)$ is called a *valuation morphism* of D.

If D is a valuation monoid and r is a module system on K such that $D = D_r$, then D is called an *r-valuation monoid.*

In particular:

- *Every valuation monoid is a* GCD*-monoid.*
- *A monoid* D *is a valuation monoid if and only if* $D^\bullet/D^\times$ *is a valuation monoid.*
- *Every divisible monoid is a valuation monoid.*

2. *Let* D *be a valuation monoid and* $w\colon K^\times \to \Gamma$ *a valuation morphism of* D.

(a) $\operatorname{Ker}(w) = D^\times$, *and* w *induces an order isomorphism*

$$w^*\colon \mathcal{G}(D) \to \Gamma\,, \quad \text{given by} \quad w^*(xD^\times) = w(x) \quad \text{for all} \ \ x \in K^\times.$$

In particular, $w^*(D^\bullet/D^\times) = \Gamma_+$.

(b) *If* $w_1\colon K^\times \to \Gamma_1$ *is another valuation morphism of* D*, then there exists a unique order isomorphism* $\varphi\colon \Gamma \to \Gamma_1$ *such that* $\varphi{\circ}w = w_1$.

(c) *If* $E \in \mathbb{P}_{\mathsf f}(K)$ *and* $E^\bullet \neq \emptyset$*, then there exists some* $a \in E$ *such that* $ED = aD$*, and for every such* $a \in E$ *we have* $w(a) = \min w(E^\bullet)$.

3. *If* D *is a valuation monoid and* V *is a monoid such that* $D \subset V \subset K$*, then* V *is a valuation monoid,* $V \setminus D \subset V^\times$, $P = D \setminus V^\times \in s\text{-spec}(D)$*, and* $V = D_P = (V^\times \cap D)^{-1}D$.

4. *Let* $(V_\lambda)_{\lambda\in\Lambda}$ *be a chain of valuation monoids such that* $\mathsf{q}(V_\lambda) = K$ *for all* $\lambda \in \Lambda$*. Then*

$$V^* = \bigcup_{\lambda\in\Lambda} V_\lambda \quad \text{and} \quad V_* = \bigcap_{\lambda\in\Lambda} V_\lambda$$

are valuation monoids of K.

Proof. 1. (a) $\Rightarrow$ (b) Let $J \in \mathcal{I}_{s,\mathsf f}(D)^\bullet$. Then $J = E_s$, where $\emptyset \neq E \in \mathbb{P}_{\mathsf f}(D^\bullet)$, and we proceed by induction on $|E|$. If $|E| = 1$, there is nothing to do. Thus suppose that $E = E' \cup \{a\}$, where $a \in E \setminus E'$, and that $E'_s = bD$. Then $J = bD \cup aD$. If $a \in bD$, then $J = bD$. If $a \notin bD$, then $b \in aD$, and $J = aD$.

(b) $\Rightarrow$ (c) Let $z = a^{-1}b \in K \setminus D$, where $a,\, b \in D^\bullet$ and $b \notin aD$. By assumption, there exists some $u \in D$ such that $aD \cup bD = uD$, and thus $u \in aD$ or $u \in bD$. If $u \in aD$, then $aD = uD \supset bD$ and $a^{-1}b = z \in D$. If $u \in bD$, then $bD = uD \supset aD$ and $b^{-1}a = z^{-1} \in D$.

(c) $\Rightarrow$ (d) If $x,\, y \in K^\times$, then either $x^{-1}y \in D$ or $y^{-1}x \in D$, and therefore either $xD^\times \leq yD^\times$ or $yD^\times \leq xD^\times$. Hence $\mathcal{G}(D)$ is totally ordered.

(d) $\Rightarrow$ (e) Let $w\colon K^\times \to \mathcal{G}(D)$ be the canonical epimorphism.

(e) $\Rightarrow$ (f) Let $w\colon K^\times \to \Gamma$ be an epimorphism onto a totally ordered abelian group Γ such that $D^\bullet = w^{-1}(\Gamma_+)$. Let $M,\, N \in \mathcal{M}_s(K)$, $M \not\subset N$, $a \in M \setminus N$, and let $b \in N^\bullet$ be arbitrary. Then $b^{-1}a \notin D^\bullet$, since otherwise $a = b^{-1}ab \in DN = N$. Hence $w(b^{-1}a) < 0$, $w(a^{-1}b) = -w(ab^{-1}) > 0$, hence $ab^{-1} \in D$ and therefore $b \in ab^{-1}bD = aD \subset M$. Thus it follows that $N \subset M$.

(f) $\Rightarrow$ (g) $\Rightarrow$ (a) Obvious.

2. (a) If $x \in K^\times$, then $x \in \operatorname{Ker}(w)$ if and only if $w(x) \geq 0$ and $w(x^{-1}) = -w(x) \geq 0$, that is, if and only if $x \in D$ and $x^{-1} \in D$ and thus $x \in D^\times$.

As w is an epimorphism, it induces an isomorphism $w^*\colon K^\times/D^\times = \mathcal{G}(D) \to \Gamma$, given as asserted, and we must prove that w^* is an order isomorphism. If $x,\,y \in K^\times$ and $xD^\times \le yD^\times$, then $x^{-1}y \in D$ and therefore $0 \le w(x^{-1}y) = -w(x) + w(y)$, which implies $w(x) \le w(y)$.

(b) Let $w^*\colon \mathcal{G}(D) \to \Gamma$ and $w_1^*\colon \mathcal{G}(D) \to \Gamma_1$ be the order isomorphisms induced by w and w_1 according to (a). Then $\varphi = w_1^* \circ w^{*-1}\colon \Gamma \to \Gamma_1$ is an order isomorphism, and it is obviously the only order isomorphism satisfying $\varphi \circ w = w_1$.

(c) The finite set $\{cD \mid c \in E^\bullet\}$ is a chain. Hence there exists some $a \in E^\bullet$ such that $cD \subset aD$ for all $c \in E$ and thus $ED = aD$. For every such $a \in E^\bullet$ we have $a^{-1}c \in D^\bullet$ for all $c \in D^\bullet$, hence $0 \le w(a^{-1}c) = -w(a) + w(c)$, and therefore $w(a) = \min w(E^\bullet)$.

3. Let $D \subset V \subset K$ be a monoid. Then $K = \mathsf{q}(V)$, and if $z \in K \setminus V$, then $z \notin D$ and $z^{-1} \in D \subset V$. Hence V is a valuation monoid of K. If $z \in V \setminus D$, then $z^{-1} \in D \subset V$ and thus $z \in V^\times$. Hence $V \setminus D \subset V^\times$. If $z \in V \setminus D$, then $z \in V^\times$, $z^{-1} \in D \cap V^\times$, and therefore $z \in (D \cap V^\times)^{-1} \subset (D \cap V^\times)^{-1}D$. Hence it follows that $V = (V \setminus D) \cup D \subset (D \cap V^\times)^{-1}D \subset V$, and equality holds.

4. Since $(V_\lambda)_{\lambda \in \Lambda}$ is a chain, it follows that V^* and V_* are submonoids of K, and by 2. V^* is a valuation monoid. If $x \in K \setminus V_*$, then $x \in K \setminus V_\mu$ for some $\mu \in \Lambda$, and consequently $x^{-1} \in V_\mu$. If $\lambda \in \Lambda$ and $V_\mu \subset V_\lambda$, then $x^{-1} \in V_\lambda$. If $\lambda \in \Lambda$ and $V_\mu \not\subset V_\lambda$, then $V_\lambda \subset V_\mu$, hence $x \notin V_\lambda$ and therefore $x^{-1} \in V_\lambda$. Thus we have proved that $x^{-1} \in V_\lambda$ for all $\lambda \in \Lambda$ and therefore $x^{-1} \in V_*$. Consequently, also V_* is a valuation monoid of K. □

Theorem 3.4.3. *Let Γ be an additive abelian group. Then the following assertions are equivalent*:

(a) *There exists an ordering $\le$ on Γ such that $(\Gamma, \le)$ is a totally ordered additive abelian group.*

(b) *Γ is torsion-free.*

(c) *There exists a subset $P \subset \Gamma$ such that $P + P \subset P$, $P \cap -P = \{0\}$ and $\Gamma = P \cup -P$.*

Proof. (a) $\Rightarrow$ (b) Let $\alpha \in \Gamma$ and $n \in \mathbb{N}$ be such that $n\alpha = 0$. Then $n(-\alpha) = 0$, and thus we may assume that $\alpha \ge 0$. If $\alpha > 0$, then it follows that $n\alpha \ge \alpha > 0$, a contradiction. Hence $\alpha = 0$ and Γ is torsion-free.

(b) $\Rightarrow$ (c) Let Ω be the set of all subsets $R \subset \Gamma$ such that $R + R \subset R$ and $R \cap -R = \{0\}$. Then $\{0\} \in \Omega$, and the union of every chain in Ω again belongs to Ω. By Zorn's Lemma, Ω contains a maximal element P, and we must prove that $\Gamma = P \cup -P$. Assume to the contrary that there is an element $\gamma \in \Gamma \setminus (P \cup -P)$. Then $\gamma \ne 0$, and we assert that either $P^+ = P \cup \mathbb{N}_0\gamma \in \Omega$ or $P^- \cup N_0(-\gamma) \in \Omega$ (which gives the desired contradiction). Assume the contrary. Then $P^+ \cap -P^+ \supsetneq \{0\}$ and $P^- \cap -P^- \supsetneq \{0\}$, and there exist $p_1,\, p_1',\, p_2,\, p_2' \in P$ and $n_1,\, n_1',\, n_2,\, n_2' \in \mathbb{N}_0$, such that $p_1 + n_1\gamma = -(p_1' + n_1'\gamma) \ne 0$ and $p_2 - n_2\gamma = -(p_2' - n_2'\gamma) \ne 0$. Since $P \cap -P = \{0\}$, we have $n_1 + n_1' > 0$ and $n_2 + n_2' > 0$, and since $(n_1 + n_1')\gamma = -(p_1 + p_1') \in -P$ and $(n_2 + n_2')\gamma = p_2 + p_2' \in P$, we obtain $(n_1 + n_1')(n_2 + n_2')\gamma \in P \cap -P$, a contradiction.

(c) $\Rightarrow$ (a) For $\alpha,\, \beta \in \Gamma$, we define $\alpha \le \beta$ if and only if $\beta - \alpha \in P$. Then $(\Gamma, \le)$ is a totally ordered additive abelian group and $\Gamma_+ = P$. □

Theorem 3.4.4. *Let D be a valuation monoid, $P \subset D$ a prime ideal, $Q \subset D$ an ideal and*

$$Q_0 = \bigcap_{n \in \mathbb{N}} Q^n\,.$$

1. *Q_0 and $\sqrt{Q}$ are prime ideals.*
2. *If Q is P-primary and $a \in D \setminus P$, then $Q = Qa$. In particular, if Q is P-primary and principal, then $P = D \setminus D^\times$.*

3. *If* $Q_1,\, Q_2 \subset D$ *are* P*-primary ideals, then* Q_1Q_2 *is* P*-primary. In particular,* P^m *is* P*-primary for all* $m \in \mathbb{N}$.
4. *If* Q *is* P*-primary and* $P \ne P^2$*, then* $Q = P^n$ *for some* $n \in \mathbb{N}$.
5. *If* $P = D \setminus D^\times$ *and* $P \ne P^2$*, then* $P = pD$ *for some* $p \in D^\bullet$.

Proof. 1. If $a,\, b \in D \setminus Q_0$, then there exist $m,\, n \in \mathbb{N}$ such that $a \notin Q^m$ and $b \notin Q^n$. Hence it follows that $Q^m \subsetneq Da$, $Q^n \subsetneq Db$, $Q^m b \subsetneq Dab$, $Q^{m+n} \subset Q^m b \subsetneq Dab$, and therefore $ab \notin Q^{m+n}$. Hence $ab \notin Q_0$, and thus Q_0 is a prime ideal.

Since $\mathcal{P}(Q)$ is a chain, it follows that $|\mathcal{P}(Q)| = 1$, and if $\mathcal{P}(Q) = \{P_0\}$, then $\sqrt{Q} = P_0$.

2. Since $a \notin P$, we obtain $Q \subset P \subset aD$, hence $A = a^{-1}Q \subset D$ and $Q = aA$. Since $a \notin P$, it follows that $A \subset Q = aA \subset A$, hence $A = Q$ and $Q = aQ$.

Assume now that $Q = qD$ for some $q \in D$. If $a \in D \setminus D^\times$, then $qD = aqD$ implies $a \in D^\times$, and therefore we obtain $P = D \setminus D^\times$.

3. By Theorem 1.3.2 we have $\sqrt{Q_1Q_2} = P$. Suppose that $a,\, b \in D$, $ab \in Q_1Q_2$ and $a \notin P$. Then $Q_1 = Q_1 a$ by 1., hence $ab \in aQ_1Q_2$ and therefore $b \in Q_1Q_2$. Hence Q_1Q_2 is P-primary.

4. We prove first that $P^m \subset Q$ for some $m \in \mathbb{N}$. Assume the contrary. Then $Q \subset P^m$ for all $m \in \mathbb{N}$, hence

$$Q \subset P_0 = \bigcap_{m \in \mathbb{N}} P^m \,.$$

Since P_0 is a prime ideal by 1., we obtain $P = \sqrt{Q} \subset P_0 \subset P^2 \subsetneq P$, a contradiction. Let now $n \in \mathbb{N}$ be minimal such that $P^n \subset Q$, and let $y \in P^{n-1} \setminus Q$. Then $Q \subset yD$ and $A = y^{-1}Q \in \mathcal{I}_s(D)$. Since $Q = yA$ and $y \notin Q$, we obtain $A \subset P$ and therefore $Q = yA \subset yP \subset P^n$. Hence $Q = P^n$.

5. If $p \in P \setminus P^2$, then $P^2 \subsetneq pD \subset P$, hence $\sqrt{pD} = P$ and thus pD is P-primary by Theorem 3.2.3.2. Hence $pD = P$ by 4. □

Theorem 3.4.5. *Let* D *be a valuation monoid,* $K \ne D$ *and* $M = D \setminus D^\times$.

1. *If* M *is not a principal ideal of* D*, then* $M^{-1} = M_v = D$.
2. *If* $\emptyset \ne X \subset D$*, then*

$$X_v = \begin{cases} aD & \text{if } X_s = aM \text{ and } M \text{ is not principal,} \\ X_s & \text{otherwise}. \end{cases}$$

3. *If* M *is principal, then* $v = s$ *is the only ideal system of* D. *If* M *is not principal, then* $v \ne s$*,* v *and* s *are the only ideal systems of* D *defined on* K*, and* $s = t$. *In any case,* $t = s$ *is the only finitary ideal system of* D *defined on* K.

Proof. 1. Suppose that there is some $z \in M^{-1} \setminus D$. Then it follows that $z^{-1} \in D \setminus D^\times = M$, hence $Mz \subset D$ and $M \subset Dz^{-1} \subset M$, which implies that $M = Dz^{-1}$ is principal. Consequently, if M is not principal, then $M^{-1} = D$ and thus $M_v = (M^{-1})^{-1} = D$.

2. If $\emptyset \ne X \subset D$, $a \in D$, $X_s = aM$ and M is not principal, then $X_v = (X_s)_v = aM_v = aD$ by 1.

Assume now that $X_s \ne X_v$. Then $X_s \subsetneq X_v$, we fix an element $a \in X_v \setminus X_s$, and we assert that $X_s = aM$ and M is not principal.

As $aD \not\subset X_s$, we obtain $X_s \subsetneq aD$, hence $a^{-1}X_s \subsetneq D$, and as $a^{-1}X_s \subset D$ is an ideal, it follows that $a^{-1}X_s \subset M$ and $X_s \subset aM$. If $X_s \subsetneq aM$, then there is some $c \in M$ such that $ac \notin X_s$, hence $X_s \subset acD$ and therefore $aD \subset X_v = (X_s)_v \subset acD$, which implies $c \in D^\times$, a contradiction. Therefore we obtain $X_s = aM$. If M were principal, say $M = pD$ for some $p \in D$, then $X_s = aM = apD$ and therefore $X_v = (X_s)_v = apD = X_s$.

3. Let $r \colon \mathbb{P}(K) \to \mathbb{P}(K)$ be an ideal system of D. Then $s \le r \le v$, and $M_r \in \{M, D\}$. We assert that $r = s$ if $M_r = M$, and $r = v$ if $M_r = D$. Indeed, let $X \subset D$ be any subset such that $X_s \ne X_v$. Then

$X_s = aM$, hence $X_r = aM_r$, and the assertion follows by 2. Consequently, if M is a principal ideal, then $M_r = M$ and therefore $r = s$. If M is not principal, then $v \neq s$ and $r \in \{s, v\}$. □

Theorem und Definition 3.4.6. *Let D be cancellative, $K \neq D$ and $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$ a finitary ideal system of D.*

1. *The following assertions are equivalent*:
   (a) *Every $q \in D \setminus D^\times$ the principal ideal qD is primary.*
   (b) *For all $a \in D \setminus D^\times$ and $b \in D^\bullet$ there is some $n \in \mathbb{N}$ such that $b \,|\, a^n$.*
   (c) *$D \setminus D^\times$ is the only non-zero prime ideal of D.*
   (d) *Every ideal $J \subsetneq D$ is primary.*

   If these conditions are fulfilled, then D is called *primary*.
2. *If D is primary, then D is r-local.*
3. *If $P \in r\text{-spec}(D)$ and $P^\bullet \neq \emptyset$, then D_P is primary if and only if $P \in \mathfrak{X}(D)$.*
4. *Let $T \subset D^\bullet$ be a multiplicatively closed subset such that $T^{-1}D$ is primary. Then $T^{-1}D = D_P$ for some $P \in \mathfrak{X}(D)$.*

Proof. 1. (a) ⇒ (b) If $a \in D \setminus D^\times$ and $b \in D^\bullet$, then $ab \notin D^\times$, hence abD is a primary ideal, $ab \in abD$ and $b \notin abD$. Hence there is some $n \in \mathbb{N}$ such that $a^{n+1} \in abD$, which implies $b \,|\, a^n$.

(b) ⇒ (c) Let $b \in D^\bullet \setminus D^\times$ and $P \in r\text{-spec}(D)$ be such that $b \in P$. Then $P \subset D \setminus D^\times$, and we assert that equality holds. If $a \in D \setminus D^\times$, then there exists some $n \in \mathbb{N}$ such that $b \,|\, a^n$, hence $a^n \in P$ and thus $a \in P$. Hence $P = D \setminus D^\times$.

(c) ⇒ (d) If $J \subsetneq D$ is an ideal, then Theorem 1.3.2.3 implies

$$\sqrt{J} = \bigcap_{P \in \mathcal{P}(J)} P = D \setminus D^\times \in s\text{-}\max(D)\,,$$

and thus J is primary by Theorem 3.2.3.2.

(d) ⇒ (a) Obvious.

2. Obvious by 1.

3. By Theorem 1.3.6.2.

4. Let $\overline{T}$ be the saturation of T. Then $P = D \setminus \overline{T} \in s\text{-spec}(D)$, $T^{-1}D = D_P$ and the assertion follows by 3. □

Theorem 3.4.7. *Let D be a valuation monoid and $K \neq D$. Then the following assertions are equivalent*:

(a) *D is primary.*
(b) *There is an additive subgroup $\Gamma \subset \mathbb{R}$ such that $D^\bullet/D^\times \cong \Gamma_+$.*
(c) *There is no monoid B such that $D \subsetneq B \subsetneq K$.*

Proof. (a) ⇒ (b) If D is primary, then $D/D^\times$ is also primary. Hence we may assume that D is reduced, and it suffices to prove that there is an additive subgroup $\Gamma \subset \mathbb{R}$ and an isomorphism $\widetilde{\Phi}\colon K^\times \to \Gamma$ such that $\Phi(D^\bullet) = \Gamma_+$.

We fix an element $a_0 \in D' = D^\bullet \setminus \{1\}$, and for $a \in D^\bullet$, we define

$$M(a) = \Big\{ \frac{m}{n} \;\Big|\; m \in \mathbb{N}_0\,,\; n \in \mathbb{N}\,,\; a_0^m \,|\, a^n \Big\} \subset \mathbb{Q}_{\ge 0}\,.$$

We assert that, for every $a \in D^\bullet$, the set $M(a)$ is bounded, $0 \in M(a)$, and $M(a) = \{0\}$ if and only if $a = 1$. Indeed, we obviously have $M(1) = \{0\}$, and $0 \in M(a)$ for all $a \in D$. Thus let $a \in D'$. As D is primary, there exist $k,\, l \in \mathbb{N}$ such that $a_0 \,|\, a^k$ and $a \,|\, a_0^l$. Hence $\frac{1}{k} \in M(a)$, and if $\frac{m}{n} \in M(a)$ for some $m,\, n \in \mathbb{N}$, then $a_0^m \,|\, a^n \,|\, a_0^{nl}$, hence $m \le nl$ and therefore $0 < \sup M(a) \le l$. Now we define

$$\Phi\colon D^\bullet \to \mathbb{R}_{\ge 0} \quad \text{by} \quad \Phi(a) = \sup M(a)\,.$$

Then $\Phi(a) = 0$ if and only if $a = 1$. We prove first that Φ is a homomorphism. Let $a_1,\, a_2 \in D$, $n \in \mathbb{N}$, and for $i \in \{1,2\}$, let $m_i \in \mathbb{N}_0$ be such that $a_0^{m_i} \,|\, a_i^n \,|\, a_0^{m_i+1}$. Then $a_0^{m_1+m_2} \,|\, (a_1a_2)^n \,|\, a_0^{m_1+m_2+2}$, hence

$$\frac{m_1}{n} \le \Phi(a_1) \le \frac{m_1+1}{n}\,, \quad \frac{m_2}{n} \le \Phi(a_2) \le \frac{m_2+1}{n} \quad \text{and} \quad \frac{m_1+m_2}{n} \le \Phi(a_1a_2) \le \frac{m_1+m_2+2}{n}\,,$$

and therefore

$$|\Phi(a_1) + \Phi(a_2) - \Phi(a_1a_2)| \le \frac{2}{n}\,.$$

As $n \to \infty$, we obtain $\Phi(a_1a_2) = \Phi(a_1) + \Phi(a_2)$. If $a_1,\, a_2 \in D$ and $a_2 \,|\, a_1$, then $a_1a_2^{-1} \in D$, and $\Phi(a_1) = \Phi(a_1a_2^{-1}) + \Phi(a_2) \ge \Phi(a_2)$.

Let $\widetilde{\Phi}\colon K^\times \to \mathbb{R}$ be the extension of Φ to a homomorphism of the quotient groups, given by $\widetilde{\Phi}(a_1a_2^{-1}) = \Phi(a_1) - \Phi(a_2)$ for all $a_1,\, a_2 \in D^\bullet$. If $a \in \operatorname{Ker}(\widetilde{\Phi}) \cap D^\bullet$, then $\Phi(a) = 0$ and thus $a = 0$. If $a \in \operatorname{Ker}(\widetilde{\Phi}) \setminus D^\bullet$, then $a^{-1} \in \operatorname{Ker}(\widetilde{\Phi}) \cap D^\bullet$ and thus again $a = 0$. Hence $\widetilde{\Phi}$ is a monomorphism, $\widetilde{\Phi}(K^\times) \subset \mathbb{R}$ is a subgroup, $\widetilde{\Phi}\colon K^\times \to \Gamma = \Phi(K^\times)$ is an isomorphism, $\Phi(D^\bullet) \subset \Gamma_+$, and it remains to prove equality. Let $a = a_1a_2^{-1} \in K^\times$ be such that $\Phi(a_1) - \Phi(a_2) = \widetilde{\Phi}(a) > 0$. Then $a_2 \,|\, a_1$ and therefore $a = a_1a_2^{-1} \in D^\bullet$.

(b) $\Rightarrow$ (c) Let $\Gamma \subset \mathbb{R}$ be a subgroup, $\Phi\colon D^\bullet/D^\times \overset{\sim}{\to} \Gamma_+$ an isomorphism and $\widetilde{\Phi}\colon K^\times/D^\times \overset{\sim}{\to} \Gamma$ its extension to an isomorphism of the quotient groups. Let B be a monoid such that $D \subsetneq B \subset K$. Then $D^\bullet/D^\times \subsetneq B^\bullet/D^\times \subset K^\times/D^\times$, and if $\Delta = \widetilde{\Phi}(B^\bullet/D^\times)$, then $\Gamma_+ \subsetneq \Delta \subset \Gamma$. It is now sufficient to prove that $\Gamma = \Delta$, for then it follows that $B^\bullet/D^\times = K^\times/D^\times$ and therefore $B = K$.

We fix an element $a \in \Delta \setminus \Gamma_+$. If $c \in \Gamma$, then $-a > 0$ implies that there is some $n \in \mathbb{N}$ such that $-c \le n(-a)$, hence $c - na \in \Gamma_+$ and $c = (c - na) + na \in \Delta$.

(c) $\Rightarrow$ (a) Let $P \subset D$ be a prime ideal such that $P^\bullet \ne \emptyset$. If $a \in P^\bullet$, then $a^{-1} \notin D_P$. Hence $D \subset D_P \subsetneq K$, which implies $D = D_P$ and therefore $P = D \setminus D^\times$. Consequently, $D \setminus D^\times$ is the only non-zero prime ideal of D, and thus D is primary. □

Theorem und Definition 3.4.8. *Let D be cancellative, $K \ne D$ and $P = D \setminus D^\times$. Then the following assertions are equivalent*:

(a) *D is factorial, and there is some $p \in D$ such that $\{p\}$ is a complete set of primes* [*equivalently*: *There is some $p \in D$ such that every $a \in D^\bullet$ has a unique representation $a = p^n u$, where $n \in \mathbb{N}_0$ and $u \in D^\bullet$*].

(b) *D is an atomic valuation monoid.*

(c) *D is atomic and P is a principal ideal.*

(d) *D is primary and contains a prime element.*

(e) *D is a valuation monoid, and*

$$\bigcap_{n \in \mathbb{N}} P^n = \{0\}\,.$$

(f) *D is an s-noetherian valuation monoid.*

(g) *D is a v-noetherian valuation monoid.*

If these conditions are fulfilled, then D is called a *discrete valuation monoid* or a *dv-monoid.*

Proof. (a) $\Rightarrow$ (b) Obvious.

(b) $\Rightarrow$ (c) If q_1, $q_2 \in D^\bullet$ are atoms, then $q_1 D \subset q_2 D$ or $q_2 D \subset q_1 D$, since D is a valuation monoid. Hence $q_1 D = q_2 D$, and thus D possesses up to associates precisely one atom. If $q \in D$ is an atom, then $D = \{q^n u \mid n \in \mathbb{N}_0\,,\ u \in D^\times\}$, and therefore $P = qD$.

(c) $\Rightarrow$ (d) Let $p \in D^\bullet$ be such that $P = pD$. Then p is a prime element. Let $Q \subset D$ be a prime ideal, $a \in Q^\bullet$ and $a = u_1 \cdot \ldots \cdot u_m$, where $m \in \mathbb{N}$ and $u_1, \ldots, u_m$ are atoms of D. For every $j \in [1, m]$ we have $u_j \in D \setminus D^\times = pD$, hence $u_j D = pD$ and $u_j = pe_j$ for some $e_j \in D^\times$. Then $e = e_1 \cdot \ldots \cdot e_m \in D^\times$ and $e^{-1} a = p^m \in Q$. Hence it follows that $p \in Q$, and therefore $Q = P$ is the only non-zero prime ideal of D.

(d) $\Rightarrow$ (e) Let $p \in D$ be a prime element, and assume to the contrary that there is some $a \in D^\bullet$ such that $a \in P^n$ for all $n \in \mathbb{N}$. As D is primary, we obtain $P = pD$, and there exists some $m \in \mathbb{N}$ such that $a \mid p^m$. Since $a \in P^{m+1} = p^{m+1}D$, it follows that $p^{m+1} \mid a \mid p^m$, a contradiction.

If a, $b \in D^\bullet$, let m, $n \in \mathbb{N}_0$ be maximal such that $a \in p^m D$ and $b \in p^n D$, say $a = p^m u$ and $b = p^n v$, where u, $v \in D^\times$, and suppose that $m \le n$. Then $b = ap^{n-m} v u^{-1} \in aD$, which implies that D is a valuation monoid.

(e) $\Rightarrow$ (f) By (e) we have $P \ne P^2$ and thus $P = pD$ for some $p \in D$ by Theorem 3.4.4.5. We prove that every ideal of D is principal. Let $\{0\} \ne J \subset D$ be an ideal, and let $n \in \mathbb{N}_0$ be maximal such that $J \subset P^n = p^n D$. If $y \in J \setminus P^{n+1}$, then $y = p^n u$, where $u \in D^\times$, hence $p^n D = yD \subset J \subset p^n D$, and $J = yD$.

(f) $\Rightarrow$ (g) Obvious.

(g) $\Rightarrow$ (a) Since $v = t = s$, it follows that every t-ideal of D is finitely generated and thus principal. Hence D is factorial, and $P = pD$ for some prime element $p \in D$. If $q \in D$ is any prime element, then $q \in D \setminus D^\times = pD$, hence $qD = pD$, and therefore $\{p\}$ is a complete set of primes. $\square$

Theorem 3.4.9. *Let D be a* GCD*-monoid, $t = t(D)$ and $V \subset K$ a submonoid.*

1. *Let V be a valuation monoid of K and r a finitary module system on K. Then the following assertions are equivalent:*
   (a) *$V = V_r$.*
   (b) *id_K is an $(r, t(V))$-homomorphism.*
   (c) *$X_r \subset XV$ for all $X \subset K$.*
2. *The following assertions are equivalent:*
   (a) *V is a t-valuation monoid.*
   (b) *V is a valuation monoid, $D \subset V$, and the inclusion map $D \hookrightarrow V$ is a* GCD*-homomorphism.*
   (c) *$V = D_P$ for some $P \in t\text{-spec}(D)$.*
3. *For every subset $X \subset K$ we have*

$$X_t = \bigcap_{P \in t\text{-spec}(D)} XD_P = \bigcap_{P \in t\text{-}\max(D)} XD_P = \bigcap_{V \in \mathcal{V}} XV\,.$$

   where $\mathcal{V}$ is the set of all t-valuation monoids of K.

Proof. 1. (a) $\Rightarrow$ (b) We must prove that $E_r \subset E_{t(V)}$ for all $E \in \mathbb{P}_{\mathfrak{f}}(K)$. If $E^\bullet = \emptyset$, this is obvious. If $E \in \mathbb{P}_{\mathfrak{f}}(K)$ and $E^\bullet \ne \emptyset$, then $E_{t(V)} = E_{s(V)} = EV = aV$ for some $a \in E$ by the Theorems 3.4.2.2 (c) and 3.4.5. Now $E \subset aV$ implies $E_r \subset (aV)_r = aV$.

(b) $\Rightarrow$ (c) If $X \subset V$, then $X_r \subset X_{t(V)} = X_{s(V)} = XV$ by Theorem 3.4.5.

(c) $\Rightarrow$ (a) $V_r \subset VV = V$ implies $V_r = V$.

2. (a) $\Rightarrow$ (b) By 1., id_K is a $(t, t(V))$-homomorphism, and thus Theorem 2.6.5 implies that $D \subset V$ and $D \hookrightarrow V$ is a GCD-homomorphism.

(b) ⇒ (c) By Theorem 2.6.5, id_K is a $(t, t(V))$-homomorphism. Since $s(V) = t(V)$, it follows that $V \setminus V^\times \in \mathcal{M}_t(K)$, hence $P = D \cap (V \setminus V^\times) \in t\text{-spec}(D)$, and obviously $D_P \subset V$. To prove the reverse inclusion, let $z = a^{-1}b \in V$, where $a, b \in D$ and $\mathrm{GCD}_D(a,b) = D^\times$. Since $D \hookrightarrow V$ is a GCD-homomorphism, we obtain $\mathrm{GCD}_V(a,b) = V^\times$ and thus either $a \in V^\times$ or $b \in V^\times$. If $a \in V^\times$, then $a \notin P$ and $z \in D_P$. If $b \in V^\times$, then $b \in aV$ implies $a \in V^\times$ and again $z \in D_P$.

(c) ⇒ (a) By Theorem 2.5.4.1 we have $(D_P)_t = (D_t)_P = D_P$. It remains to prove that D_P is a valuation monoid. Thus let $z \in K$, say $z = a^{-1}b$, where $a, b \in D$ and $\mathrm{GCD}(a,b) = D^\times$. Then $\{a,b\}_t = D$, hence $\{a,b\} \not\subset P$. If $a \notin P$, then $z \in D_P$, and if $b \notin P$, then $z^{-1} \notin P$.

3. If $P \in t\text{-spec}(D)$, then t_P is a finitary D_P-module system on K, hence $t_P = s(D_P) = s_P$ by Theorem 3.4.5, and for every subset $X \subset K$ we have $X_t \subset (X_t)_P = X_{t_P} = XD_P$. By Theorem 3.2.2 we obtain

$$X_t \subset \bigcap_{P \in t\text{-spec}(D)} XD_P \subset \bigcap_{P \in t\text{-}\max(D)} XD_P = \bigcap_{P \in t\text{-}\max(D)} (X_t)_P = X_t \,,$$

and the assertion follows. □

Theorem 3.4.10. *Let $\varepsilon \colon K \to K'$ be a homomorphism of divisible monoids, r' a module system on K' and $r = \varepsilon^* r'$.*

1. *If V' is an r'-valuation monoid of K', then $\varepsilon^{-1}(V')$ is an r-valuation monoid of K.*
2. *Let ε be surjective. Then the assignment $V \mapsto \varepsilon(V)$ defines a bijective map from the set of all r-valuation monoids of K onto the set of all r'-valuation monoids of K'.*

Proof. 1. Let V' be an r'-homomorphism of K'. If $x \in K \setminus \varepsilon^{-1}(V')$, then $\varepsilon(x) \in K' \setminus V'$, hence $\varepsilon(x^{-1}) = \varepsilon(x)^{-1} \in V'$ and $x^{-1} \in \varepsilon^{-1}(V')$. Hence $\varepsilon^{-1}(V')$ is a valuation monoid of K and, by Theorem 2.3.6, it is an r-valuation monoid.

2. Let $V \subset K$ be an r-valuation monoid and $x' \in K' \setminus \varepsilon(V)$. Then $x' = \varepsilon(x)$ for some $x \in K \setminus V$. Hence we obtain $x^{-1} \in V$ and $x'^{-1} = \varepsilon(x)^{-1} = \varepsilon(x^{-1}) \in \varepsilon(V)$. Hence $\varepsilon(V)$ is a valuation monoid of K', and since $V = V_r = \varepsilon^{-1}(\varepsilon(V)_{r'})$, it follows that $\varepsilon(V) = \varepsilon(V)_{r'}$ and $V = \varepsilon^{-1}(\varepsilon(V))$.

Conversely, if V' is an r'-valuation monoid of K', then $\varepsilon^{-1}(V')$ is an r-valuation monoid of K by 1., and $V' = \varepsilon(\varepsilon^{-1}(V'))$. □

3.5. Valuation domains

In this Section, we use the common terminology of commutative ring theory.

A domain D is called a *valuation domain* if its multiplicative monoid is a valuation monoid, and if $K = \mathsf{q}(D)$, then D is called a valuation domain of K. In this case, the totally ordered abelian group $\mathcal{G}(D) = K^\times / D^\times$ is called the *value group* of D.

Theorem 3.5.1. *A domain D is a valuation domain if and only if $d(D) = s(D)$.*

Proof. If D is a valuation domain, then $s(D) = t(D)$, and as $s(D) \le d(D) \le t(D)$, we obtain $s(D) = d(D)$. Conversely, assume that $s(D) = d(D)$, and let $a, b \in D$. Then it follows that

$$a + b \in \mathcal{I}_{d(D)}(D) = \mathcal{I}_{s(D)}(D) = aD \cup bD \,,$$

say $a + b \in aD$. Consequently, $a + b = ax$ for some $x \in D$, and $b = a(x-1) \in aD$. Hence D is a valuation domain. □

Remarks and Definition 3.5.2. Let $(\Gamma, \le)$ be a totally ordered additive abelian group. We consider the extension $\Gamma \uplus \{\infty\}$, where $\alpha \le \infty = \alpha + \infty$ for all $\alpha \in \Gamma$.

1. Let K be a field. A *valuation* of K (with value group Γ) is a surjective map $v\colon K \to \Gamma \cup \{\infty\}$, such that for all $a,\, b \in K$ the following assertions hold:

   V1. $v(a) = \infty$ if and only if $a = 0$.

   V2. $v(ab) = v(a) + v(b)$.

   V3. $v(a+b) \ge \min\{v(a), v(b)\}$.

   Consequences: *If $a,\, b \in K$, then $v(-a) = v(a)$, $v(a-b) \ge \min\{v(a),\, v(b)\}$, and if $v(a) < v(b)$, then $v(a+b) = v(a)$.*

   Proof: $v \,|\, K^\times \colon K^\times \to \Gamma$ is a homomorphism. Hence $2v(-1) = v((-1)^2) = v(1) = 0$, $v(-1) = 0$, $v(-a) = v(-1) + v(a) = v(a)$, and $v(a-b) \ge \min\{v(a), v(-b)\} = \min\{v(a), v(b)\}$. If $v(a) < v(b)$, then $v(a) = v((a+b)-b) \ge \min\{v(a+b), v(b)\} = v(a+b)$.

   If v is a valuation of K, then $\mathcal{O}_v = \{a \in K \mid v(a) \ge 0\}$ is a valuation domain with maximal ideal $\mathfrak{p}_v = \{a \in K \mid v(a) > 0\} = \mathcal{O}_v \setminus O_v^\times$, and v induces an isomorphism $K^\times/\mathcal{O}_v^\times \overset{\sim}{\to} \Gamma$.

   We call (K, v) a *valued field*, $\mathcal{O}_v$ the *valuation domain*, $\mathfrak{p}_v$ the *valuation ideal* and $\mathcal{O}_v/\mathfrak{p}_v$ the *residue field* of (K, v).

2. Let D be a valuation domain, $K = \mathsf{q}(D)$ and $w\colon K^\times \to \Gamma$ a valuation morphism of D. We set $w(0) = \infty$. Then $w\colon K \to \Gamma \cup \{\infty\}$ is a valuation of K, and $\mathcal{O}_w = D$.

   Proof: Since $D = \{x \in K \mid w(x) \ge 0\}$, it suffices to prove that $w(x+y) \ge \min\{w(x), w(y)\}$ for all $x,\, y \in K$. Thus let $x,\, y \in K$, and assume that $w(x) \ge w(y)$. If $y = 0$, then $x = 0$, and there is nothing to do. If $y \ne 0$, then $w(y^{-1}x) = -w(y) + w(x) \ge 0$, hence $y^{-1}x \in D$ and therefore also $1 + y^{-1}x \in D$. But this implies $w(x+y) = w(y(1+y^{-1}x)) = w(y) + w(1+y^{-1}x) \ge w(y)$. □

3. Let D be a ring and $v_0\colon D \to \Gamma_+ \cup \{\infty\}$ a surjective map satisfying **V1**, **V2**, **V3** for all $a,\, b \in D$. Then D is a domain. If $K = \mathsf{q}(D)$, then there exists a unique valuation $v\colon K \to \Gamma \cup \{\infty\}$ such that $v \,|\, D = v_0$. It is given by $v(a^{-1}b) = v_0(b) - v_0(a)$ for all $a \in D^\bullet$ and $b \in D$.

Theorem und Definition 3.5.3. *Let K be a field, $K[X]$ a polynomial domain and $v\colon K \to \Gamma \cup \{\infty\}$ a valuation. Then there is a unique valuation $v^*\colon K(X) \to \Gamma \cup \{\infty\}$ such that, for all $f \in K[X]$,*

$$f = \sum_{i \ge 0} a_i X^i \ \ (\,\textit{where}\ \ a_i \in K,\ \ a_i = 0\ \ \textit{for almost all}\ \ i \ge 0\,) \quad \textit{implies} \quad v^*(f) = \min\{v(a_i) \mid i \ge 0\}\,.$$

v^* is called the trivial extension of v.

Proof. It suffices to prove that $v^* \,|\, K[X]$ satisfies **V1**, **V2**, **V3** for all $f,\, g \in K[X]$. **V1** is obvious. Suppose that

$$f = \sum_{i \ge 0} a_i X^i \quad \text{and} \quad g = \sum_{i \ge 0} b_i X^i\,, \quad \text{where} \quad a_i,\, b_i \in K\,, \quad a_i = b_i = 0 \quad \text{for almost all} \quad i \ge 0.$$

V2. By definition,

$$\begin{aligned} v^*(f+g) &= \min\{v(a_i + b_i) \mid i \ge 0\} \ge \min\{\min\{v(a_i),\, v(b_i)\} \mid i \ge 0\} \\ &= \min\bigl\{\min\{v(a_i) \mid i \ge 0\}\,, \min\{v(b_i) \mid i \ge 0\}\bigr\} = \min\{v^*(f), v^*(g)\}\,. \end{aligned}$$

V3. We may assume that $fg \ne 0$ and $k,\, l \in \mathbb{N}_0$ are such that $v^*(f) = v(a_k) < v(a_i)$ for all $i > k$, and $v^*(g) = v(b_l) < v(b_i)$ for all $i > l$. Then we have $v(a_i) \ge v(a_k)$ for all $i \ge 0$ and $v(b_i) \ge v(b_l)$ for all $i \ge 0$. We set

$$fg = \sum_{i \ge 0} c_i X^i\,, \quad \text{where} \quad c_i = \sum_{\nu=0}^{i} a_\nu b_{i-\nu}\,, \quad \text{and in particular} \quad c_{k+l} = a_k b_l + \sum_{\nu=1}^{k+l} a_\nu b_{k+l-\nu}\,.$$

Hence $v(c_i) \ge \min\{v(a_\nu) + v(b_{i-\nu}) \mid \nu \in [0,i]\} \ge v(a_k) + v(b_l)$ for all $i \ge 0$, and $v(c_{k+l}) = v(a_k) + v(b_l)$, since $v(a_\nu b_{k+l-\nu}) = v(a_\nu) + v(b_{k+l-\nu}) > v(a_k + v(b_l)$ for all $\nu \in [1, k+l]$. Therefore we obtain $v^*(fg) = v(a_k) + v(b_l) = v^*(f) + v^*(g)$. □

Theorem 3.5.4. *Let k be a field and $(\Gamma, \le)$ an ordered additive abelian group. Then there exists a valued field (K, v) with value group Γ and residue field k.*

Proof. We consider the semigroup ring $D = k[\Gamma_+, X]$, consisting of all sums

$$a = \sum_{\gamma \in \Gamma_+} a_\gamma X^\gamma\,, \quad \text{where} \quad a_\gamma \in k\,,\ a_\gamma = 0 \ \text{ for almost all } \ \gamma \in \Gamma_+\,,$$

and we set

$$v_0(a) = \min\{\gamma \in \Gamma_+ \mid a_\gamma \ne 0\} \in \Gamma_+ \ \text{ if } \ a \ne 0\,, \quad \text{and} \quad v_0(0) = \infty\,.$$

Then $v_0 \colon D \to \Gamma_+ \cup \{\infty\}$ is a surjective map satisfying **V1**, **V2** **V3** for all $a, b \in D$. By 3.4.2.3, D is a domain. If $K = \mathsf{q}(D)$, then there exists a unique valuation $v \colon K \to \Gamma \cup \{\infty\}$ such that $v \mid D = v_0$. It remains to prove that k is the residue field of (K, v).

If $\mathfrak{p} = \{a \in D \mid v(a) > 0\}$, then $\mathfrak{p} \in \operatorname{spec}(D)$ and $D = k + \mathfrak{p}$. Every $z \in K^\times$ has a representation

$$z = X^\gamma \frac{a+p}{1+q}\,, \quad \text{where} \quad \gamma \in \Gamma\,,\ a \in k\,,\ p, q \in \mathfrak{p}\,, \quad \text{and then} \quad v(z) = \gamma\,.$$

In particular, we have $z \in \mathcal{O}_v$ if and only if $\gamma \ge 0$, and therefore $\mathcal{O}_v = D_\mathfrak{p}$. Hence $\mathfrak{p}_v = \mathfrak{p} D_\mathfrak{p}$, and $\mathcal{O}_v / \mathfrak{p}_v = D/\mathfrak{p} = k$. □

Theorem 3.5.5. *Let K be a field, $D \subset K$ a subring and $P \subset D$ a prime ideal. Then there exists a valuation domain V of K such that $D \subset V$ and $P = D \setminus V^\times$.*

The proof requires the following Lemma from Commutative Algebra.

Lemma 3.5.6 (The (u, u^{-1})-Lemma)**.** *Let $R \subset S$ be rings, $u \in S^\times$, $I \lhd R$ and $b \in IR[u] \cap IR[u^{-1}]$. Then there exist some $k \in \mathbb{N}$ and $r_0, \ldots, r_{k-1} \in I$ such that $b^k + r_{k-1}b^{k-1} + \ldots + r_1 b + r_0 = 0$. In particular, if $I \ne R$, then $IR[u] \ne R[u]$ or $IR[u^{-1}] \ne R[u^{-1}]$.*

Proof of the Lemma. Suppose that $b = a_0 + a_1 u + \ldots + a_n u^n = c_0 + c_1 u^{-1} + \ldots + c_m u^{-m}$, where $m, n \in \mathbb{N}$ and $a_0, \ldots, a_n, c_0, \ldots, c_m \in I$. We set $M = R + Ru + \ldots + Ru^{n+m-1}$, and we assert that $bM \subset IM$. Indeed,

$$bu^l = \sum_{i=0}^n a_i u^{i+l} \ \text{ for } \ l \in [0, m-1]\,, \quad \text{and} \quad bu^l = \sum_{j=0}^m c_j u^{-j+l} \ \text{ for } \ l \in [m, m+n-1]\,.$$

In particular, for every $i \in [0, m+n-1]$, there is a relation of the form

$$bu^i = \sum_{j=0}^{m+n-1} d_{i,j} u^j\,, \quad \text{where} \quad d_{i,j} \in I\,, \quad \text{and therefore} \quad \sum_{j=0}^{m+n-1} (b\delta_{i,j} - d_{i,j}) u^j = 0\,,$$

which implies $\det(b\delta_{i,j} - d_{i,j})_{i,j \in [0, m+n-1]} u^l = 0$ for all $l \in [0, m+n-1]$, and therefore, as $u \in S^\times$, $0 = \det(b\delta_{i,j} - d_{i,j})_{i,j \in [0,m+n-1]} = b^{m+n-1} + r_{m+n-2} b^{m+n-2} + \ldots + r_1 b + r_0$, where $r_i \in I$ for all $i \in [0, m+n-2]$.

If $IR[u] = R[u]$ and $IR[u^{-1}] = R[u^{-1}]$, then $1 \in IR[u] \cap IR[u^{-1}]$, and the above relation implies $1 \in I$ and thus $I = R$. □

Proof of the Theorem. Let Ω be the set of all domains W satisfying $D_P \subset W \subset K$ such that $PW \neq W$. Then $D_P \in \Omega$ and the union of every chain in Ω belongs to Ω. Indeed, let $(W_\lambda)_{\lambda\in\Lambda}$ be a chain in Ω,

$$W = \bigcup_{\lambda\in\Lambda} W_\lambda\,, \quad \text{and assume that} \quad 1 \in PW\,.$$

Then $1 = p_1w_1 + \ldots + p_nw_n$ for some $n \in \mathbb{N}$, $p_1, \ldots, p_n \in P$ and $w_1, \ldots, w_n \in W$. Hence there is some $\lambda \in \Lambda$ such that $\{w_1, \ldots, w_n\} \subset W_\lambda$ and $1 \in PW_\lambda$, a contradiction.

By Zorn's Lemma, Ω contains a maximal element V, and we assert that V is a valuation domain of K such that $D \setminus P = D \cap V^\times$. Thus suppose that $z \in K \setminus V$. Then $V[z] \supset V$, and as V is maximal in Ω it follows that $PV[z] = V[z]$. By the (u, u^{-1})-Lemma we obtain $PV[z^{-1}] \neq V[z^{-1}]$ and therefore $z^{-1} \in V$. Hence V is a valuation domain of K, and

$$P = PD_P \cap D \subset PV \cap D \subset D \setminus V^\times = (V \setminus V^\times) \cap D_P \cap D \subset PD_P \cap D = P\,.$$

Hence $P = D \setminus V^\times$. $\square$

CHAPTER 4

Invertibility, Cancellation and Integrality

4.1. Invertibility and class groups

Definition 4.1.1. Let D be a cancellative monoid, $K = \mathsf{q}(D)$ and $r \colon \mathbb{P}(K) \to \mathbb{P}(K)$ an ideal system of D. A fractional r-ideal $J \in \mathcal{F}_r(D)$ is called *r-invertible* if $J \in \mathcal{F}_r(D)^\times$ (equivalently, $J \cdot_r J' = D$ for some $J' \in \mathcal{F}_r(D)$).

If D is a domain, we use the common terminology of Commutative Algebra. In particular, we set $\mathcal{F}(D) = \mathcal{F}_{d(D)}(D)$ and $\mathcal{I}(D) = \mathcal{I}_{d(D)}(D)$. In this case, (fractional) $d(D)$-ideals are called (*fractional*) *ideals*, and they are called *invertible* if they are $d(D)$-invertible.

Theorem 4.1.2. *Let D be a cancellative monoid, $K = \mathsf{q}(D) \ne D$, $r \colon \mathbb{P}(K) \to \mathbb{P}(K)$ an ideal system of D, $v = v(D)$, $t = t(D)$, and for $X \subset K$, let $X^{-1} = (D \colon X)$.*

1. *Let $X, Y \subset K$ be such that $(XY)_r = D$. Then $Y_r = X^{-1} = X_r^{-1}$.*
2. *If $J \in \mathcal{F}_r(D)^\times$, then $J \cdot_r J^{-1} = D$ [hence J^{-1} is the inverse of J in $\mathcal{F}_r(D)$].*
3. *If q is an ideal system of D defined on K such that $r \le q$, then $\mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)^\times$ is a subgroup. In particular, every r-invertible fractional r-ideal is v-invertible, and $\mathcal{F}_r(D)^\times \subset \mathcal{F}_v(D)^\times$ is a subgroup.*
4. *If r is finitary, then $\mathcal{F}_r(D)^\times = \mathcal{F}_{r,\mathsf{f}}(D)^\times$, and $\mathcal{F}_r(D)^\times \subset \mathcal{F}_t(D)^\times$ is a subgroup. In particular, if J is r-invertible, then both J and J^{-1} are r-finitely generated.*
5. *$\mathcal{F}_{r,\mathsf{f}}(D)^\times = \mathcal{F}_{r_\mathsf{f}}(D)^\times$.*
6. *$\mathcal{F}_v(D)^\times = \{J \in \mathcal{F}_v(D)^\bullet \mid (J \colon J) = D\}$.*

Proof. 1. Clearly, $X^{-1} = (D \colon X) = (D \colon X_r) = X_r^{-1}$. Since $XY \subset (XY)_r = D$, it follows that $Y \subset X^{-1}$ and therefore $Y_r \subset X^{-1}$, since $X^{-1} \in \mathcal{M}_v(K) \subset \mathcal{M}_r(K)$. On the other hand, we have $X^{-1} = X^{-1}(XY)_r \subset (X^{-1}XY)_r \subset (DY)_r = Y_r$.

2. Let $J' \in \mathcal{F}_r(D)$ be such that $J \cdot_r J' = (JJ')_r = D$. Then $J' = J'_r = J^{-1}$ by 1.

3. Let q be an ideal system of D such that $r \le q$. If $J \in \mathcal{F}_r(D)^\times$, then $J = (J^{-1})^{-1} = J_v$ and thus $J \in \mathcal{F}_v(D) \subset \mathcal{F}_q(D)$. As $JJ^{-1} \subset D$ and $D = (JJ^{-1})_r \subset (JJ^{-1})_q \subset D$, it follows that $(JJ^{-1})_q = D$ whence $J \in \mathcal{F}_q(D)^\times$. Hence $\mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)^\times$, and it remains to prove that it is a subgroup. Thus let $I, J \in \mathcal{F}_r(D)^\times$. Then $(IJ)_r = I \cdot_r J \in \mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)^\times$, and therefore $I \cdot_q J = (IJ)_q = ((IJ)_r)_q = (I \cdot_r J)_q = I \cdot_r J$.

4. Let r be finitary. Then $r \le t$, and thus $\mathcal{F}_r(D)^\times \subset \mathcal{F}_t(D)^\times$ is a subgroup by 3. As $\mathcal{F}_{r,\mathsf{f}}(D) \subset \mathcal{F}_r(D)$ is a submonoid, it follows that $\mathcal{F}_{r,\mathsf{f}}(D)^\times \subset \mathcal{F}_r(D)^\times$. Thus let $J \in \mathcal{F}_r(D)^\times$. Then

$$1 \in D = J \cdot_r J^{-1} = \Big(\bigcup_{E \in \mathbb{P}_\mathsf{f}(J)} E_r \Big) \cdot_r J^{-1} = \Big(\bigcup_{E \in \mathbb{P}_\mathsf{f}(J)} E_r \cdot_r J^{-1} \Big)_r = \bigcup_{E \in \mathbb{P}_\mathsf{f}(J)} E_r \cdot_r J^{-1},$$

since $\{E_r \cdot_r J^{-1} \mid E \in \mathbb{P}_\mathsf{f}(J)\}$ is directed. Hence there exists some $E \in \mathbb{P}_\mathsf{f}(J)$ such that $1 \in E_r \cdot_r J^{-1} \subset D$ and therefore $E_r \cdot_r J^{-1} = D$, which implies $E_r = (J^{-1})^{-1} = J_v = J \in \mathcal{F}_{r,\mathsf{f}}(D)$. The same argument, applied for J^{-1} instead of J, shows that $J^{-1} \in \mathcal{F}_{r,\mathsf{f}}(D)$, and consequently $J \in \mathcal{F}_{r,\mathsf{f}}(D)^\times$.

5. $\mathcal{F}_{r_{\mathsf{f}}}(D)^{\times} = \mathcal{F}_{r_{\mathsf{f}},\mathsf{f}}(D)^{\times} = \mathcal{F}_{r,\mathsf{f}}(D)^{\times}$.

6. If $J \in \mathcal{F}_v(D)$, then $J = X^{-1}$ for some $X \subset K$, and $(J:J) = (XX^{-1})^{-1} = (X \cdot_v X^{-1})^{-1}$ by Theorem 2.6.2. Hence $(J:J) = D$ if and only if $X \cdot_v X^{-1} = D$. □

Theorem 4.1.3. *Let D be a cancellative monoid, $K = \mathsf{q}(D) \neq D$, $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$ an ideal system of D, and for $X \subset K$, let $X^{-1} = (D:X)$.*

For $I \in \mathcal{F}_r(D)^{\bullet}$, the following assertions are equivalent:

(a) *$I \in \mathcal{F}_r(D)^{\times}$.*

(b) *$I \cdot_r J = (J:I^{-1})$ for all $J \in \mathcal{F}_r(D)$.*

(c) *For all $J \in \mathcal{F}_r(D)$ satisfying $J \subset I$ there exists some $C \in \mathcal{I}_r(D)$ such that $J = I \cdot_r C$.*

Proof. (a) $\Rightarrow$ (b) Let $J \in \mathcal{F}_r(D)$. From $I^{-1}(I \cdot_r J) \subset (I^{-1}IJ)_r = ((I^{-1}I)_r J)_r = J$ we obtain $I \cdot_r J \subset (J:I^{-1})$. Conversely, if $z \in (J:I^{-1})$, then $z \in zD = I \cdot_r zI^{-1} \subset I \cdot_r J$.

(b) $\Rightarrow$ (a) With $J = I^{-1}$, we obtain $1 \in (I^{-1}:I^{-1}) = I \cdot_r I^{-1} \subset D$ and therefore $I \cdot_r I^{-1} = D$.

(a) $\Rightarrow$ (c) Set $C = I^{-1} \cdot_r J \in \mathcal{F}_r(D)$. Then $I \cdot_r C = I \cdot_r I^{-1} \cdot_r J = J$, and since $C \subset I^{-1} \cdot I = D$, we obtain $C \in \mathcal{I}_r(D)$.

(c) $\Rightarrow$ (a) If $a \in I^{\bullet}$, then $aD \subset I$, and there exists some $C \in \mathcal{I}_r(D)$ such that $aD = I \cdot_r C$. Then $a^{-1}C \in \mathcal{F}_r(D)$, and $I \cdot_r (a^{-1}C) = D$, whence $I \in \mathcal{F}_r(D)^{\times}$. □

Theorem 4.1.4. *Let D be a cancellative monoid, $K = \mathsf{q}(D) \neq D$, $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$ a finitary ideal system of D and $t = t(D)$.*

1. *Let D be r-local and $X \subset K$ a D-fractional subset such that X_r is r-invertible. Then there exists some $a \in X$ such that $X_r = aD$. In particular, every r-invertible fractional r-ideal is principal.*
2. *If $J \in \mathcal{F}_r(D)^{\times}$ and $T \subset D^{\bullet}$ is a multiplicatively closed subset, then $T^{-1}J \in \mathcal{F}_{T^{-1}r}(T^{-1}D)^{\times}$.*
3. *For $J \in \mathcal{F}_r(D)^{\bullet}$, the following assertions are equivalent:*

   (a) *J is r-invertible.*

   (b) *$J \in \mathcal{F}_{r,\mathsf{f}}(D)$ and J_P is principal for all $P \in r\text{-}\max(D)$.*

   (c) *$J_t \in \mathcal{F}_{t,\mathsf{f}}(D)$ and J_P is principal for all $P \in r\text{-}\max(D)$.*

Proof. 1. By Corollary 3.1.5 $M = D \setminus D^{\times}$ is he only r-maximal r-ideal of D. Let $X \subset K$ be a D-fractional subset such that X_r is r-invertible. Then $X \not\subset (XM)_r$. Indeed, otherwise it follows that $X_r \subset X_r \cdot_r M$ and therefore $D = X_r^{-1} \cdot_r X_r \subset X_r^{-1} \cdot_r X_r \cdot_r M = M$, a contradiction. If $a \in X \setminus (XM)_r$, then $aX^{-1} \in \mathcal{I}_r(D)$, and we assert that $aX^{-1} \not\subset M$. Indeed, otherwise $a \in aD = a(X^{-1}X)_r \subset (XM)_r$, a contradiction. Hence $aX^{-1} = D$, and $X_r = aX^{-1} \cdot_r X_r = a(X^{-1}X)_r = aD$.

2. Obvious, since the map $\mathcal{F}_r(D) \to \mathcal{F}_{T^{-1}r}(T^{-1}D)$, $J \mapsto T^{-1}J$, is a monoid homomorphism.

3. (a) $\Rightarrow$ (b) If J is r-invertible, then J is r-finitely generated by Theorem 4.1.2.4, J_P is r_P-invertible by 2. and thus J_P principal by 1.

(b) $\Rightarrow$ (c) If $J = E_r$ for some $E \in \mathbb{P}_{\mathsf{f}}(D)$, then $J_t = E_t$.

(c) $\Rightarrow$ (a) Assume that $J \in \mathcal{F}_{t,\mathsf{f}}(D)$ and that for all $P \in r\text{-}\max(D)$ there is some $a_P \in D_P^{\bullet}$ such that $J_P = a_P D_P$. Since $J \in \mathcal{F}_{t,\mathsf{f}}(D)$, we obtain $(J^{-1})_P = (J_P)^{-1} = a_P^{-1} D_P$, and therefore $(J \cdot_r J^{-1})_P = J_P \cdot_{r_P} J_P^{-1} = (a_P D_P) \cdot_{r_P} (a_P^{-1} D_P) = D_P$. Hence $J \cdot_r J^{-1} = D$ by Theorem 3.2.2. □

Remarks and Definition 4.1.5. Let D be a cancellative monoid, $K = \mathsf{q}(D)$, $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$ an ideal system of D, $v = v(D)$ and $t = t(D)$.

1. The map $\partial_r\colon K^\times \to \mathcal{F}_r(D)^\times$, defined by $\partial_r(a) = aD$, is a group homomorphism with kernel $\mathrm{Ker}(\partial_r) = D^\times$. Its cokernel
$$\mathcal{C}_r(D) = \mathcal{F}_r(D)^\times/\partial_r(K^\times)$$
is called the *r-class group* of D, and it is usually written additively. It gives rise to an exact sequence
$$1 \to K^\times/D^\times \to \mathcal{F}_r(D)^\times \to \mathcal{C}_r(D) \to \mathbf{0}\,.$$
2. Let $q\colon \mathbb{P}(K) \to \mathbb{P}(K)$ be an ideal system of D such that $r \le q$. Then $\mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)^\times$ by Theorem 4.1.2.3, and thus also $\mathcal{C}_r(D) \subset \mathcal{C}_q(D)$. In particular, it follows that $\mathcal{C}_r(D) \subset \mathcal{C}_v(D)$, and if r is finitary, then $\mathcal{C}_r(D) \subset \mathcal{C}_t(D)$.
3. Let D be a domain and $d = d(D)$. Then $\mathrm{Pic}(D) = \mathcal{C}_d(D)$ is called the *Picard group* and $\mathcal{C}(D) = \mathcal{C}_t(D)$ is called the *divisor class group* of D.

   By 2. we have $\mathrm{Pic}(D) \subset \mathcal{C}(D)$. The factor group $\mathcal{G}(D) = \mathcal{C}(D)/\mathrm{Pic}(D)$ is called the *local class group* of D. By definition, $\mathcal{G}(D) \cong \mathcal{F}_t(D)^\times/\mathcal{F}(D)^\times$.

Theorem 4.1.6. *Let D be a domain.*
1. *If D is semilocal, then* $\mathrm{Pic}(D) = \mathbf{0}$ [*every invertible ideal is principal*].
2. *Suppose that $\mathcal{C}(D_M) = \mathbf{0}$ for all $M \in \max(D)$. Then $\mathcal{G}(D) = \mathbf{0}$.*

,

Proof. 1. Let $\max(D) = \{M_1, \dots, M_r\}$, and for $i \in [1,r]$, let
$$M_i^* = \bigcap_{\substack{j=1\\ j\ne i}}^{r} M_j\,, \quad \text{whence} \quad M_i^* \lhd D \quad \text{and} \quad M_i^* \not\subset M_i\,.$$
If $J \in \mathcal{F}(D)^\times$ and $i \in [1,r]$, then $JM_i^* \not\subset JM_i$, we fix an element $a_i \in JM_i^* \setminus JM_i$, and we set $a = a_1 + \ldots + a_r$. Then $a \in J \setminus JM_i$ for all $i \in [1,r]$, hence $aJ^{-1} \lhd D$ and $aJ^{-1} \not\subset JM_i$ for all $i \in [1,r]$, which implies $aJ^{-1} = D$ and $J = aD$.

2. Let $J \in \mathcal{F}_t(D)^\times$. If $M \in \max(D)$, then $J_M \in \mathcal{F}_{t_M}(D_M)^\times \subset \mathcal{F}_{t(D_M)}(D_M)^\times$ and thus J_M is principal. Since $J \in \mathcal{F}_{t,\mathfrak{f}}(D)$, it follows that $J \in \mathcal{F}(D)^\times$ by Theorem 4.1.4.2. □

4.2. Cancellation

Throughout this section, let K be a monoid, and $\mathbb{P}_\mathfrak{f}^*(K) = \{X \in \mathbb{P}_\mathfrak{f}(K) \mid X \cap K^* \ne \emptyset\}$.

Definition 4.2.1. Let r be a weak module system on K.
1. An r-module $A \in \mathcal{M}_r(K)$ is called (*r-finitely*) *r-cancellative* if, for all (r-finitely generated) r-modules $M, N \in \mathcal{M}_r(K)$, $A \cdot_r M = A \cdot_r N$ implies $M = N$.

   In particular, $A \in \mathcal{M}_r(K)$ is r-cancellative if and only if $A \in \mathcal{M}_r(K)^*$, and then A is r-finitely r-cancellative. If $A \in \mathcal{M}_{r,\mathfrak{f}}(K)$, then A is r-finitely r-cancellative if and only if $A \in \mathcal{M}_{r,\mathfrak{f}}(K)^*$.
2. r is called *cancellative* or *arithmetisch brauchbar* if every $A \in \mathcal{M}_r(K) \cap \mathbb{P}_\mathfrak{f}^*(K)$ is r-cancellative.

   If $\mathcal{M}_r(K)$ is a cancellative monoid, then r is cancellative, and the converse is true if K itself is cancellative.
3. r is called *finitely cancellative* or *endlich arithmetisch brauchbar* if every $A \in \mathcal{M}_{r,\mathfrak{f}}(K) \cap \mathbb{P}_\mathfrak{f}^*(K)$ is r-finitely r-cancellative.

   If $\mathcal{M}_{r,\mathfrak{f}}(K)$ is a cancellative monoid, then r is finitely cancellative, and the converse is true if K itself is cancellative.

Theorem 4.2.2. *Let r be a weak module system on K and $A \in \mathcal{M}_r(K)$.*

1. *The following assertions are equivalent:*
   (a) *A is (r-finitely) r-cancellative.*
   (b) *For all (r-finitely generated) r-modules $M,\, N \in \mathcal{M}_r(K)$, $A \cdot_r M \subset A \cdot_r N$ implies $M \subset N$.*
   (c) *For all (finite) subsets $M,\, N \subset K$, $AM \subset (AN)_r$ implies $M \subset N_r$.*
   (d) *For all (r-finitely generated) r-modules $N \in \mathcal{M}_r(K)$ and all $c \in K$, $cA \subset A \cdot_r N$ implies $c \in N$.*
   (e) *For all (r-finitely generated) r-modules $N \in \mathcal{M}_r(K)$ we have $(A \cdot_r N : A) \subset N$.*
2. *Let r be finitary, and let A be r-finitely generated and r-finitely cancellative.*
   (a) *A is r-cancellative.*
   (b) *If $T \subset K$ is a multiplicatively closed subset, then $T^{-1}A$ is $T^{-1}r$-cancellative.*
3. *If A is r-finitely r-cancellative, then $(A : A) \subset \{1\}_r$.*
4. *r is finitely cancellative if and only if $((EF)_r : E) \subset F_r$ for all $E \in \mathbb{P}_{\mathsf{f}}^*(K)$ and $F \in \mathbb{P}_{\mathsf{f}}(K)$.*

PROOF. 1. We prove the equivalence under the additional specification of r-finiteness.

(a) $\Rightarrow$ (b) If $M,\, N \in \mathcal{M}_{r,\mathsf{f}}(K)$ and $A \cdot_r M \subset A \cdot_r N$, then $A \cdot_r (M \cup N)_r = [(A \cdot_r M) \cup (A \cdot_r N)]_r = A \cdot_r N$, and as $(M \cup N)_r \in \mathcal{M}_{r,\mathsf{f}}(DK)$, it follows that $M \subset (M \cup N)_r = N$.

(b) $\Rightarrow$ (c) If $M,\, N \in \mathbb{P}_{\mathsf{f}}(K)$ and $AM \subset (AN)_r$, then $A \cdot_r M_r = (AM)_r \subset (AN)_r = A \cdot_r N_r$ and $M_r,\, N_r \in \mathcal{M}_{r,\mathsf{f}}(K)$. Hence it follows that $M \subset M_r \subset N_r$.

(c) $\Rightarrow$ (d) Obvious, setting $M = \{c\}$.

(d) $\Rightarrow$ (e) Obvious.

(e) $\Rightarrow$ (a) Let $M,\, N \in \mathcal{M}_{r,\mathsf{f}}(K)$ be such that $A \cdot_r M = A \cdot_r N$. If $x \in M$, then $Ax \subset A \cdot_r M = A \cdot_r N$ and therefore $x \in (A \cdot N : A) \subset N$. Hence $M \subset N$, and by symmetry equality follows.

2. Suppose that $A = E_r$, where $E \in \mathbb{P}_{\mathsf{f}}(K)$.

(a) By 1. we must prove that, for all subsets $N \subset K$ and $c \in K$, $cE \subset (EN)_r$ implies $c \in N_r$. Thus let $N \subset K$, $c \in K$ and $cE \subset (EN)_r$. If $e \in E$, then $ce \in (EN)_r$, and as r is finitary, there exists some $F \in \mathbb{P}_{\mathsf{f}}(N)$ such that $ce \in (EF_e)_r$. If

$$F = \bigcup_{e \in E} F_e\,, \quad \text{then} \quad F_e \in \mathbb{P}_{\mathsf{f}}(N) \quad \text{and} \quad cE \in \bigcup_{e \in E} (EF_e)_r \subset (EF)_r\,,$$

and therefore $c \in F_r \subset N_r$, since $A = E_r$ is r-finitely r-cancellative.

(b) By 1. we must prove that $(T^{-1}A \cdot_{T^{-1}r} \overline{N} : T^{-1}A) \subset \overline{N}$ for every $\overline{N} \in \mathcal{M}_{T^{-1}r,\mathsf{f}}(T^{-1}K)$. If $\overline{N} \in \mathcal{M}_{T^{-1}r,\mathsf{f}}(T^{-1}K)$, then $\overline{N} = T^{-1}N$ for some $N \in \mathcal{M}_{r,\mathsf{f}}(K)$, and

$$(T^{-1}A \cdot_{T^{-1}r} T^{-1}N : T^{-1}A) = (T^{-1}(A \cdot_r N) : T^{-1}E) = T^{-1}(A \cdot_r N : E) = T^{-1}(A \cdot_r N : A) \subset T^{-1}N\,.$$

3. If A is r-finitely r-cancellative, then $A \subset A \cdot_r \{1\}_r$ implies $(A : A) \subset (A \cdot_r \{1\}_r : A) \subset \{1\}_r$ by 1.(d).

4. Let r be finitely cancellative, $E \in \mathbb{P}_{\mathsf{f}}^*(K)$ and $F \in \mathbb{P}_{\mathsf{f}}(K)$. Then E_r is r-finitely r-cancellative, and as $F_r \in \mathcal{M}_{r,\mathsf{f}}(K)$, it follows that $((EF)_r : E) = (E_r \cdot_r F_r : E_r) \subset F_r$.

Conversely, assume that $((EF)_r : E) \subset F_r$ for all $E \in \mathbb{P}_{\mathsf{f}}^*(K)$ and $F \in \mathbb{P}_{\mathsf{f}}(K)$. If $A \in \mathcal{M}_{r,\mathsf{f}}(K) \cap \mathbb{P}_{\mathsf{f}}^*(K)$, then $A = E_r$ for some $E \in \mathbb{P}_{\mathsf{f}}^*(K)$, and $(A \cdot_r N : A) \subset N$ for all $N \in \mathcal{M}_{r,\mathsf{f}}(K)$. Indeed, if $N \in \mathcal{M}_{r,\mathsf{f}}(K)$, then $N = F_r$ for some $F \in \mathbb{P}_{\mathsf{f}}(K)$, and $(A \cdot_r N : A) = ((EF)_r : E_r) = ((EF)_r : E) \subset F_r = N$. □

Theorem 4.2.3. *Let D be a cancellative monoid, $K = \mathsf{q}(D)$, $r \colon \mathbb{P}(K) \to \mathbb{P}(K)$ an ideal system of D and $J \in \mathcal{F}_r(D)$.*

1. *If J is r-finitely r-cancellative, then $(J : J) = D$.*
2. *If J is r-invertible, then J is r-cancellative.*

Proof. 1. If J is r-finitely r-cancellative, then $D \subset (J:J) \subset \{1\}_r = D$ by Theorem 4.2.2.3, and therefore $(J:J) = D$.

2. Let J be r-invertible and $M,\, N \in \mathcal{M}_r(K)$ such that $J \cdot_r M = J \cdot_r N$. Then $M = J^{-1} \cdot_r J \cdot_r M = J^{-1} \cdot_r J \cdot_r N = N$. □

Theorem 4.2.4. *Let D be a ring and $I \lhd D$.*

1. *Then the following assertions are equivalent:*
   (a) *I is (d-)cancellative.*
   (b) *For every $M \in \max(D)$ there exists some $a_M \in D_M^*$ such that $I_M = a_M D_M$.*
   If I is finitely generated, then there is also equivalent:
   (a′) *I is (d-)finitely (d-)cancellative.*
2. *Let D be a domain, and let I be finitely generated. Then I is cancellative if and only if I is invertible.*

Proof. 1. (a) $\Rightarrow$ (a′) Obvious.

(a′) $\Rightarrow$ (a) By Theorem 4.2.2.2 (a).

(b) $\Rightarrow$ (a) Let $B,\, C \lhd D$ be such that $IB = IC$. For $M \in \max(D)$, this implies $I_M B_M = I_M C_M$, hence $a_M B_M = a_M C_M$ and therefore $A_M = B_M$, since $a_M \in D_M^*$. Now $B = C$ follows by Theorem 3.2.2.

(a) $\Rightarrow$ (b) We prove first: If $I = (a, b, A)$, where $a,\, b \in D$, $A \lhd D$, $M \in \max(D)$ and $MI \subset A$, then $I = (a, A)$ or $I = (b, A)$.

We consider the ideal $J = (A^2, a^2 + b^2, ab) \lhd D$ and calculate

$$\begin{aligned} I^2 J &= (a^2,\, b^2,\, ab,\, aA,\, bA,\, A^2)(A^2,\, a^2 + b^2,\, ab) \\ &= (a^2A^2,\, b^2A^2,\, abA^2,\, aA^3,\, bA^3,\, A^4,\, a^4 + a^2b^2,\, a^2b^2 + b^4,\, a^3b + ab^3, \\ &\qquad (a^3 + ab^2)A,\, (a^2b + b^3)A,\, (a^2 + b^2)A^2,\, a^3b,\, ab^3,\, a^2b^2,\, a^2bA,\, ab^2A) \\ &= (a^2A^2,\, b^2A^2,\, abA^2,\, aA^3,\, bA^3,\, A^4,\, a^4,\, b^4,\, a^3A,\, b^3A,\, a^3b,\, ab^3,\, a^2b^2,\, a^2bA,\, ab^2A) = I^4\,. \end{aligned}$$

Hence it follows that $I^2 = J$ and therefore $a^2 \in J$, say $a^2 = \lambda(a^2 + b^2) + z$, where $\lambda \in D$ and $z \in (A^2,\, ab)$. If $\lambda \in M$, then $\lambda a \in MI \subset A$, and $a^2 = (\lambda a)a + \lambda b^2 + z \in (A^2,\, b^2,\, ab,\, aA)$, and therefore

$$I(b, A) = (b^2,\, ab,\, aA,\, bA,\, A^2) = (a^2,\, b^2,\, ab,\, aA,\, bA,\, A^2) = I^2\,, \quad \text{which implies} \quad I = (b, A)\,.$$

If $\lambda \notin M$, then $D = (M, \lambda)$, say $1 = m + \lambda u$ for some $m \in M$ and $u \in D$. Since $mb^2 = (mb)b \in MIb \subset bA$ and $\lambda b^2 = (1 - \lambda)a^2 - z \in (a^2,\, ab,\, A^2)$, we obtain $b^2 = mb^2 + \lambda b^2 u \in (a^2,\, ab,\, bA,\, A^2)$, and therefore

$$I(a, A) = (a^2,\, ab,\, aA,\, bA,\, A^2) = (a^2,\, b^2,\, ab,\, aA,\, bA,\, A^2) = I^2\,, \quad \text{which implies} \quad I = (a, A)\,.$$

Now we can do the actual proof.

Let $M \in \max(D)$ and $\pi\colon I \to I/MI$ the canonical epimorphism. Let $B \subset I$ be a subset such that $\pi \,|\, B$ is injective and $\pi(B)$ is a D/M-basis of M/IM. Then $I = (B) + MI$, and $I \supsetneq (B') + MI$ for every subset $B' \subsetneq B$. We assert that $|B| = 1$. Indeed, suppose the contrary. Then $B = \{a,\, b\} \cup B'$, where $a \neq b$ and $\{a,\, b\} \cap B' = \emptyset$, and if $A = (B') + MI \lhd D$, then $I = (a, b, A)$. By **A** we obtain $I = (a, A)$ or $I = (b, A)$, a contradiction. Hence $|B| = 1$ and $I = bD + MI$ for some $b \in D$.

We assert that $I_M = (bD)_M = \frac{b}{1} D_M$, and for this we must prove that $\frac{c}{1} \in (bD)_M$ for all $c \in I$. If $c \in I$, then $cI = bcD + cMI \subset I(bD + cM)$, which implies $c \in bD + cM$, say $c = bu + cm$ for some $u \in D$ and $m \in M$. Hence $c(1 - m) = bu$ and

$$\frac{c}{1} = \frac{bu}{1 + m} \in (bD)_M\,.$$

It remains to prove that $\frac{b}{1}$ is not a zero divisor in D_M. Let $c \in D$ and $s \in D \setminus M$ be such that $\frac{c}{s}\frac{b}{1} = \frac{0}{1} \in D_M$. Then $tcb = 0$ for some $t \in D \setminus M$, and we obtain $(tcI)_M = \frac{tcb}{1} D_M = \{0\}_M = (tcMI)_M$. For $N \in \max(D) \setminus \{M\}$ we have $M_N = D_N$ and therefore $(tcMI)_N = (tcI)_N$. By Theorem 3.2.2 we obtain $tcI = tcMI$, which implies $tc \in tcM$, say $tc = tcm$ for some $m \in M$. Consequently,

$$\frac{c}{s} = \frac{tc(1-m)}{st(1-m)} = \frac{0}{1} \in D_M\,.$$

2. By Theorem 4.1.4. □

Theorem und Definition 4.2.5. *Let r be a finitary weak module system on K. Then there exists a unique finitary weak module system $r_{\mathsf a}$ on K such that*

$$X_{r_{\mathsf a}} = \bigcup_{B \in \mathbb{P}_{\mathsf f}^*(K)} ((XB)_r : B) \quad \textit{for all finite subsets } X \subset K\,. \tag{$*$}$$

If K is cancellative and r is a module system, then $r_{\mathsf a}$ is a module system.

$r_{\mathsf a}$ is called the *completion* of r. *It has the following properties:*

1. *$r \le r_{\mathsf a}$, and $(*)$ holds for all subsets $X \subset K$.*
2. *$r_{\mathsf a}$ is finitely cancellative, and if q is any finitely cancellative finitary weak module system on K such that $r \le q$, then $r_{\mathsf a} \le q$. In particular, $(r_{\mathsf a})_{\mathsf a} = r_{\mathsf a}$, and r is finitely cancellative if and only if $r = r_{\mathsf a}$.*
3. *Let $D \subset K$ be a submonoid. Then $r[D]_{\mathsf a} = r_{\mathsf a}[D]$. In particular, if r is a weak D-module system, then so is $r_{\mathsf a}$.*
4. *If $T \subset K^\bullet$ is a multiplicatively closed subset, then $T^{-1}r_{\mathsf a} = (T^{-1}r)_{\mathsf a}$.*
5. *Let D be a* GCD-*monoid, $L = \mathsf q(D)$ and $t = t(D)\colon \mathbb{P}(L) \to \mathbb{P}(L)$. Then t is finitely cancellative, and $\mathrm{Hom}_{(r,t)}(K,L) = \mathrm{Hom}_{(r_{\mathsf a},t)}(K,L)$.*
   In particular, if K is divisible, then every r-valuation monoid of K is an $r_{\mathsf a}$-valuation monoid.

Proof. Note that for every subset $X \subset K$, the system $\{((XB)_r : B) \mid B \in \mathbb{P}_{\mathsf f}^*(K)\}$ is directed. Indeed, if $B,\, B' \in \mathbb{P}_{\mathsf f}^*(K)$, then $((XB)_r : B) \subset ((XBB')_r : BB')$.

By Theorem 2.2.2 we must check the conditions $\mathbf{M1}_{\mathsf f}$, $\mathbf{M2}_{\mathsf f}$ and $\mathbf{M3}_{\mathsf f}$. Suppose that $X,\, Y \in \mathbb{P}_{\mathsf f}(K)$ and $c \in K$.

$\mathbf{M1}_{\mathsf f}$ If $B \in \mathbb{P}_{\mathsf f}^*(K)$, then $XB \cup \{0\} \subset (XB)_r$ implies $X \cup \{0\} \subset ((XB)_r : B) \subset X_{r_{\mathsf a}}$.

$\mathbf{M2}_{\mathsf f}$ Suppose that $X \subset Y_{r_{\mathsf a}}$ and $z \in X_{r_{\mathsf a}}$. Then there is some $F \in \mathbb{P}_{\mathsf f}^*(K)$ such that $z \in ((XF)_r : F)$. As $\{((YB)_r : B) \mid B \in \mathbb{P}_{\mathsf f}^*(K)\}$ is directed, there exists some $B \in \mathbb{P}_{\mathsf f}^*(K)$ such that $X \subset ((YB)_r : B)$. Then $zFB \subset (XF)_rB \subset (XBF)_r \subset [(YB)_rF]_r = (YFB)_r$ and thus $z \in ((YFB)_r : FB) \subset Y_{r_{\mathsf a}}$, since $FB \in \mathbb{P}_{\mathsf f}^*(K)$.

$\mathbf{M3}_{\mathsf f}$ We have

$$cX_{r_{\mathsf a}} = \bigcup_{B \in \mathbb{P}_{\mathsf f}^*(K)} c\big((XB)_r : B\big) \;\subset \bigcup_{B \in \mathbb{P}_{\mathsf f}^*(K)} \big(c(XB)_r : B\big) \;\subset \bigcup_{B \in \mathbb{P}_{\mathsf f}^*(K)} \big((cXB)_r : B\big) = (cX)_{r_{\mathsf a}}\,.$$

Here the first inclusion becomes an equality if K is cancellative, and the second one becomes an equality if r is a module system. Consequently, $r_{\mathsf a}$ is a module system if K is cancellative and r is a module system.

1. If $X \in \mathbb{P}_{\mathsf f}(K)$ and $B \in \mathbb{P}_{\mathsf f}^*(K)$, then $X_rB \subset (XB)_r$, hence $X_r \subset ((XB)_r : B) \subset X_{r_{\mathsf a}}$ and therefore $r \le r_{\mathsf a}$. For every subset $X \subset K$, we have

$$X_{r_{\mathsf a}} = \bigcup_{B \in \mathbb{P}_{\mathsf f}^*(K)} \Big(\big(\bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} EB\big)_r : B\Big) = \bigcup_{B \in \mathbb{P}_{\mathsf f}^*(K)} \; \bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} ((EB)_r : B) = \bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} E_{r_{\mathsf a}}\,.$$

If r is a module system, then $\mathbf{M3}_{\mathsf f}$ holds for $r_{\mathsf a}$, and thus $r_{\mathsf a}$ is also a module system.

2. By Theorem 4.2.2.4 we must prove that $((EF)_{r_{\mathsf{a}}} : E) \subset F_{r_{\mathsf{a}}}$ for all $E \in \mathbb{P}_{\mathsf{f}}^*(K)$ and $F \in \mathbb{P}_{\mathsf{f}}(K)$. Thus let $E \in \mathbb{P}_{\mathsf{f}}^*(K)$, $F \in \mathbb{P}_{\mathsf{f}}(K)$ and $z \in ((EF)_{r_{\mathsf{a}}} : E)$. Since $zE \subset (EF)_{r_{\mathsf{a}}}$, there exists some $B \in \mathbb{P}_{\mathsf{f}}^*(K)$ such that $zE \subset ((EFB)_r : B)$. Hence it follows that $zEB \subset (EFB)_r$ and $z \in ((EFB)_r : EB) \subset F_{r_{\mathsf{a}}}$, since $EB \in \mathbb{P}_{\mathsf{f}}^*(K)$.

Let now q be any finitely cancellative finitary weak module system on K such that $r \le q$. If $X \in \mathbb{P}_{\mathsf{f}}(K)$ and $B \in \mathbb{P}_{\mathsf{f}}^*(K)$, Theorem 4.2.2 implies $((XB)_r : B) \subset ((XB)_q : B) \subset X_q$, and thus $r_{\mathsf{a}} \le q$ by Theorem 2.3.2.1.

3. For $X \subset K$, we obtain

$$X_{r_{\mathsf{a}}[D]} = (XD)_{r_{\mathsf{a}}} = \bigcup_{B \in \mathbb{P}_{\mathsf{f}}^*(K)} ((XDB)_r : B) = \bigcup_{B \in \mathbb{P}_{\mathsf{f}}^*(K)} ((XB)_{r[D]} : B) = X_{r[D]_{\mathsf{a}}}\,.$$

4. By Theorem 2.4.1 we must prove that $j_T(E)_{(T^{-1}r)_{\mathsf{a}}} = T^{-1}E_{r_{\mathsf{a}}}$ for all $E \in \mathbb{P}_{\mathsf{f}}(K)$. Thus assume that $E = \{a_1, \ldots, a_n\}$, where $n \in \mathbb{N}_0$ and $a_1, \ldots, a_n \in K$. Then

$$j_T(E)_{(T^{-1}r)_{\mathsf{a}}} = \bigcup_{\overline{B} \in \mathbb{P}_{\mathsf{f}}^*(T^{-1}K)} \big((j_T(E)\overline{B})_{T^{-1}r} : \overline{B}\,\big)\,.$$

Suppose that

$$\overline{B} = \Big\{\frac{b_1}{t_1}, \ldots, \frac{b_m}{t_m}\Big\} \in \mathbb{P}_{\mathsf{f}}^*(T^{-1}K)\,,$$

where $m \in \mathbb{N}$, $b_1, \ldots, b_m \in K$ and $t_1, \ldots, t_m \in T$. Then $B = \{b_1, \ldots, b_m\} \in \mathbb{P}_{\mathsf{f}}^*(K)$,

$$\big(j_T(E)\overline{B}\big)_{T^{-1}r} = \Big\{\frac{a_ib_j}{t_j} \;\Big|\; i \in [1,n]\,,\ j \in [1,m]\Big\}_{T^{-1}r} = (T^{-1}EB)_{T^{-1}r} = T^{-1}(EB)_r\,,$$

and $\big((j_T(E)\overline{B})_{T^{-1}r} : \overline{B}\,\big) = \big(T^{-1}(EB)_r : T^{-1}B\big) = T^{-1}\big((EB)_r : B\big)$. Hence it follows that

$$j_T(E)_{(T^{-1}r)_{\mathsf{a}}} = T^{-1}\Big(\bigcup_{\substack{B \in \mathbb{P}_{\mathsf{f}}(K)\\ B \cap K^* \ne \emptyset}} ((EB)_r : B)\Big) = T^{-1}E_{r_{\mathsf{a}}}\,.$$

5. By Theorem 1.5.3, every t-finitely generated t-ideal of D is principal. Hence it follows that $\mathcal{M}_{t,\mathsf{f}}(L)^\bullet = \{a^{-1}J \mid J \in \mathcal{I}_{t,\mathsf{f}}(D)^\bullet,\ a \in D^\bullet\} = \{zD \mid z \in L^\times\}$ is cancellative, and thus t is finitely cancellative.

Since $r \le r_{\mathsf{a}}$, every (r_{a}, t)-homomorphism is an (r,t)-homomorphism. If $\varphi \colon K \to L$ is an (r,t)-homomorphism, then by Proposition 2.3.6.2 we must prove that $\varphi(X_{r_{\mathsf{a}}}) \subset \varphi(X)_t$ for all $X \in \mathbb{P}_{\mathsf{f}}(K)$. If $X \in \mathbb{P}_{\mathsf{f}}(K)$, $z \in X_{r_{\mathsf{a}}}$ and $B \in \mathbb{P}_{\mathsf{f}}^*(K)$ are such that $zB \subset (XB)_r$, then

$$\varphi(z)\varphi(B) \subset \varphi((XB)_r) \subset \varphi(XB)_t = [\varphi(X)\varphi(B)]_t$$

and therefore $\varphi(z) \in \big([\varphi(X)\varphi(B)]_t : \varphi(B)\big) \subset \varphi(X)_t$ by Theorem 4.2.2.

Let K be divisible, $V \subset K$ is a valuation monoid and $t = t(V)$. It follows by Theorem 3.4.9 that V is an r- (resp. r_{a}-)valuation monoid if and only if id_K is an (r,t)- [resp. (r_{a}, t)]-homomorphism. Hence every r-valuation monoid is an r_{a}-valuation monoid. □

Theorem 4.2.6. *Let $D \subset K$ be a submonoid and $s = s(D) \colon \mathbb{P}(K) \to \mathbb{P}(K)$. If $X \subset K$, $X \cap K^* \ne \emptyset$ and $z \in K$, then $z \in X_{s_{\mathsf{a}}}$ if and only if there exist some $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$ such that $z^{k+l} \in z^kX^lD$.*

Proof. Note that $z \in X_{s_{\mathsf{a}}}$ holds if and only if $zB \subset (XB)_s = XBD$ for some $B \in \mathbb{P}_{\mathsf{f}}^*(K)$.

Suppose that $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$ are such that $z^{k+l} \in z^kX^lD$, and let $X_0 \subset X$ be a finite subset such that $X_0 \cap K^* \ne \emptyset$ and $z^{k+l} \in z^kX_0^lD$. Then

$$B = \bigcup_{\nu=0}^{k+l-1} X_0^\nu z^{k+l-\nu-1} \in \mathbb{P}_{\mathsf{f}}(K)\,, \quad X_0^{k+l-1} \subset B\,, \quad \text{and therefore} \quad B \in \mathbb{P}_{\mathsf{f}}^*(K)\,,$$

$$zB = \bigcup_{\nu=1}^{k+l-1} X_0^\nu z^{k+l-\nu} \cup \{z^{k+l}\} \subset X_0\Big(\bigcup_{\nu=0}^{k+l-2} X_0^\nu z^{k+l-\nu-1} \ \cup\ z^k X_0^{l-1} D\Big) \subset X_0 BD \subset XBD\,,$$

and therefore it follows that $z \in X_{s_{\mathsf{a}}}$.

Assume now that $z \in X_{s_{\mathsf{a}}}$, and let $B = \{b_1, \ldots, b_n\} \in \mathbb{P}_{\mathsf{f}}(K)$ be such that $n \ge 1$, $b_1 \in K^*$ and $zB \subset XBD$. Then there exist $x_1, \ldots, x_n \in X$ and a map $\sigma\colon [1,n] \to [1,n]$ such that $zb_i \in b_{\sigma(i)} x_i D$ for all $i \in [1,n]$. Let $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$ be such that $\sigma^{k+l}(1) = \sigma^k(1)$. Then

$$z^{k+l} b_1 \ \in\ b_{\sigma^{k+l}(1)} \prod_{\mu=0}^{k+l-1} x_{\sigma^\mu(1)}\, D \ =\ b_{\sigma^k(1)} \prod_{\mu=0}^{k-1} x_{\sigma^\mu(1)} \prod_{\mu=k}^{k+l-1} x_{\sigma^\mu(1)}\, D \subset z^k b_1 X^l D$$

and therefore $z^{k+l} \in z^k X^l D$. □

Theorem 4.2.7. *Let R be a ring, $D \subset R$ a subring, $d = d(D)\colon \mathbb{P}(R) \to \mathbb{P}(R)$, $X \subset R$, $X \cap R^* \ne \emptyset$ and $z \in R$. Then $z \in X_{d_{\mathsf{a}}}$ if and only if z satisfies an equation $z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0$, where $n \in \mathbb{N}$ and $a_i \in (X^i)_d$ for all $i \in [1,n]$.*

Proof. Note that $z \in X_{d_{\mathsf{a}}}$ holds if and only if $zB \subset (XB)_d$ for some $B \in \mathbb{P}_{\mathsf{f}}^*(R)$.

Suppose that $z \in R$ satisfies an equation $z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0$, where $n \in \mathbb{N}$ and $a_i \in (X^i)_d$ for all $i \in [1,n]$. Let $X_0 \subset X$ be a finite subset such that $X_0 \cap R^* \ne \emptyset$ and $a_i \in (X_0^i)_d$ for all $i \in [1,n]$. If

$$B = \bigcup_{\nu=0}^{n-1} X_0^\nu z^{n-\nu-1} \ \in \mathbb{P}_{\mathsf{f}}(R)\,, \quad \text{then} \quad X_0^{n-1} \subset B\,, \quad \text{hence} \quad B \in \mathbb{P}_{\mathsf{f}}^*(R)\,, \quad \text{and}$$

$$zB = \{z^n\} \cup \bigcup_{\nu=0}^{n-2} X_0^{\nu+1} z^{n-\nu-1} \subset \{z^n\} \cup X_0 B.$$

Since

$$z^n = -\sum_{\nu=0}^{n-1} a_{\nu+1} z^{n-\nu-1} \ \in \Big(\bigcup_{\nu=0}^{n-1} X_0^{\nu+1} z^{n-\nu-1}\Big)_d \subset (X_0 B)_d \subset (XB)_d\,,$$

it follows that $zB \subset (XB)_d$ and thus $z \in X_{d_{\mathsf{a}}}$.

Assume now that $z \in X_{d_{\mathsf{a}}}$, and let $B = \{b_1, \ldots, b_n\} \in \mathbb{P}_{\mathsf{f}}(R)$ be such that $n \ge 1$, $b_1 \in R^*$ and $zB \subset (XB)_d$. Then there exist elements $x_{i,j} \in X_d$ such that

$$zb_i = \sum_{j=1}^n x_{i,j} b_j \quad \text{and therefore} \quad \sum_{j=1}^n (\delta_{i,j} z - x_{i,j}) b_j = 0 \quad \text{for all} \ \ i \in [1,n].$$

Hence it follows that $\det(\delta_{i,j} z - x_{i,j})_{i,\,j \in [1,n]} b_1 = 0$ and consequently $\det(\delta_{i,j} z - x_{i,j})_{i,\,j \in [1,n]} = 0$, which gives the desired equation for z. □

4.3. Integrality

Throughout this section, let K be a monoid, and $\mathbb{P}_{\mathsf{f}}^*(K) = \{X \in \mathbb{P}_{\mathsf{f}}(K) \mid X \cap K^* \ne \emptyset\}$.

Remarks and Definition 4.3.1. Let r be a finitary weak module system on K.

1. Let $X \subset K$. An element $x \in K$ is called *r-integral* over X if

$$x \in X_{r_{\mathsf{a}}} = \bigcup_{B \in \mathbb{P}_{\mathsf{f}}^*(K)} ((XB)_r : B)$$

[equivalently: There exists some $B \in \mathbb{P}_{\mathsf{f}}^*(K)$ such that $xB \subset (XB)_r$].

2. Let $D \subset K$ be a submonoid and r a weak D-module system on K. Then

$$D_{r_{\mathsf{a}}} = \bigcup_{\substack{J \in \mathcal{M}_{r,\mathsf{f}}(K) \\ J \cap K^* \neq \emptyset}} (J : J)$$

[an element $x \in K$ is r-integral over D if and only if there is some $J \in \mathcal{M}_{r,\mathsf{f}}(K)$ such that $J \cap K^* \neq \emptyset$ and $x \in (J : J)$].

Proof. By definition, $x \in D_{r_{\mathsf{a}}}$ if and only if $xB \subset (DB)_r = B_r$ and thus $xB_r \subset B_r$ for some $B \in \mathbb{P}^*_{\mathsf{f}}(K)$, and this holds if and only if $xJ \subset J$ for some $J \in \mathcal{M}_{r,\mathsf{f}}(K)$ such that $J \cap K^* \neq \emptyset$. □

3. Let $D \subset B \subset K$ be submonoids.
   - $\mathrm{cl}_r^B(D) = D_{r_{\mathsf{a}}} \cap B$ is called the r-(*integral*) *closure* of D in B.
   - B is called *r-integral over D* if $\mathrm{cl}_r^B(D) = B$.
   - D is called *r-(integrally) closed in B* if $\mathrm{cl}_r^B(D) = D$.

   By definition, B is r-integral over D if and only if $B \subset D_{r_{\mathsf{a}}}$, and D is r-integrally closed in B if and only if $D_{r_{\mathsf{a}}} \cap B = D$.

4. If K is a ring, $D \subset B \subset K$ are subrings and $r = d = d(K)$, then (by Theorem 4.2.7) the above definitions coincide with the usual ones in ring theory as follows.
   - $z \in K$ is called *integral* over D if z is d-integral over D [equivalently, $z \in D_{d_{\mathsf{a}}}$].
   - $\mathrm{cl}^B(D) = D_{d_{\mathsf{a}}} \cap B$ is called the *integral closure* of D in B.
   - B is called *integral over D* if $\mathrm{cl}^B(D) = B$.
   - D is called *integrally closed in B* if $\mathrm{cl}^B(D) = D$.

   By definition, B is integral over D if and only if $B \subset D_{d_{\mathsf{a}}}$, and D is integrally closed in B if and only if $D_{d_{\mathsf{a}}} \cap B = D$.

5. Let D be cancellative, $K = \mathsf{q}(D)$ and $r \colon \mathbb{P}(K) \to \mathbb{P}(K)$ a finitary ideal system of D. Then $\mathrm{cl}_r(D) = \mathrm{cl}_r^K(D) = D_{r_{\mathsf{a}}}$ is called the r-(*integral*) *closure of* D, and D is called r-(*integrally*) *closed* if $\mathrm{cl}_r(D) = D$. By 2. we have

$$\mathrm{cl}_r(D) = \bigcup_{J \in \mathcal{I}_{r,\mathsf{f}}(D)^\bullet} (J : J),$$

   and consequently D is r-closed if and only if $(J : J) = D$ for all $J \in \mathcal{I}_{r,\mathsf{f}}(D)^\bullet$.

   [Indeed, $\{J \in \mathcal{M}_{r,\mathsf{f}}(D) \mid J \cap K^* \neq \emptyset\} = \mathcal{F}_{r,\mathsf{f}}(D)^\bullet = \{c^{-1}J \mid c \in D^\bullet,\ J \in \mathcal{I}_{r,\mathsf{f}}(D)^\bullet\}$, and if $c \in D^\bullet$ and $J \in \mathcal{I}_{r,\mathsf{f}}(D)^\bullet$, then $(c^{-1}J : c^{-1}J) = (J : J)$].

   In particular:

   (a) If $s = s(D) \colon \mathbb{P}(K) \to \mathbb{P}(K)$, then $\mathrm{cl}_s(D) = \{z \in K \mid z^n \in D \text{ for some } n \in \mathbb{N}\}$ by Theorem 4.2.6. $\mathrm{cl}_s(D)$ is called the *root closure* of D, and if $D = \mathrm{cl}_s(D)$, then D is called *root-closed*.

   (b) If D is a domain, and $d = d(D) \colon \mathbb{P}(K) \to \mathbb{P}(K)$, then D is called *integrally closed* if it is d-integrally closed.

Theorem 4.3.2. *Let D be a cancellative monoid, $K = \mathsf{q}(D)$, and let $r, q \colon \mathbb{P}(K) \to \mathbb{P}(K)$ be finitary ideal systems of D such that $r \le q$.*

1. *If r is finitely cancellative, then D is r-closed.*
2. *$\mathrm{cl}_r(D) \subset \mathrm{cl}_q(D)$, and if D is q-closed, then D is r-closed and, in particular, D is root-closed.*

Proof. 1. By Theorem 4.2.3 we have $(J : J) = D$ for all $J \in \mathcal{I}_{r,\mathsf{f}}(D)$. Hence D is r-closed by Remark 4.3.1.4.

2. If $x \in \mathrm{cl}_r(D)$, then there exists some $J \in \mathcal{I}_{r,\mathsf{f}}(D)^\bullet$ such that $x \in (J : J)$. Then $J_q \in \mathcal{I}_{q,\mathsf{f}}(D)^\bullet$ and $zJ_q = (zJ)_q \subset J_q$ implies $z \in (J_q : J_q) \subset \mathrm{cl}_q(D)$. If D is q-closed, then $D = \mathrm{cl}_q(D) \supset \mathrm{cl}_r(D) \supset D$. Hence D is r-closed, and since $s(D) \le r$, it is also root-closed by Remark 4.3.1.5. □

Theorem 4.3.3. *Let D be an integrally closed domain, $K = \mathsf{q}(D)$ and $d = d(D) \colon \mathbb{P}(K) \to \mathbb{P}(K)$. Then d_{a} is a finitary ideal system of D, d_{a}-$\max(D) = d$-$\max(D)$, and if $X \subset D$, then $X_{d_{\mathsf{a}}} = D$ if and only if $X_d = D$.*

Proof. d_{a} is a finitary D-module system on K, and as $D_{d_{\mathsf{a}}} = D$, it is even an ideal system of D.

If $X \subset D$, then $X_d \subset X_{d_{\mathsf{a}}} \subset D$, and therefore $X_d = D$ implies $X_{d_{\mathsf{a}}} = D$. Conversely, if $X_{d_{\mathsf{a}}} = D$, then $1 \in X_{d_{\mathsf{a}}}$, and thus there is an equation $1 + a_1 + \ldots + a_n = 0$, where $n \in \mathbb{N}$ and $a_i \in (X^i)_d$ for all $i \in [1, n]$. Since $X_d \triangleleft D$ and $(X^i)_d = (X_d)^i \subset X_d$ for all $i \in [1, n]$, it follows that $1 \in X_d$ and therefore $X_d = D$.

If $M \in d$-$\max(D)$, then $M \subset M_{d_{\mathsf{a}}} \subsetneq D$, and there is some $M^* \in d_{\mathsf{a}}$-$\max(D)$ such that $M_{d_{\mathsf{a}}} \subset M^*$. But $M^* \in \mathcal{I}_d(D)$, and therefore $M = M^* \in d_{\mathsf{a}}$-$\max(D)$. Conversely, if $M \in d_{\mathsf{a}}$-$\max(D)$, then $M \in \mathcal{I}_d(D)$, and there exists some $\overline{M} \in d$-$\max(D)$ such that $M \subset \overline{M}$. Since $\overline{M}_{d_{\mathsf{a}}} \subsetneq D$, we obtain $M = \overline{M}_{d_{\mathsf{a}}}$ and therefore $M = \overline{M} \in d$-$\max(D)$. □

Theorem 4.3.4. *Let $D \subset B \subset K$ be submonoids and r a finitary weak module system on K.*

1. *Let B be an r-monoid and $B' = \mathrm{cl}_r^B(D) \subset B$. Then B' is an r-monoid which is r-closed in B.*
2. *Let B be r-integral over D. If $z \in K$ is r-integral over B, then z is r-integral over D.*
3. *If $T \subset D^\bullet$ is a multiplicatively closed subset, then $\mathrm{cl}_{T^{-1}r}^{T^{-1}K}(T^{-1}D) = T^{-1}\mathrm{cl}_r^K(D)$.*
4. *For $P \in r_D$-$\max(D)$ let $j_P \colon K \to K_P$ be the natural embedding. Then*
$$\mathrm{cl}_r^K(D) = \bigcap_{P \in r_D\text{-}\max(D)} j_P^{-1}\big(\mathrm{cl}_{r_P}^{K_P}(D_P)\big)\,.$$
In particular:
   (a) *An element $z \in K$ is r-integral over D if and only if, for all $P \in r_D$-$\max(D)$, the element $\frac{z}{1} \in K_P$ is r_P-integral over D_P.*
   (b) *If $D^\bullet \subset K^\times$, then $D_P \subset K_P = K$ for all $P \in r_D$-$\max(D)$, and*
$$\mathrm{cl}_r^K(D) = \bigcap_{P \in r_D\text{-}\max(D)} \mathrm{cl}_{r_P}^K(D_P)\,.$$
   (c) *If D is cancellative and $K = \mathsf{q}(D)$, then D is r-closed if and only if, for all $P \in r$-$\max(D)$, D_P is r_P-closed.*

Proof. 1. Since $r \le r_{\mathsf{a}}$, it follows that $D_{r_{\mathsf{a}}}$ is an r-monoid. Hence $B' = \mathrm{cl}_r^B(D) = D_{r_{\mathsf{a}}} \cap B$ is an r-monoid, and $\mathrm{cl}_r^B(B') = B'_{r_{\mathsf{a}}} \cap B = (D_{r_{\mathsf{a}}} \cap B)_{r_{\mathsf{a}}} \cap B = D_{r_{\mathsf{a}}} \cap B$.

2. If B is r-integral over D, then $B \subset D_{r_{\mathsf{a}}}$, and therefore $B_{r_{\mathsf{a}}} = D_{r_{\mathsf{a}}}$.

3. If $T \subset D^\bullet$ is multiplicatively closed, then $(T^{-1}D)_{(T^{-1}r)_{\mathsf{a}}} = (T^{-1}D)_{T^{-1}r_{\mathsf{a}}} = T^{-1}D_{r_{\mathsf{a}}}$ by the Theorems 4.2.5.4 and 2.4.1.

4. Since $D_{r_{\mathsf{a}}}$ is a D-module, Theorem 3.2.2 implies
$$\mathrm{cl}_r^K(D) = D_{r_{\mathsf{a}}} = \bigcap_{P \in r_D\text{-}\max(D)} j_P^{-1}\big((D_{r_{\mathsf{a}}})_P\big)\,.$$
If $P \in r_D$-$\max(D)$, then $(D_{r_{\mathsf{a}}})_P = (D_P)_{(r_{\mathsf{a}})_P} = (D_P)_{(r_P)_{\mathsf{a}}} = \mathrm{cl}_{r_P}^{K_P}(D_P)$.

If $D^\bullet \subset K^\times$, then $D_P \subset K_P = K$, $j_P = \mathrm{id}_K$, and $(D_P)_{(r_\mathsf{a})_P} = \mathrm{cl}_r^K(D_P)$ by Theorem 2.5.4. $\square$

We reformulate Theorem 4.3.4 for the classical case of integral ring extensions.

Theorem 4.3.5. *Let $D \subset B \subset K$ be rings.*

1. *$B' = \mathrm{cl}^B(D)$ is a subring of B which is integrally closed in B.*
2. *If B is integral over D and $z \in K$ is integral over B, then z is integral over D.*
3. *If $T \subset D^\bullet$ is a multiplicatively closed subset, then $\mathrm{cl}^{T^{-1}K}(T^{-1}D) = T^{-1}\mathrm{cl}^K(D)$.*
4. *For $P \in \max(D)$ let $j_P \colon K \to K_P$ be the natural embedding. Then*

$$\mathrm{cl}^K(D) = \bigcap_{P \in \max(D)} j_P^{-1}\big(\mathrm{cl}^{K_P}(D_P)\,.$$

In particular:

(a) *An element $z \in K$ is integral over D if and only if, for all $P \in \max(D)$, the element $\frac{z}{1} \in K_P$ is integral over D_P.*

(b) *If $D^\bullet \subset K^\times$, then $D_P \subset K_P = K$ for all $P \in \max(D)$, and*

$$\mathrm{cl}^K(D) = \bigcap_{P \in \max(D)} \mathrm{cl}^K(D_P)\,.$$

(c) *If D is a domain and $K = \mathsf{q}(D)$, then D is integrally closed if and only if D_P is integrally closed for all $P \in \max(D)$.*

Proof. By Theorem 4.3.4, observing that $T^{-1}d = d(T^{-1}D)$ for every multiplively closed subset $T \subset D^\bullet$, and that $d_D = d \,|\, \mathbb{P}(D)$. $\square$

4.4. Lorenzen monoids

Remarks and Definition 4.4.1. Let D be a cancellative monoid, $K = \mathsf{q}(D)$, r a finitary module system on K and $D \subset \{1\}_{r_\mathsf{a}}$ (then $D_{r_\mathsf{a}} = \{1\}_{r_\mathsf{a}}$).
By Theorem 4.2.5.2, the monoid $\mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)$ is cancellative, and $\mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)^\bullet = \{C \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(D) \mid C^\bullet \ne \emptyset\}$. We define

$$\Lambda_r(K) = \mathsf{q}(\mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K))\,, \quad \text{and we call} \quad \Lambda_r(K)^\times = \Lambda_r(K)^\bullet \quad \text{the } \textit{Lorenzen } r\textit{-group} \text{ of } K.$$

For an element $X \in \Lambda_r(K)^\bullet$, we denote by $X^{[-1]}$ its inverse in $\Lambda_r(K)$. Then we obtain

$$\Lambda_r(K) = \{C^{[-1]}A \mid A,\, C \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)\,,\ C^\bullet \ne \emptyset\}\,.$$

If $A, A' \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)$ and $C, C' \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)^\bullet$, then $C^{[-1]}A = C'^{[-1]}A'$ if and only if $(AC')_{r_\mathsf{a}} = (A'C)_{r_\mathsf{a}}$, and multiplication in $\Lambda_r(K)$ is given by the formula $(C^{[-1]}A) \cdot (C'^{[-1]}A') = (CC')_{r_\mathsf{a}}{}^{[-1]}(AA')_{r_\mathsf{a}}$. In particular, $D_{r_\mathsf{a}} = \{1\}_{r_\mathsf{a}}$ is the unit element and $\{0\}$ is the zero element of $\Lambda_r(K)$. The submonoid

$$\Lambda_r^+(K) = \{C^{[-1]}A \mid A,\, C \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)\,,\ C^\bullet \ne \emptyset\,,\ A \subset C\} \ \subset\ \Lambda_r(K)$$

is called the *Lorenzen r-monoid* of K.
The map $\tau_r \colon K \to \Lambda_r(K)$ is defined by $\tau_r(a) = \{a\}_{r_\mathsf{a}} = aD_{r_\mathsf{a}} \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K) \subset \Lambda_r(K)$ for all $a \in K$, is a monoid homomorphism, called the *Lorenzen r-homomorphism*.
By definition, $\tau_r(D) \subset \tau_r(D_{r_\mathsf{a}}) \subset \Lambda_r^+(K)$, and $\tau_r \,|\, K^\times \colon K^\times \to \Lambda_r(K)^\times$ is a group homomorphism satisfying $\mathrm{Ker}(\tau_r \,|\, K^\times) = D_{r_\mathsf{a}}^\times$.

Theorem 4.4.2. *Let D be a cancellative monoid, $K = \mathsf{q}(D)$, r a finitary module system on K, $D \subset \{1\}_{r_\mathsf{a}}$ and $t = t(\Lambda_r^+(K)) \colon \mathbb{P}(\Lambda_r(K)) \to \mathbb{P}(\Lambda_r(K))$.*

1. $\Lambda_r(K) = \mathsf{q}(\Lambda_r^+(K))$.
2. *If $A,\, C \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)$ and $C^\bullet \ne \emptyset$, then $C^{[-1]}A \in \Lambda_r^+(K)$ if and only if $A \subset C$. In particular, $A \in \Lambda_r^+(K)$ holds if and only if $A \subset D_{r_\mathsf{a}}$.*
3. *$\Lambda_r^+(K)$ is a reduced* GCD*-monoid. If $X,\, Y \in \Lambda_r^+(K)$, then there exist $A,\, B,\, C \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)$ such that $C^\bullet \ne \emptyset$, $A \cup B \subset C$, $X = C^{[-1]}A$ and $Y = C^{[-1]}B$. In this case, we have $X \,|\, Y$ if and only if $B \subset A$, and $\gcd(X,Y) = C^{[-1]}(A \cup B)_{r_\mathsf{a}}$.*
4. *If $E \in \mathbb{P}_\mathsf{f}(D_{r_\mathsf{a}})$, then $E_{r_\mathsf{a}} = \gcd(\tau_r(E)) \in \Lambda_r^+(K)$. In particular, for every $X \in \Lambda_r(K)$ there exist $E,\, E' \in \mathbb{P}_\mathsf{f}(D)$ such that $E'^\bullet \ne \emptyset$, $X = E'^{[-1]}_{r_\mathsf{a}} E_{r_\mathsf{a}} = \gcd(\tau_r(E'))^{[-1]} \gcd(\tau_r(E))$, and then we have $X \in \Lambda_r^+(K)$ if and only if $E \subset E'_{r_\mathsf{a}}$.*
5. *$r_\mathsf{a} = \tau_r^* t$. In particular, τ_r is an (r_a, t)-homomorphism and thus also an (r,t)-homomorphism, $X_{r_\mathsf{a}} = \tau_r^{-1}[\,\tau_r(X)_t]$ for all $X \subset K$, and $D_{r_\mathsf{a}} = \tau_r^{-1}(\Lambda_r^+(K))$.*

Proof. We will thorough use the fact that r_a is finitely cancellative and apply Theorem 4.2.2.

1. If $X = C^{[-1]}A \in \Lambda_r(K)$, where $A,\, C \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(D)$ and $C^\bullet \ne \emptyset$, then $(C \cup A)_{r_\mathsf{a}}^{[-1]} C \in \Lambda_r^+(K)$, $(C \cup A)_{r_\mathsf{a}}^{[-1]} A \in \Lambda_r^+(K)$, and $X = [\,(C \cup A)_{r_\mathsf{a}}^{[-1]} C\,]^{[-1]} [\,(C \cup A)_{r_\mathsf{a}}^{[-1]} A\,]$.

2. Let $A,\, C \in \mathcal{M}_{r,\mathsf{f}}(K)$ and $C^\bullet \ne \emptyset$. If $A \subset C$, then $C^{[-1]}A \in \Lambda_r^+(K)$ by definition. Thus suppose that $C^{[-1]}A \in \Lambda_r^+(K)$, say $C^{[-1]}A = C_1^{[-1]}A_1$ for some $A_1,\, C_1 \in \mathcal{M}_{r,\mathsf{f}}(K)$ such that $C_1^\bullet \ne \emptyset$ and $A_1 \subset C_1$. Then $(C_1A)_{r_\mathsf{a}} = (CA_1)_{r_\mathsf{a}} \subset (CC_1)_{r_\mathsf{a}}$, and thus $A \subset C$.

3. We prove first that $\Lambda_r^+(K)$ is reduced. Let $X \in \Lambda_r^+(K)^\times$, say $X = C^{[-1]}A$ and $X^{[-1]} = C_1^{[-1]}A_1$, where $A,\, A_1,\, C,\, C_1 \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)$, $C^\bullet \ne \emptyset$, $C_1^\bullet \ne \emptyset$, $A \subset C$ and $A_1 \subset C_1$. Then $(CC_1)_{r_\mathsf{a}}^{-1}(AA_1)_{r_\mathsf{a}} = D_{r_\mathsf{a}}$, hence $A_1^\bullet \ne \emptyset$ and $(AA_1)_{r_\mathsf{a}} = (CC_1)_{r_\mathsf{a}} \supset (CA_1)_{r_\mathsf{a}}$. Now it follows again that $A \supset C$, hence $A = C$ and $X = D_{r_\mathsf{a}}$.

Now let $X,\, Y \in \Lambda_r^+(K)$. As $\Lambda_r^+(K) \subset \mathsf{q}(\mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)$, there exist $A,\, B,\, C \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)$ such that $C^\bullet \ne \emptyset$, $X = C^{[-1]}A$ and $Y = C^{[-1]}B$, and by 2. we obtain $A \cup B \subset C$.

Assume that $X \,|\, Y$, say $Y = X \boldsymbol{\cdot} Z$, where $Z = W^{[-1]}U \in \Lambda_r^+(K)$ for some $U,\, W \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)$ such that $W^\bullet \ne \emptyset$ and $U \subset W$. Therefore we obtain $C^{[-1]}B = C^{[-1]}A \boldsymbol{\cdot} W^{[-1]}U = (CW)_{r_\mathsf{a}}^{[-1]}(AU)_{r_\mathsf{a}}$, which implies $(BCW)_{r_\mathsf{a}} = (CAU)_{r_\mathsf{a}}$, hence $(BW)_{r_\mathsf{a}} = (AU)_{r_\mathsf{a}} \subset (AW)_{r_\mathsf{a}}$ and $B \subset A$ by cancelation.

Assume now that $B \subset A$. If $B^\bullet = \emptyset$, then $B = (BA)_{r_\mathsf{a}} \in \Lambda_r^+(K)$, $Y = C^{[-1]}B = (C^{[-1]}A) \boldsymbol{\cdot} B = X \boldsymbol{\cdot} B$ and therefore $X \,|\, Y$. If $B^\bullet \ne \emptyset$, then $A^\bullet \ne \emptyset$, hence $U = A^{[-1]}B \in \Lambda_r^+(K)$ and $Y = X \boldsymbol{\cdot} U$, which again implies $X \,|\, Y$.

To prove the assertion concerning the gcd, set $Z = C^{[-1]}(A \cup B)_{r_\mathsf{a}}$. Then $Z \,|\, X$ and $Z \,|\, Y$. We assume that $Z_1 \in \Lambda_r^+(K)$ is another element such that $Z_1 \,|\, X$ and $Z_1 \,|\, Y$. We must prove that $Z_1 \,|\, Z$. By 1., there exist $A_1,\, B_1,\, C_1,\, U,\, W \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)$ such that $C_1^\bullet \ne \emptyset$, $A_1 \cup B_1 \cup U \cup W \subset C_1$, $X = C_1^{[-1]}A_1$, $Y = C_1^{[-1]}B_1$, $Z = C_1^{[-1]}U$ and $Z_1 = C_1^{[-1]}W$. Then it follows that $A_1 \cup B_1 \subset W$, $(CA_1)_{r_\mathsf{a}} = (C_1A)_{r_\mathsf{a}}$, $(CB_1)_{r_\mathsf{a}} = (C_1B)_{r_\mathsf{a}}$ and $(CU)_{r_\mathsf{a}} = (C_1(A \cup B))_{r_\mathsf{a}}$. Moreover, we obtain

$$(C(A_1 \cup B_1))_{r_\mathsf{a}} = ((CA_1)_{r_\mathsf{a}} \cup (CB_1)_{r_\mathsf{a}})_{r_\mathsf{a}} = ((C_1A)_{r_\mathsf{a}} \cup (C_1B)_{r_\mathsf{a}})_{r_\mathsf{a}} = (C_1(A \cup B))_{r_\mathsf{a}} = (CU)_{r_\mathsf{a}}\,,$$

and therefore $U = (A_1 \cup B_1)_{r_\mathsf{a}} \subset W$, which implies $Z_1 \,|\, Z$.

4. If $E \in \mathbb{P}_\mathsf{f}(D_{r_\mathsf{a}})$, then $E_{r_\mathsf{a}} \in \Lambda_r^+(K)$, $\tau_r(E) \subset \Lambda_r^+(K)$, and 2. implies

$$E_{r_\mathsf{a}} = \Big(\bigcup_{e \in E} \{e\}_{r_\mathsf{a}}\Big)_{r_\mathsf{a}} = \Big(\bigcup_{e \in E} \tau_r(e)\Big)_{r_\mathsf{a}} = \gcd\big(\{\tau_r(e) \mid e \in E\}\big) = \gcd(\tau_r(E))\,.$$

If $X \in \Lambda_r(K)$, then $X = C^{[-1]}A$, where $A,\, C \in \mathcal{M}_{r_\mathsf{a},\mathsf{f}}(K)$, $A \subset C$ and $C^\bullet \ne \emptyset$. Then there exist $E,\, E' \in \mathbb{P}_\mathsf{f}(D)$ and $c \in D^\bullet$ such that $C = (c^{-1}E')_{r_\mathsf{a}}$ and $A = (c^{-1}E)_{r_\mathsf{a}}$, and it follows that $E'^\bullet \ne \emptyset$ and $X = (c^{-1}E')_{r_\mathsf{a}}^{[-1]} E_{r_\mathsf{a}} = \gcd(\tau_r(E'))^{[-1]} \gcd(\tau_r(E))$

5. Since t is finitary, it suffices to prove that $Z_{r_{\mathsf{a}}} = Z_{\tau_r^* t} = \tau_r^{-1}(\tau_r(Z)_t)$ for all $Z \in \mathbb{P}_{\mathsf{f}}(K)$. Let $Z \in \mathbb{P}_{\mathsf{f}}(K)$ and $a \in D^\bullet$ such that $E = aZ \subset D$. Then $E_{r_{\mathsf{a}}} = \gcd(\tau_r(E))$ by 4., and therefore it follows that $\tau_r(E)_t = E_{r_{\mathsf{a}}}\Lambda_r^+(K)$. For $c \in K$, we obtain (observing that r_{a} is a module system)

$$\begin{aligned} c \in Z_{r_{\mathsf{a}}} &\iff ac \in aZ_{r_{\mathsf{a}}} = E_{r_{\mathsf{a}}} \iff \tau_r(ac) = \{ac\}_{r_{\mathsf{a}}} \subset E_{r_{\mathsf{a}}} \iff E_{r_{\mathsf{a}}}^{[-1]}\tau_r(ac) \in \Lambda_r^+(K) \\ &\iff \tau_r(a)\tau_r(c) = \tau_r(ac) \in E_{r_{\mathsf{a}}}\Lambda_r^+(K) = \tau_r(E)_t = \tau_r(aZ)_t = \tau_r(a)\tau_r(Z)_t \\ &\iff \tau_r(c) \in \tau_r(Z)_t \iff c \in \tau_r^{-1}(\tau_r(Z)_t)\,. \end{aligned}$$

The remaining assertions are obvious. □

Theorem 4.4.3 (Universal property of the Lorenzen monoid)**.** *Let D be a cancellative monoid, $K = \mathsf{q}(D)$, r a finitary module system on K, $D \subset \{1\}_{r_{\mathsf{a}}}$ and $t = t(\Lambda_r^+(K)) \colon \mathbb{P}(\Lambda_r(K)) \to \mathbb{P}(\Lambda_r(K))$.*

1. *Let G be a reduced* GCD*-monoid, $L = \mathsf{q}(G)$ and $t' = t(G) \colon \mathbb{P}(L) \to \mathbb{P}(L)$. Then there is a bijective map*

$$\mathrm{Hom}_{(t,t')}(\Lambda_r(K), L) \;\to\; \mathrm{Hom}_{(r,t')}(K, L)\,, \quad \textit{given by} \quad \Phi \;\mapsto \Phi\circ\tau_r\,.$$

2. *Let $\mathcal{V}$ be the set of all r-valuation monoids of K and $\mathcal{W}$ the set of all t-valuation monoids of $\Lambda_r(K)$.*
   - (a) *Suppose that $W \in \mathcal{W}$, and let $w \colon \Lambda_r(K)^\times \to \Gamma$ be a valuation morphism of W. Then $V = \tau_r^{-1}(W) \in \mathcal{V}$, and $w\circ\tau_r \,|\, K^\times \colon K^\times \to \Gamma$ is a valuation morphism of V. If $E \in \mathbb{P}_{\mathsf{f}}^*(K)$, then $w(E_{r_{\mathsf{a}}}) = \min\{w\circ\tau_r(E^\bullet)\}$.*
   - (b) *The assignment $W \to \tau_r^{-1}(W)$ defines a bijective map $\widetilde{\tau}_r \colon \mathcal{W} \to \mathcal{V}$.*

Proof. 1. If $\Phi \colon \Lambda_r(K) \to L$ is a (t,t')-homomorphism, then $\Phi\circ\tau_r \colon K \to L$ is an (r,t')-homomorphism, since τ_r is an (r,t)-homomorphism. We prove that for every (r,t')-homomorphism $\varphi \colon K \to L$ there is a unique (t,t')-homomorphism $\Phi \colon \Lambda_r(K) \to L$ such that $\Phi\circ\tau_r = \varphi$.

Thus let $\varphi \in \mathrm{Hom}_{(r,t')}(K,L) = \mathrm{Hom}_{(r_{\mathsf{a}},t')}(K,L)$ (see Theorem 4.2.5.5). By Theorem 2.6.5, the map $\mathrm{Hom}_{(t,t')}(\Lambda_r(K),L) \to \mathrm{Hom}_{\mathrm{GCD}}(\Lambda_r^+(K),G)$, $\Phi \mapsto \Phi \,|\, \Lambda_r^+(K)$, is bijective, and if $\Phi \in \mathrm{Hom}_{(t,t')}(\Lambda_r(K),L)$, then $\varphi = \Phi\circ\tau_r$ if and only if $\varphi \,|\, D = (\Phi \,|\, \Lambda_r^+(K))\circ\tau_r \,|\, D$. Hence it suffices to prove that there exists a unique $\psi \in \mathrm{Hom}_{\mathrm{GCD}}(\Lambda_r^+(K),G)$ such that $\psi\circ\tau_r(a) = \varphi(a)$ for all $a \in D^\bullet$.

Uniqueness : Let $\psi \in \mathrm{Hom}_{\mathrm{GCD}}(\Lambda_r^+(K),G)$ be such that $\psi\circ\tau_r(a) = \varphi(a)$ for all $a \in D^\bullet$, and assume that $X \in \Lambda_r^+(K)$, say $X = \gcd(\tau_r(E'))^{[-1]}\gcd(\tau_r(E))$, where $E,\, E' \in \mathbb{P}_{\mathsf{f}}^\bullet(D)$, $E'^\bullet \ne \emptyset$ and $E_{r_{\mathsf{a}}} \subset E'_{r_{\mathsf{a}}}$. Then it follows that $\psi(X) = \gcd[\,\psi(\tau_r(E'))\,]^{-1}\gcd[\,\psi(\tau_r(E))\,] = \gcd[\,\varphi(E')\,]^{-1}\gcd[\,\varphi(E)\,]$, and thus ψ is uniquely determined by φ.

Existence : Define ψ provisionally by

$$\psi(X) = \gcd(\varphi(E'))^{-1}\gcd(\varphi(E)) \in L \quad \text{if} \quad X = \gcd(\tau_r(E'))^{[-1]}\gcd(\tau_r(E)) = E'^{[-1]}_{r_{\mathsf{a}}}E_{r_{\mathsf{a}}} \in \Lambda_r^+(K)\,,$$

where $E,\, E' \in \mathbb{P}_{\mathsf{f}}(D)$, $E'^\bullet \ne \emptyset$, and $E \subset E'_{r_{\mathsf{a}}}$. We must prove : **1)** $\psi(X) \in G$; **2)** the definition is independent of the choice of E and E' ; **3)** ψ is a GCD-homomorphism.

If this is done and $a \in D$, then (putting $E' = \{1\}$ and $E = \{a\}$) we obtain $\psi\circ\tau_r(a) = \psi(\{a\}_{r_{\mathsf{a}}}) = \varphi(a)$.

1) Since φ is an (r_{a},t')-homomorphism, we obtain $\varphi(E) \subset \varphi(E'_{r_{\mathsf{a}}}) \subset \varphi(E')_{t'}$, and therefore $\gcd(\varphi(E))G = \varphi(E)_{t'} \subset \varphi(E')_{t'} = \gcd(\varphi(E')G$. Hence $\psi(X) = \gcd(\varphi(E'))^{-1}\gcd(\varphi(E)) \in G$.

2) Suppose that $X = E'^{[-1]}_{r_{\mathsf{a}}}E_{r_{\mathsf{a}}} = F'^{[-1]}_{r_{\mathsf{a}}}F_{r_{\mathsf{a}}}$, where $E,\, E',\, F,\, F' \in \mathbb{P}_{\mathsf{f}}(D)$, $E'^\bullet \ne \emptyset$, $F'^\bullet \ne \emptyset$, $E \subset E'_{r_{\mathsf{a}}}$ and $F \subset F'_{r_{\mathsf{a}}}$. Then $(EF')_{r_{\mathsf{a}}} = (E'F)_{r_{\mathsf{a}}}$, and since φ is an (r_{a},t')-homomorphism, we obtain

$$\varphi(EF') \subset \varphi((EF')_{r_{\mathsf{a}}}) = \varphi((E'F)_{r_{\mathsf{a}}} \subset \varphi(E'F)_{t'} \quad \text{and} \quad \varphi(EF')_{t'} \subset \varphi(E'F)_{t'}\,.$$

Similarly, $\varphi(E'F)_{t'} \subset \varphi(EF')_{t'}$, and thus equality holds. Therefore it follows that

$$\begin{aligned}\gcd(\varphi(E))\gcd(\varphi(F'))G &= \gcd(\varphi(EF'))G = \varphi(EF')_{t'} \\ &= \varphi(E'F)_{t'} = \gcd(\varphi(E'F))G = \gcd(\varphi(E'))\gcd(\varphi(F))G\,,\end{aligned}$$

hence $\gcd(\varphi(E))\gcd(\varphi(F')) = \gcd(\varphi(E'))\gcd(\varphi(F))$ (since G is reduced), which finally implies that $\gcd(E')^{-1}\gcd(E) = \gcd(F')^{-1}\gcd(F)$.

3) Let $X_1,\, X_2 \in \Lambda_r^+(K)$ and $E,\, E_1,\, E_2 \in \mathbb{P}_{\mathsf{f}}(D)$ be such that $E^\bullet \ne \emptyset$, $E_1 \cup E_2 \subset E_{r_{\mathsf{a}}}$ and $X_i = E_{r_{\mathsf{a}}}^{[-1]}(E_i)_{r_{\mathsf{a}}}$ for $i \in \{1,2\}$. Then $\gcd(X_1,X_2) = E_{r_{\mathsf{a}}}^{[-1]}(E_1 \cup E_2)_{r_{\mathsf{a}}}$,

$$\begin{aligned}\psi(X_1 \cdot X_2) &= \psi\big((E^2)_{r_{\mathsf{a}}}^{[-1]}(E_1E_2)_{r_{\mathsf{a}}}\big) = \gcd(\varphi(E^2))^{-1}\gcd(\varphi(E_1E_2)) \\ &= [\,\gcd(\varphi(E))^{-1}\gcd(\varphi(E_1))\,]\,[\,\gcd(\varphi(E))^{-1}\gcd(\varphi(E_2))\,] = \psi(X_1)\psi(X_2)\end{aligned}$$

and

$$\begin{aligned}\psi\big(\gcd(X)\big) &= \gcd(\varphi(E))^{-1}\gcd(\varphi(E_1 \cup E_2)) = \gcd(\varphi(E))^{-1}\gcd\big[\gcd(\varphi(E_1)),\gcd(\varphi(E_2))\,\big] \\ &= \gcd\big[\gcd(\varphi(E))^{-1}\gcd(\varphi(E_1)),\, \gcd(\varphi(E))^{-1}\gcd(\varphi(E_2))\,\big] = \gcd\big(\psi(X_1),\, \psi(X_2)\big)\,.\end{aligned}$$

2. (a) If $W \in \mathcal{W}$, then $\tau_r^{-1}(W)$ is an r_{a}-valuation monoid (and hence also an r-valuation monoid) by Theorem 3.4.10, and therefore $\tau_r^{-1}(W) \in \mathcal{V}$.

If $E \in \mathbb{P}_{\mathsf{f}}^*(D)$, then $E_{r_{\mathsf{a}}} = \gcd(\tau_r(E))$. Hence it follows that $E_{r_{\mathsf{a}}}\Lambda_r^+(K) = \tau_r(E)_t$, $E_{r_{\mathsf{a}}}W = \tau_r(E)W$ and $w(E_{r_{\mathsf{a}}}) = \min\{w(\tau_r(E)\} \in w\circ\tau_r(K^\times)$ by Theorem 3.4.2.2, and $w(\mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)^\bullet) = w\circ\tau_r(K^\times) \subset \Gamma$ is a subgroup. Since $\Lambda_r(K)^\times = \mathsf{q}(\mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)^\bullet)$, we obtain $\Gamma = \mathsf{q}(w\circ\tau_r(K^\times)) = w\circ\tau_r(K^\times)$. By definition, $V = \tau_r^{-1}(W) = (w\circ\tau_r)^{-1}(\Gamma_+)$, and since $w\circ\tau_r \,|\, K^\times \colon K^\times \to \Gamma$ is surjective, it is a valuation morphism of V.

(b) By (a) we must prove that $\widetilde{\tau}_r$ is bijective.

$\widetilde{\tau}_r$ is injective: For $i \in \{1,2\}$, let $W_i \in \mathcal{W}$ be such that $\tau_r^{-1}(W_i) = V \in \mathcal{V}$, and let $w_i \colon \Lambda_r(K)^\times \to \Gamma_i$ be a valuation morphism of W_i. Then $w_i\circ\tau_r \,|\, K^\times \colon K^\times \to \Gamma_i$ is a valuation morphism of V, and by Theorem 3.4.2.2 there exists an order isomorphism $\varphi\colon \Gamma_1 \to \Gamma_2$ such that $\varphi\circ w_1\circ\tau_r \,|\, K^\times = w_2\circ\tau_r \,|\, K^\times$. If $X \in \Lambda_r(K)^\times$, then $X = E_{r_{\mathsf{a}}}'^{[-1]}E_{r_{\mathsf{a}}}$ for some $E,\, E' \in \mathbb{P}_{\mathsf{f}}^*(D)$. Hence we obtain

$$\begin{aligned}w_2(X) &= w_2(E_{r_{\mathsf{a}}}) - w_2(E'_{r_{\mathsf{a}}}) = \min\{w_2\circ\tau_r(E^\bullet)\} - \min\{w_2\circ\tau_r(E'^\bullet)\} \\ &= \min\{\varphi\circ w_1\circ\tau_r(E^\bullet)\} - \min\{\varphi\circ w_1\circ\tau_r(E'^\bullet)\} = \varphi\big(\min\{w_1\circ\tau_r(E^\bullet)\} - \min\{w_1\circ\tau_r(E'^\bullet)\}\big) \\ &= \varphi\big(w_1(E_{r_{\mathsf{a}}}) - w_1(E'_{r_{\mathsf{a}}})\big) = \varphi\circ w_1(X)\,.\end{aligned}$$

Therefore $w_2(X) \ge 0$ holds if and only if $w_1(X) \ge 0$, and consequently $W_1 = W_2$.

$\widetilde{\tau}_r$ is surjective: Let $V \in \mathcal{V}$, and let $\varepsilon\colon K \to K/V^\times$ be the natural epimorphism. By the Theorems 2.3.7 and 3.4.10, V/V^* is an $\varepsilon(r)$-monoid of $K/V^\times$, and if $t^* = t(V/V^\times)$, then $\varepsilon(r) = t^*$, since $\varepsilon(r)$ is finitary, and ε is an (r,t^*)-homomorphism.

By 1. the map $\mathrm{Hom}_{(t,t^*)}(\Lambda_r(K), K/V^\times) \to \mathrm{Hom}_{(r,t^*)}(K, K/V^\times)$, given by $\Phi \mapsto \Phi\circ\tau_r$, is bijective. Hence there exists a unique (t,t^*)-homomorphism $\Phi\colon \Lambda_r(K) \to K/V^\times$ such that $\Phi\circ\tau_r = \varepsilon$, and we set $W = \Phi^{-1}(V/V^\times) \subset \Lambda_r(K)$. Then $\tau_r^{-1}(W) = (\Phi\circ\tau_r)^{-1}(V/V^\times) = \varepsilon^{-1}(V/V^\times) = V$, and since Φ is a (t,t^*)-homomorphism, Theorem 3.4.10 implies $W \in \mathcal{W}$. □

Theorem 4.4.4. *Let D be a cancellative monoid, $K = \mathsf{q}(D)$, r a finitary module system on K, $D \subset \{1\}_{r_{\mathsf{a}}}$ and $\mathcal{V}_r$ the set of all r-valuation monoids of K. Then $\mathcal{V}_r = \mathcal{V}_{r_{\mathsf{a}}}$, and for all $X \subset K$ we have*

$$X_{r_{\mathsf{a}}} = \bigcap_{V \in \mathcal{V}_r(D)} XV\,.$$

In particular, $\mathrm{cl}_r(D) = D_{r_{\mathsf{a}}}$ is the intersection of all r-valuation monoids of K.

PROOF. By Theorem 4.2.5.5 we have $\mathcal{V}_r = \mathcal{V}_{r_{\mathsf{a}}}$. Let $\tau_r \colon K \to \Lambda_r(K)$ be the Lorenzen r-homomorphism, $t = t(\Lambda_r^+(K))$ and $\mathcal{W}$ the set of all t-valuation monoids of $\Lambda_r(K)$. Then $\tau_r^* t = r_{\mathsf{a}}$ and $\mathcal{V}_r = \{\tau_r^{-1}(W) \mid W \in \mathcal{W}\}$. If $X \subset K$, then $\tau_r^{-1}(\tau_r(X)) = XD_{r_{\mathsf{a}}}^\times$, and therefore, using Theorem 3.4.9.3,

$$X_{r_{\mathsf{a}}} = \tau_r^{-1}(\tau_r(X)_t) = \tau_r^{-1}\Big(\bigcap_{W\in\mathcal{W}} \tau_r(X)W\Big) = \bigcap_{W\in\mathcal{W}} \tau_r^{-1}(\tau_r(X))\tau_r^{-1}(W) = \bigcap_{V\in\mathcal{V}_r(D)} XD_{r_{\mathsf{a}}}^\times V = \bigcap_{V\in\mathcal{V}_r(D)} XV\,. \ \square$$

Corollary 4.4.5. *Let D be a domain, $K = \mathsf{q}(D)$ and $d = d(D) \colon \mathbb{P}(K) \to \mathbb{P}(K)$. Let r be a finitary module system on K such that $d \le r$.*

1. *Let $V \subset K$ be a subset.*
   (a) *V is an r-valuation monoid of K if and only if V is a valuation domain satisfying $V_r = V$. If this is the case, then $D \subset D_r \subset V$.*
   (b) *V is a d-valuation monoid of K if and only if V is a valuation domain satisfying $D \subset V$.*
2. *The r-closure $\mathrm{cl}_r(D)$ of D is the intersection of all valuation domains V of K satisfying $V_r = V$. In particular, the integral closure $\mathrm{cl}_d(D)$ of D is the intersection of all valuation domains V of K containing D.*

PROOF. Obvious by the Theorems 4.4.3 and 4.4.4. $\square$

CHAPTER 5

Complete integral closures

Throughout this Chapter, let D be a cancellative monoid, $K = \mathsf{q}(D) \ne D$, $v = v(D)$ and $t = t(D)$.

5.1. Strong ideals

Theorem und Definition 5.1.1.

1. *For an ideal $I \subset D$, the following assertions are equivalent*:
   (a) $I^{-1} \subset (I:I)$.
   (b) $I^{-1} = (I:I)$.
   (c) I^{-1} *is an overmonoid of* D.
   (d) *There exists an overmonoid $T \supset D$ such that $I = T^{-1} = (D:T)$.*
   (e) $I_v = (II^{-1})_v$.

   A non-zero ideal $I \subset D$ satisfying these conditions is called *strong* (in D).
2. *Let D be a Mori domain and $\{0\} \ne P \in v\text{-spec}(D)$.*
   (a) *P is not strong if and only if D_P is a dv-monoid (and then $P \in \mathfrak{X}(D)$).*
   (b) *If $P \in v\text{-max}(D)$, then P is not strong if and only if P is v-invertible.*
   (c) *If $T \subset D^\bullet$ is a multiplicatively closed subset, then P is strong if and only if $T^{-1}P$ is strong in $T^{-1}D$.*

Proof. 1. (a) $\Rightarrow$ (b) $(I:I) \subset (D:I) = I^{-1}$.

(b) $\Rightarrow$ (c) $(I:I) \supset D$ is an overmonoid.

(c) $\Rightarrow$ (d) Obvious.

(d) $\Rightarrow$ (e) Let $T \supset D$ be an overmonoid such that $I = T^{-1}$. Then $I^{-1} = T_v \supset T$ is a monoid, and by Theorem 2.6.2.2 we obtain $(II^{-1})^{-1} = (I^{-1}:I^{-1}) = (T_v:T_v) = T_v = I^{-1}$. Hence $(II^{-1})_v = I_v$.

(e) $\Rightarrow$ (a) $(I:I) = (II^{-1})^{-1} = (II^{-1})_v^{-1} = I_v^{-1} = I^{-1}$ (by Theorem 2.6.2.2, applied with $X = I^{-1}$).

2. (a) If P is not strong and $a \in P^{-1} \setminus (P:P)$, then $aP \subset D$ and $aP \not\subset P$, which implies that $aP_P = D_P$. Since D_P is a Mori monoid, it satisfies the ascending chain condition on principal ideals. Hence it is atomic by Theorem 1.5.5, and by Theorem 3.4.8, it is a dv-monoid.

If P is strong, then $(D:P) = (P:P)$ implies $(D_P:P_P) = (P_P:P_P)$, and therefore D_P is not a dv-monoid.

(b) Assume that $P \in v\text{-max}(D)$. If P is strong, then $(PP^{-1})_v = P$ by 1., and therefore P is not v-invertible. If P is not strong, then D_P is a dv-monoid and P_P is a principal ideal of D_P. If $M \in v\text{-max}(D) \setminus \{P\}$, then $P_M = D_M$. Hence P is t-invertible (and thus v-invertible) by Theorem 4.1.4.

(c) By Theorem 1.3.8 we have $D_P = (T^{-1}D)_{T^{-1}P}$, and thus the assertion follows by (a). □

Theorem 5.1.2. *Let $I \subset D$ be a strong ideal, $C = (D:I) = (I:I)$ and $Q \subset C$ a prime ideal such that $I = {}_C\sqrt{I} \subset Q$. Then $(Q:Q) = C$.*

Proof. It suffices to prove that $(Q:Q) \subset (I:I)$. Indeed, then $C \subset (Q:Q) \subset (I:I) = C$, hence $(Q:Q) = C$, and if Q is strong, then $(C:Q) = C$ and $Q_{v(C)} = C \neq Q$.

Thus assume that $x \in (Q:Q)$ and $y \in I$. We must prove that $xy \in I$, and since

$$I = {}_C\sqrt{I} = \bigcap_{P \in \mathcal{P}_C(D)} P\,,$$

it suffices to prove that $xy \in P$ for all $P \in \mathcal{P}_C(I)$. If $Q \in \mathcal{P}_C(I)$, then $xy \in (Q:Q)I \subset (Q:Q)Q \subset Q$. If $P \in \mathcal{P}_C(I)$ and $P \neq Q$, then $Q \not\subset P$ and $xyQ \subset I(Q:Q)Q \subset IQ \subset I \subset P$, which implies $xy \in P$. □

Theorem 5.1.3. *Let $I \subset D$ be a strong ideal, $C = (D:I) = (I:I)$ and $v^* = v(C)$.*

1. *If D is a Mori monoid, then C is also a Mori monoid, and $\mathcal{F}_{v^*}(C) \subset \mathcal{F}_v(D)$.*
2. *The assignment $P \mapsto (P:I)$ defines a bijective map*

$$\Phi\colon \{P \subset D \mid P \text{ is a prime ideal, } I \not\subset P\} \to \{Q \subset C \mid Q \text{ is a prime ideal, } I \not\subset Q\}\,,$$

whose inverse is given by $Q \mapsto Q \cap D$.

3. *Let $P \subset D$ be a prime ideal such that $I \not\subset P$ and $Q = (P:I)$.*
   (a) $D_P = C_Q$.
   (b) *If $J \subset D$ and $J^* \subset C$ are ideals such that $J^* \cap D = J \subset P$, then $J^* \subset Q$.*
   (c) *If $P \in v$-spec(D), then $Q \in v^*$-spec(C).*
   (d) *If D is a Mori monoid and $P \in v$-max(D), then $Q \in v^*$-max(C).*

Proof. 1. Since $(D:I) \in \mathcal{F}_v(D) \subset \mathcal{F}_t(D)$, Theorem 2.6.6.3 implies that C is a Mori monoid, and $\mathcal{F}_{v^*}(C) = \mathcal{F}_{t(C)}(C) \subset \mathcal{F}_t(D) = \mathcal{F}_v(D)$.

2. Let $P \subset D$ be a prime ideal, $I \not\subset P$ and $Q = (P:I)$.

Clearly, $Q \subset (D:I) = C$, and $CQI \subset QI \subset P$ implies $CQ \subset (P:I) = Q$. Hence $Q \subset C$ is an ideal, and we prove that it is a prime ideal of C. Suppose that $x, y \in C$, $xy \in Q$ and $x \notin Q$. Then $xyI^2 \subset (P:I)I^2 \subset PI \subset P$, and since $xI \not\subset P$, we obtain $yI \subset P$ and $y \in (P:I) = Q$.

Next we prove that $Q \cap D = P$. Clearly, $PI \subset P$ implies $P \subset (P:I) \cap D = Q \cap D$. Conversely, if $z \in Q \cap D$, then $zI \subset P$ and $I \not\subset P$ implies $z \in P$.

It remains to prove that Φ is surjective. Thus let $R \subset C$ be a prime ideal ideal such that $I \not\subset R$. Then $R \cap D \subset D$ is a prime ideal, $I \not\subset R \cap D$, and we assert that $R = (R \cap D:I)$. If $x \in R$, then $R \subset C = (D:I)$ implies $xI \subset R \cap D$ and $x \in (R \cap D:I)$. Conversely, if $x \in (R \cap D:I$, then $xI \subset R$ and $I \not\subset R$ implies $x \in R$.

3. (a) Since $D \setminus P \subset C \setminus Q$, we obtain $D_P \subset C_Q$. Thus let $z = s^{-1}c \in C_Q$, where $c \in C$ and $s \in C \setminus Q$. If $y \in I \setminus P$, then $cy \in CI = I \subset D$, and $sI \not\subset P$ implies $sy \in CI \setminus P \subset D \setminus P$. Hence it follows that $z = (sy)^{-1}cy \in D_P$.

(b) Let $J \subset D$ and $J^* \subset C$ be ideals such that $J \subset P$ and $J^* \cap D = J$. Then it follows that $J^*I \subset J^* \cap CI \subset J^* \cap D = J \subset P$, and therefore $J^* \subset (P:I) = Q$.

(c) Suppose that $P \in v$-spec(D). We must prove that $(P:I)_{v^*} \subset (P:I)$. We have

$$I(P:I)_{v^*} = I\big(C:(C:(P:I))\big) = I\big(I^{-1}:((I:I):(P:I))\big) \subset \big(II^{-1}:(I:I(P:I))\big) \subset \big(D:(I:I(P:I))\big)\,,$$

and we shall prove that $P^{-1} \subset (I:I(P:I))$. If this is done, then $I(P:I)_{v^*} \subset (I:I(P:I))^{-1} \subset P_v = P$, and therefore $(P:I)_{v^*} \subset (P:I)$. If $z \in P^{-1}$, then $zI(P:I) \subset I(zP:I) \subset II^{-1} \subset I(I:I) \subset I$.

(d) Suppose that D is a Mori monoid and $P \in v$-max(D). Then $Q \in v^*$-spec(C), and since C is a Mori monoid, there exists some $M \in v^*$-max(C) such that $M \supset Q$. Then $M \cap D \in \mathcal{F}_v(D)$ is a prime ideal of D, hence $M \cap D \in v$-spec(D), and $P \subset M \cap D$. Hence $P = M \cap D$, $I \not\subset M$, and by 1. it follows that $Q = M \in v^*$-max(D). □

5.2. Complete integral closures and Krull monoids

Definition 5.2.1.

1. An element $a \in K$ is called *almost integral* over D if there exists some $c \in D^\bullet$ such that $ca^n \in D$ for all $n \in \mathbb{N}$.
2. The set $\widehat{D} = \{a \in K \mid a \text{ is almost integral over } D\}$ is called the *complete integral closure* of D, and $F_D = (D\!:\!\widehat{D})$ is called the *conductor* of D.
3. D is called *completely integrally closed* if $D = \widehat{D}$.
4. D is called a *Krull monoid* if D is a completely integrally closed Mori monoid.

Theorem 5.2.2. *Let r be an ideal system on D.*

1. *$\widehat{D}$ is a submonoid of K,*

$$\widehat{D} \;=\; \bigcup_{\substack{I \in \mathcal{I}_v(D) \\ I \text{ strong}}} I^{-1} = \bigcup_{J \in \mathcal{F}_v(D)^\bullet} (J\!:\!J) \;=\; \bigcup_{J \in \mathcal{F}_r(D)^\bullet} (J\!:\!J) \;=\; \bigcup_{J \in \mathcal{I}_r(D)^\bullet} (J\!:\!J)$$

and if r is finitary, then $\widehat{D}_r = \widehat{D}$.

In particular, if D is a domain, then $\widehat{D}$ is also a domain.

2. $\widehat{D/D^\times} = \widehat{D}/D^\times$. *In particular, D is completely integrally closed if and only if $D/D^\times$ is completely integrally closed.*
3. $\mathrm{cl}_r(D) \subset \widehat{D}$, *and if D is r-noetherian, then $\widehat{D} = \mathrm{cl}_r(D)$. In particular, if D is completely integrally closed, then D is r-closed, and the converse holds if D is r-noetherian.*
4. *F_D is the intersection of all strong v-ideals of D.*
5. *$F_D^\bullet \ne \emptyset$ if and only if D contains a smallest strong v-ideal F. If F is the smallest strong v-ideal of D, then $F_D = F$, $\widehat{D} = F^{-1} \in \mathcal{F}_v(D)$, and $\widehat{D}$ is completely integrally closed.*

Proof. 1. We show that

$$\widehat{D} \;\subset\; \bigcup_{\substack{I \in \mathcal{I}_v(D) \\ I \text{ strong}}} I^{-1} \;\subset\; \bigcup_{J \in \mathcal{F}_v(D)^\bullet} (J\!:\!J) \;\subset\; \bigcup_{J \in \mathcal{F}_r(D)^\bullet} (J\!:\!J) \;=\; \bigcup_{J \in \mathcal{I}_r(D)^\bullet} (J\!:\!J) \;\subset \widehat{D}\,.$$

If $x \in \widehat{D}$, then there is some $c \in D^\bullet$ such that $X = \{cx^n \mid n \in \mathbb{N}\} \subset D$. By Theorem 5.1.1.1 (b), $I = (X_v\!:\!X_v)^{-1} \in \mathcal{I}_v(D)$ is strong, and since $xX \subset X$, it follows that $xX_v \subset X_v$ and thus $x \in I^{-1}$.

The two following inclusions are obvious. If $J \in \mathcal{F}_r(D)^\bullet$ and $c \in D^\bullet$ is such that $cJ \subset D$, then $(J\!:\!J) = (cJ\!:\!cJ)$. If $J \in \mathcal{I}_r(D)^\bullet$, $c \in J^\bullet$ and $x \in (J\!:\!J)$, then $x^n \in (J\!:\!J)$ and therefore $cx^n \in J \subset D$ for all $n \in \mathbb{N}$, which implies $x \in \widehat{D}$.

If $J,\, J' \in \mathcal{I}_r(D)^\bullet$, then $((JJ')_r\!:\!(JJ')_r) \supset (J\!:\!J)$. Therefore $\{(J\!:\!J) \mid J \in \mathcal{I}_r(D)^\bullet\}$ is a directed set of r-monoids. Hence $\widehat{D}$ is a monoid, and if r is finitary, then $\widehat{D}_r = D$.

If D is a domain, then $\widehat{D}_{d(D)} = \widehat{D}$. Hence $\widehat{D}$ is a D-module and therefore itself a domain.

2. By definition, $\mathsf{q}(D/D^\times) = K/D^\times$, and if $x \in K$, then $x \in \widehat{D}$ if and only if $xD^\times \in \widehat{D/D^\times}$. Hence $\widehat{D/D^\times} = \widehat{D}/D^\times$, and $D = \widehat{D}$ if and only if $D/D^\times = \widehat{D/D^\times}$.

3. By Theorem 4.3.3 we have

$$\mathrm{cl}_r(D) = \bigcup_{J \in \mathcal{I}_{r,\mathsf{f}}(D)} (J\!:\!J)\,.$$

Hence $\mathrm{cl}_r(D) \subset \widehat{D}$. If D is r-noetherian, then $\mathcal{I}_r(D) = \mathcal{I}_{r,\mathsf{f}}(D)$, and therefore equality holds.

4. By 1., we obtain

$$F_D = \widehat{D}^{-1} = \Big(\bigcup_{\substack{I \in \mathcal{I}_v(D) \\ I \text{ strong}}} I^{-1} \Big)^{-1} = \bigcap_{\substack{I \in \mathcal{I}_v(D) \\ I \text{ strong}}} I \,.$$

5. If $F_D^\bullet \ne \emptyset$, then F_D is a strong v-ideal by Theorem 5.1.1, and by 4. it is the smallest strong v-ideal of D. Conversely, if F is the smallest strong v-ideal of D, then $F = F_D$ by 4. Hence $\widehat{D}_v = F^{-1} \subset \widehat{D}$, and therefore $F^{-1} = \widehat{D} \in \mathcal{F}_v(D)$. In particular, if $\mathcal{F}(D)$ resp. $\mathcal{F}(\widehat{D})$ denotes the set of all fractional semigroup ideals of D resp. $\widehat{D}$, then $\mathcal{F}(\widehat{D}) \subset \mathcal{F}(D)$, hence

$$\widehat{\widehat{D}} = \bigcup_{J \in \mathcal{F}(\widehat{D})} (J \colon J) \subset \bigcup_{J \in \mathcal{F}(D)} (J \colon J) \subset \widehat{D}\,,$$

and therefore equality follows. □

Theorem 5.2.3. *The following assertions are equivalent:*

(a) *D is completely integrally closed.*

(b) *$\mathcal{F}_v(D)^\bullet = \mathcal{F}_v(D)^\times$ [equivalently: every non-zero (fractional) v-ideal of D is v-invertible].*

(c) *D is the only strong v-ideal of D.*

Proof. (a) ⇒ (b) If $J \in \mathcal{F}_v(D)^\bullet$, then $(J:J) \subset \widehat{D} = D$, hence $(J:J) = D$, and therefore $J \in \mathcal{F}_v(D)^\times$ by Theorem 4.1.2.

(b) ⇒ (c) If $J \in \mathcal{F}_v(D)^\bullet$ is strong and invertible, then $J = J \cdot_v J^{-1} = D$.

(c) ⇒ (a) By Theorem 5.2.2.1. □

Theorem 5.2.4. *Let D be a Mori monoid.*

1. *If $F_D^\bullet \ne \emptyset$, then $\widehat{D}$ is a Krull monoid.*
2. *Let $T \subset D^\bullet$ be a multiplicatively closed subset. Then $\widehat{T^{-1}D} = T^{-1}\widehat{D}$. In particular, if D is a Krull monoid, then $T^{-1}D$ is a Krull monoid.*
3. *$\widehat{D}^\times \cap D = D^\times$.*

Proof. 1. If $F_D^\bullet \ne \emptyset$, then $\widehat{D}$ is completely integrally closed by Theorem 5.2.2.5, and $\widehat{D}$ is a Mori monoid by Theorem 5.1.1.2.

2. Observe that $T^{-1}t = t(T^{-1}D)$, and $\widehat{T^{-1}D} = \mathrm{cl}_{T^{-1}t}T^{-1}D) = T^{-1}\mathrm{cl}_t(D) = T^{-1}\widehat{D}$ by the Theorems 2.6.6.2 and 4.3.4.3.

3. Obviously, $D^\times \subset \widehat{D}^\times \cap D$. If $a \in \widehat{D}^\times \cap D$, then there is some $c \in D^\bullet$ such that $ca^{-n} \in D$ for all $n \in \mathbb{N}$. Hence it follows that $c \in a^nD$ for all $n \in \mathbb{N}$, and therefore the set $\{a^nD \mid n \in \mathbb{N}\} \subset \mathcal{I}_v(D)$ has a smallest element. Consequently, there is some $n \in \mathbb{N}$ such that $a^nD = a^{n+1}D$, which implies $D = aD$ and $a \in D^\times$. □

Theorem 5.2.5.

1. *The following assertions are equivalent:*
   (a) *D is a Krull monoid.*
   (b) *$\mathcal{F}_t(D)^\bullet = \mathcal{F}_t(D)^\times$ [equivalently: every non-zero (fractional) t-ideal of D is t-invertible].*
   (c) *D is a Mori monoid, and for every $M \in t\text{-max}(D)$, D_M is a dv-monoid.*

   In particular, if D is a Krull monoid, then $t\text{-max}(D) = \mathfrak{X}(D)$, and therefore D_P is a discrete valuation monoid for every non-zero prime t-ideal.

2. *D is factorial if and only if D is a Krull monoid and $\mathcal{C}_v(D) = \mathbf{0}$.*
3. *D is a dv-monoid if and only if D is a t-local Krull monoid.*

Proof. 1. (a) $\Rightarrow$ (b) $v = t$, and by Theorem 5.2.3 we have $\mathcal{F}_v(D)^\bullet = \mathcal{F}_v(D)^\times$.

(b) $\Rightarrow$ (c) Since every non-zero t-ideal is t-invertible and hence t-finitely generated, it follows that D is a Mori monoid. If $M \in t\text{-max}(D)$, then D_M is t-noetherian, hence atomic, and M_M is a principal ideal. By Theorem 3.4.8, D_P is a dv-monoid.

(c) $\Rightarrow$ (a) If $J \in \mathcal{F}_v(D)^\bullet = \mathcal{F}_t(D)^\bullet$, then J_M is principal for all $M \in t\text{-max}(D)$. Hence J is t-invertible, and as $t = v$, D is completely integrally closed by Theorem 5.2.3.

In particular, if D is a Krull monoid and $P \in t\text{-max}(D)$, then P is t-invertible and thus $P \in \mathfrak{X}(D)$ by Theorem 5.1.1.4.

2. By Theorem 2.6.3.2, D is factorial if and only if every non-zero t-ideal is principal. However, this holds if and only if ever $J \in \mathcal{I}_t(D)^\bullet$ is t-invertible and principal. By 1., the assertion follows.

3. Obvious by 1.(c). □

Theorem 5.2.6. *Let D be a Krull monoid. Then $\Lambda_t(K) = \mathcal{F}_{t,\mathsf{f}}(D)$, and $\Lambda_t^+(K) = \mathcal{I}_{t,\mathsf{f}}(D)$ is free with basis t-max(D).*

Proof. Since $\mathcal{M}_{t,\mathsf{f}}(K) = \mathcal{F}_{t,\mathsf{f}}(D)$ and $\mathcal{F}_{t,\mathsf{f}}(D)^\bullet$ is a group, it follows that t is finitely cancellative, hence $t = t_{\mathsf{a}}$, $\Lambda_t(K) = \mathsf{q}(\mathcal{M}_{t,\mathsf{f}}(K)) = \mathcal{F}_{t,\mathsf{f}}(D)$, and $\Lambda_t^+(K) = \{C \in \mathcal{F}_{t,\mathsf{f}}(D) \mid C \subset D_{t_{\mathsf{a}}} = D\} = \mathcal{I}_{t,\mathsf{f}}(D)$ is a reduced GCD-monoid by Theorem 4.4.2. Moreover, for all $I,\, J \in \mathcal{I}_{t,\mathsf{f}}(D)$ we have $I \mid J$ in $\mathcal{I}_{t,\mathsf{f}}(D)$ if and only if $J \subset I$. Hence $\Lambda_t^+(D)$ satisfies the ACC for principal ideals, and as it is a reduced GCD-monoid, it is factorial and therefore free with the set of prime elements as a basis. An element $P \in \mathcal{I}_{t,\mathsf{f}}(D) \setminus \{D\}$ is a prime element if and only if it is maximal with respect to inclusion, that is, if and only if it is a t-maximal t-ideal. □

Definition 5.2.7. A domain D is called a

- *Krull domain* if it is a Krull monoid;
- *Dedekind domain* if it is a Krull domain, and $d(D) = t$ [equivalently, every ideal is divisorial].

Theorem 5.2.8. *For a domain D, the following assertions are equivalent:*

(a) *D is a Dedekind domain.*
(b) *D is a Krull domain and* $\dim(D) = 1$ [*equivalently, every non-zero prime ideal of D is maximal*].
(c) *Every non-zero ideal of D is invertible.*
(d) *D is noetherian, and for every non-zero prime ideal P, D_P is a discrete valuation domain.*
(e) *D is noetherian, integrally closed, and* $\dim(D) = 1$ [*equivalently, every non-zero prime ideal of D is maximal*].

Proof. Set $d = d(D)$.

(a) $\Rightarrow$ (b) If $P \in \operatorname{spec}(D) = t\text{-spec}(D)$ and $P^\bullet \ne \emptyset$, then P is not strong by Theorem 5.2.3, and thus $P \in \mathfrak{X}(D)$ by Theorem 5.1.1.4.

(b) $\Rightarrow$ (c) Let $J \in \mathcal{I}(D)^\bullet$ be a non-zero ideal. Then $J_t \in \mathcal{I}_{t,\mathsf{f}}(D)$, and by Theorem 4.1.4 we must prove that J_P is principal for all $P \in \max(D)$. If $P \in t\text{-max}(D)$, then D_P is a discrete valuation domain and therefore J_P is principal. However, $\max(D) = \mathfrak{X}(D)$ by assumption, and by the Theorems 3.1.6.4 and 5.2.5 it follows that $\max(D) = t\text{-max}(D)$.

(c) $\Rightarrow$ (a) Every non-zero ideal of D is invertible, hence a t-ideal by Theorem 4.1.2. Therefore $t = d$, and D is a Krull domain by Theorem 5.2.5.

(a) $\Rightarrow$ (d) Obvious by Theorem 5.2.5.

(d) $\Rightarrow$ (e) If $P \in \operatorname{spec}(D)$ and $P^\bullet \neq \emptyset$, then D_P is a discrete valuation domain, hence primary, and therefore $P \in \mathfrak{X}(D)$ by Theorem 3.4.6.3. Hence $\dim(D) = 1$. Moreover, for all non-zero $P \in \operatorname{spec}(D)$, D_P is a Krull domain and thus (completely) integrally closed. Hence D is integrally closed by Theorem 4.3.4.4.

(e) $\Rightarrow$ (a) It suffices to prove that $\mathcal{I}(D) \subset \mathcal{I}_t(D)$. Since $\dim(D) = 1$, we have $\max(D) = t\text{-}\max(D)$, and we assert that, for every $P \in \max(D)$, D_P is a discrete valuation domain. If $P \in \max(D)$, then D_P is noetherian and integrally closed, hence v-noetherian and completely integrally closed and therefore a Krull domain. Being t-local, D_P is a discrete valuation domain, and $t_P = s(D_P)$. Thus, if $J \in \mathcal{I}(D)$, then $(J_t)_P = (J_P)_{t_P} = J_P$, and therefore (using Theorem 3.2.2),

$$J_t = \bigcap_{P \in t\text{-}\max(D)} (J_t)_P = \bigcap_{P \in \max(D)} J_P = J \in \mathcal{I}_t(D)\,. \qquad \square$$

The following example shows that the complete integral closure need not be completely integrally closed.

Example 5.2.9. Let K be a field,

$$R = K[\,\{X^{2n+1}Y^{n(2n+1)} \mid n \in \mathbb{N}_0\,\}\,] \quad \text{and} \quad S = K[\,\{XY^n \mid n \in \mathbb{N}_0\,\}\,]\,.$$

Then $R \subset S \subset K[X,Y] = \mathsf{q}(R)$, $S = \widehat{R}$ and $K[X,Y] = \widehat{S}$.

Proof. By definition, $R \subset S \subset K[X,Y]$, and for all $n \in \mathbb{N}_0$, $(XY^n)^{2n+1} \in R$. Hence S is integral over R, and therefore $S \subset \widehat{R}$. Since $\{X,\, X^3Y^3,\, X^5Y^{10}\} \subset R$, we obtain $Y = X^4(X^3Y^3)^{-3}(X^5Y^{10}) \in \mathsf{q}(R)$ and therefore $\mathsf{q}(R) = K[X,Y]$. Since $XY^n \in S$ for all $n \in \mathbb{N}_0$, it follows that $K[X,Y] \subset \widehat{S}$. On the other hand, $K[X,Y]$ is factorial, hence a Krull domain and therefore completely integrally closed. Thus we obtain $\widehat{S} \subset \widehat{K[X,Y]} = K[X,Y]$, and it remains to prove that $\widehat{R} \subset S$. We show the following two assertions:

A. $K[X,Y] = S + K[Y]$.

B. $K[Y] \cap \widehat{R} = K$.

Suppose that **A** and **B** hold, and let $f \in \widehat{R} \subset K[X,Y]$. By **A** we have $f = g + h$, where $g \in S$ and $h \in K[Y]$. Since $S \subset \widehat{R}$, it follows that $h = f - g \in K[Y] \cap \widehat{R} = K$ and therefore $f = g + h \in S$.

Proof of **A**. It suffices to prove that $X^iY^j \in S + K[Y]$ for all $i,\, j \in \mathbb{N}_0$. This is obvious for $i = 0$, and if $i \geq 1$, then $X^iY^j = X^{i-1}(XY^j) \in S$, since $X \in S$. $\square$[**A.**]

Proof of **B.** Assume to the contrary, that there is some $f \in K[Y] \cap \widehat{R}$ such that $\deg(f) = n \geq 1$, and let $a \in K^\times$ be the leading coefficient of f. Then there exists some $g \in R^\bullet$ such that $gf^k \in R$ for all $k \in \mathbb{N}$. Suppose that $g = (bX^l + h_1)Y^r + g_0$, where $l,\, r \in \mathbb{N}_0$, $b \in K^\times$, $h_1 \in K[X]$, $\deg(h_1) < l$, $g_0 \in K[X,Y]$ and $\deg_Y(g_0) < r$. Let $k \in \mathbb{N}$ be such that

$$r + nk > l \sum_{i=0}^{l} i(2i+1)\,.$$

Then $gf^k = ba^kX^lY^{r+nk} + g_k$, where $g_k \in K[X,Y]$ and $\deg_Y(g_k) < r + nk$. A K-basis of R is given by the set of all products of the form

$$\prod_{\nu=0}^{N} X^{2s_\nu+1}Y^{s_\nu(2s_\nu+1)}\,, \quad \text{where} \quad N \in \mathbb{N}_0 \ \text{ and } \ s_0, \ldots, s_N \in \mathbb{N}_0\,.$$

Hence there exist some $N \in \mathbb{N}$ and $s_0, \ldots, s_N \in \mathbb{N}_0$ such that

$$X^l Y^{r+nk} = \prod_{\nu=0}^{N} X^{2s_\nu+1} Y^{s_\nu(2s_\nu+1)}\,.$$

For $i \in \mathbb{N}_0$, we define $r_i = |\{\nu \in [0,N] \mid s_\nu = i\}|$, and then we obtain

$$l = \sum_{\nu=0}^{N}(2s_\nu+1) = \sum_{i\ge 0} r_i(2i+1) \quad \text{and} \quad r+nk = \sum_{\nu=0}^{N} s_\nu(2s_\nu+1) = \sum_{i\ge 0} r_i i(2i+1)\,.$$

Hence it follows that $r_i \le l$ for all $i \ge 0$, and

$$r+nk \le l\sum_{i\ge 0} i(2i+1) < r+nl\,, \quad \text{a contradiction.}$$ □

5.3. Overmonoids of Mori monoids

Theorem 5.3.1. *Let $(D_\lambda)_{\lambda\in\Lambda}$ be a family of monoids such that $D \subset D_\lambda \subset K$ for all $\lambda \in \Lambda$,*

$$D' = \bigcap_{\lambda\in\Lambda} D_\lambda\,,$$

and assume that, for every $a \in D^\bullet$, the set $\{\lambda \in \Lambda \mid a \notin D_\lambda^\times\}$ is finite.

1. *If $T \subset D^\bullet$ is a multiplicatively closed subset, then*

$$T^{-1}D' = \bigcap_{\lambda\in\Lambda} T^{-1}D_\lambda\,.$$

2. *If $(D_\lambda)_{\lambda\in\Lambda}$ is a family of Mori monoids, then D' is a Mori monoid.*

Proof. 1. Obviously, $T^{-1}D' \subset T^{-1}D_\lambda$ for all $\lambda \in \Lambda$. Thus suppose that

$$x \in \bigcap_{\lambda\in\Lambda} T^{-1}D_\lambda\,, \quad \text{say} \quad x = a^{-1}b\,, \quad \text{where} \quad a \in D^\bullet \quad \text{and} \quad b \in D\,.$$

The set $\Delta = \{\lambda \in \Lambda \mid a \notin D_\lambda^\times\}$ is finite, and if $\lambda \in \Lambda \setminus \Delta$, then $x \in D_\lambda$. For each $\lambda \in \Delta$, there exist $a_\lambda \in D_\lambda$ and $t_\lambda \in T$ such that $x = t_\lambda^{-1}a_\lambda$, and we set

$$t = \prod_{\lambda\in\Delta} t_\lambda\,.$$

Then it follows that $t \in T$, $tx \in D'$ and $x = t^{-1}(tx) \in T^{-1}D'$.

2. For every subset $X \subset D'$, we set

$$X' = \bigcap_{\lambda\in\Lambda} X_{v(D_\lambda)}\,, \quad \text{and we assert that} \quad X \subset X' \subset X_{v(D')}.$$

Obviously, $X \subset X'$, and if $c \in K$ is such that $X \subset D'c$, then $X_{v(D_\lambda)} \subset D'_{v(D_\lambda)}c \subset D_\lambda c$ for all $\lambda \in \Lambda$, and therefore $X' \subset D'c$. Hence it follows that

$$X' \subset \bigcap_{\substack{c\in K\\ X\subset D'c}} D'c = X_{v(D')}\,.$$

We prove that for every subset $X \subset D'$ there exists some $E \in \mathbb{P}_{\mathfrak{f}}(X)$ such that $X \subset E_{v(D')}$. Thus let $X \subset D'$. We may assume that $X^\bullet \ne \emptyset$, and we fix some $a \in X^\bullet$. Then the set $\Delta = \{\lambda \in \Lambda \mid a \notin D_\lambda^\times\}$ is finite, and for every $\lambda \in \Delta$, there is some $E_\lambda \in \mathbb{P}_{\mathsf{f}}(X)$ such that $a \in E_\lambda$ and $X_{v(D_\lambda)} = (E_\lambda)_{v(D_\lambda)}$. Now we consider the set

$$E = \bigcup_{\lambda \in \Delta} E_\lambda \in \mathbb{P}_{\mathsf{f}}(X)\,.$$

If $\lambda \in \Delta$, then $E_{v(D_\lambda)} \supset (E_\lambda)_{v(D_\lambda)} = X_{v(D_\lambda)}$, and if $\lambda \in \Lambda \setminus \Delta$, then $E_{v(D_\lambda)} = D_\lambda = X_{v(D_\lambda)} = D_\lambda$. Hence we obtain

$$E_{v(D')} \supset E' = \bigcap_{\lambda \in \Lambda} E_{v(D_\lambda)} \supset \bigcap_{\lambda \in \Lambda} X_{v(D_\lambda)} = X' \supset X\,. \qquad \square$$

Definition 5.3.2. Let D be a Mori monoid. We define

$$\mathcal{S}(D) = \{P \in v\text{-}\max(D) \mid P \text{ strong}\} \quad \text{and} \quad \mathcal{R}(D) = \{P \in v\text{-}\max(D) \mid P \text{ not strong}\}\,,$$

$$\widetilde{D} = \bigcap_{P \in \mathcal{R}(D)} D_P \cap \bigcap_{P \in \mathcal{S}(D)} (D_P : P_P)\,, \quad \widetilde{v} = v(\widetilde{D}) \quad \text{and} \quad \widetilde{t} = t(\widetilde{D})\,.$$

If $P \in v$-max(D), then Theorem 5.1.1.2 implies that $P \in \mathcal{R}(D)$ if and only if D_P is a dv-monoid, and $P \in \mathcal{S}(D)$ if and only if D_P is not a dv-monoid. In particular, Theorem 5.2.5 implies that D is a Krull monoid if and only if $\mathcal{S}(D) = \emptyset$.

Theorem 5.3.3. *Let D be a Mori monoid.*

1. *$\widetilde{D} \in \mathcal{M}_t(K)$ is a Mori monoid, and $\widetilde{D} \subset \mathrm{cl}_t(D) \subset \widehat{D}$.*
2. *If $Q \in \mathcal{S}(D)$, then $\widetilde{D}_Q = (D_Q : Q_Q) = (Q_Q : Q_Q)$.*
3. *If $R \in \widetilde{v}$-spec$(\widetilde{D})$, then $R \cap D \in v$-spec(D), and if R is strong, then $R \cap D$ is strong, too.*

Proof. 1. If $P \in \mathcal{S}(D)$, then $(D_P : P_P) = (D : P)_P = (P : P)_P \subset D$ is an overmonoid, and therefore $\widetilde{D} \supset D$ is an overmonoid. If $P \in v$-max(D), then $(D_P)_t = D_P \in \mathcal{M}_t(K)$ by Theorem 2.5.4, hence $(D_P : P_P) \in \mathcal{M}_t(K)$, and therefore it follows that $\widetilde{D} \in \mathcal{M}_t(K)$. By Theorem 2.6.6 it follows that D_P is a Mori monoid for all $P \in \mathcal{R}(D)$, and that $(D_P : P_P) = (P : P)_P$ is a Mori monoid for all $P \in \mathcal{S}(D)$. If $a \in D^\bullet$, then the set $\{P \in v\text{-spec}(D) \mid a \in P\}$ is finite by Theorem 3.2.7.2. If $P \in \mathcal{R}(D)$ and $a \notin P$, then $a \in D_P^\times$. If $P \in \mathcal{S}(D)$ and $a \notin P$; then $a^{-1} \in (P : P)_P = (D_P : P_P)$, and therefore $a \in (D_P : P_P)^\times$. By Theorem 5.3.1.2 it follows that $\widetilde{D}$ is a Mori monoid.

If $P \in \mathcal{S}(D)$, then $(D_P : P_P) = (D : P)_P = (P : P)_P \subset \mathrm{cl}_t(D)_P$, and therefore

$$\widetilde{D} \subset \bigcap_{P \in v\text{-}\max(D)} \mathrm{cl}_t(D)_P = \mathrm{cl}_t(D) \subset \widehat{D}\,.$$

2. Assume that $Q \in \mathcal{S}(D)$. If $P \in v$-max(D) and $P \neq Q$, then $(D : Q) \subset D_P$ by Theorem 1.3.9.1, and therefore $(D_Q : Q_Q) = (D : Q)_Q \subset (D_P)_Q \subset (D_P : P_P)_Q$. If $P \in \mathcal{R}(D)$, then D_P is a dv-monoid, and since $P \not\subset Q$, Theorem 1.3.9.2 implies that $D_P \subsetneq (D_P)_Q$. By Theorem 3.4.8, D_P is primary, and by Theorem 3.4.6 we obtain $(D_P)_Q = K$. Collecting these arguments, we obtain, using Theorem 5.3.1.1,

$$\widetilde{D}_Q = \bigcap_{P \in \mathcal{R}(D)} (D_P)_Q \cap \bigcap_{P \in \mathcal{S}(D)} (D_P : P_P)_Q = \bigcap_{\substack{P \in \mathcal{S}(D) \\ P \neq Q}} (D_P : P_P)_Q \cap (D_Q : Q_Q) = (D_Q : Q_Q)\,.$$

Finally, $(D_Q : Q_Q) = (Q_Q : Q_Q)$, since Q_Q is strong in D_Q.

3. By Theorem 2.5.2.4, $t[\widetilde{D}]$ is an ideal system of $\widetilde{D}$, and therefore $t \le t[\widetilde{D}] \le \widetilde{t}$. If $R \in \widetilde{v}$-spec$(\widetilde{D})$, then $(R \cap D)_v = (R \cap D)_t \subset (R \cap D)_{\widetilde{t}} \subset R_{\widetilde{t}} = R$, hence $(R \cap D)_v \subset R \cap D$ and therefore $R \cap D \in v$-spec(D).

If $R \cap D$ is not strong, then $D_{R \cap D}$ is a dv-monoid, and since $D_{R \cap D} \subset \widetilde{D}_R \subsetneq K$, it follows that $\widetilde{D}_R = D_{R \cap D}$. Hence R is not strong. $\square$

Theorem 5.3.4. *Let D be a Mori monoid and $P \in v\text{-spec}(D) \setminus \mathcal{S}(D)$. Then there exists a unique $\widetilde{P} \in \widetilde{v}\text{-spec}(\widetilde{D})$ such that $\widetilde{P} \cap D = P$, and the following assertions hold:*

- $D_P = \widetilde{D}_{\widetilde{P}}$.
- *P is strong if and only if $\widetilde{P}$ is strong.*
- *If $P \in \mathcal{R}(D)$, then $\widetilde{P} \in \mathcal{R}(\widetilde{D})$.*
- *If $I \in \mathcal{I}_v(D)$, $\widetilde{I} \in \mathcal{I}_{\widetilde{v}}(\widetilde{D})$ and $\widetilde{I} \cap D = I \subset P$, then $\widetilde{I} \subset \widetilde{P}$, and $\widetilde{I}_{\widetilde{P}} = I_P$.*

Proof. We assume first that all statements of the Theorem except the equality $\widetilde{I}_{\widetilde{P}} = I_P$ in in the last assertion hold, and we show how this equality follows. Since $D_P = \widetilde{D}_{\widetilde{P}}$, we obtain $\widetilde{P}_{\widetilde{P}} = P_P \subset \widetilde{P}_P$, and since $D \setminus P \subset \widetilde{D} \setminus \widetilde{P}$, it follows that $\widetilde{P}_P \subset \widetilde{P}_{\widetilde{P}}$ and therefore $\widetilde{P}_P = P_P$. Let now $I \in \mathcal{I}_v(D)$ and $\widetilde{I} \in \mathcal{I}_{\widetilde{v}}(\widetilde{D})$ be such that $\widetilde{I} \cap D = I \subset P$ and $\widetilde{I} \subset \widetilde{P}$. Then $P \cap \widetilde{I} = I$, and $\widetilde{I}_{\widetilde{P}} = \widetilde{I}\widetilde{D}_{\widetilde{P}} = \widetilde{I}D_P = \widetilde{I}_P = \widetilde{P}_P \cap \widetilde{I}_P = P_P \cap \widetilde{I}_P = (P \cap \widetilde{I})_P = I_P$.

For the main part of the proof we distinguish two cases. Since $P \in v\text{-spec}(D) \setminus \mathcal{S}(D)$, it follows that either $P \in \mathcal{R}(D)$, or that P is not v-maximal. In this second case, there is some $M \in v\text{-max}(D)$ such that $P \subsetneq M$, and then necessarily $M \in \mathcal{S}(D)$.

CASE 1: $P \in \mathcal{R}(D)$.

In this case, D_P is a dv-monoid, $\widetilde{D} \subset D_P$, and we set $\widetilde{P} = P_P \cap \widetilde{D}$. Then $\widetilde{P} \subset \widetilde{D}$ is a prime ideal, and $\widetilde{P} \cap D = P_P \cap D = P$. Suppose now that $P' \subset \widetilde{D}$ is another prime ideal satisfying $P' \cap D = P$. Then $D_P \subset \widetilde{D}_{P'} \subsetneq K$, hence $D_P = \widetilde{D}_{P'}$, and $P_P = P'_{P'}$ is a principal ideal. Therefore it follows that $\widetilde{P} = P_P \cap \widetilde{D} = P'_{P'} \cap \widetilde{D} = P' \in \widetilde{v}\text{-spec}(\widetilde{D})$ by Theorem 2.6.6.2 (c). Since $\widetilde{D}_{\widetilde{P}} = D_P$ is a dv-monoid, $\widetilde{P}$ is not strong, and we assert that $\widetilde{P} \in \widetilde{v}\text{-max}(\widetilde{D})$. Indeed, if $\overline{P} \in \widetilde{v}\text{-spec}(\widetilde{D})$ is such that $\widetilde{P} \subset \overline{P}$, then $P = \widetilde{P} \cap D \subset \overline{P} \cap D$, and since $\overline{P} \cap D \in v\text{-spec}(D)$ by Theorem 5.3.3.3, it follows that $\overline{P} \cap D = P$ and therefore $\widetilde{P} = \overline{P} \in \widetilde{v}\text{-max}(\widetilde{D})$ by the uniqueness of $\widetilde{P}$.

Assume finally that $I \in \mathcal{I}_v(D)$, $\widetilde{I} \in \mathcal{I}_{\widetilde{v}}(\widetilde{D})$ and $\widetilde{I} \cap D = I \subset P$. We must prove that $\widetilde{I} \subset \widetilde{P}$, and we may assume that $\widetilde{I}^\bullet \ne \emptyset$. Then Theorem 3.2.7.2 implies that $\{P' \in \widetilde{v}\text{-max}(\widetilde{D}) \mid \widetilde{I} \subset P'\} = \{P'_1, \ldots, P'_n\}$ for some $n \in \mathbb{N}$. For $i \in [1,n]$, we set $P_i = P'_i \cap D$, and then we obtain

$$P \supset I = \widetilde{I} \cap D = \widetilde{I} = \bigcap_{P' \in \widetilde{v}\text{-}\max(\widetilde{D})} \widetilde{I}_{P'} \cap D = \widetilde{I}_{P'_1} \cap \ldots \cap \widetilde{I}_{P'_n} \cap D \supset I_{P_1} \cap \ldots \cap I_{P_n} \cap D\,.$$

Hence there exists some $i \in [1,n]$ such that $I_{P_i} \cap D \subset P$, and therefore

$$P \supset \sqrt{I_{P_i} \cap D} \supset \sqrt{I_{P_i}} \cap D = \sqrt{I_{P_i}} \cap D = \bigcap_{Q \in \mathcal{P}(I_{P_i})} Q \cap D\,.$$

Hence it follows that $Q \cap D \subset P$ for some $Q \in \mathcal{P}(I_{P_i}) \subset v_{P_i}\text{-spec}(D_{P_i})$, and since $Q \cap D \in v\text{-spec}(D)$ and $P \in \mathfrak{X}(D)$, we obtain $P = Q \cap D \subset (P_i)_{P_i} \cap D = P_i$. As $P \in v\text{-max}(D)$, we get $P = P_i$ and (by the uniqueness of $\widetilde{P}$) $\widetilde{P} = P'_i \supset \widetilde{I}$.

CASE 2: There is some $M \in \mathcal{S}(D)$ is such that $P \subsetneq M$.

In this case, $\widetilde{D}_M = (D_M : M_M) = (M_M : M_M)$ by Theorem 5.3.3, and $P_M \in v_M\text{-spec}(D_M)$. By Theorem 5.1.3, applies to the extension $D_M \subset \widetilde{D}_M$, there exists a unique $P^* \in \widetilde{v}_M\text{-spec}(\widetilde{D}_M)$ such that $P^* \cap D_M = P_M$, and the following assertions hold:

- $(\widetilde{D}_M)_{P^*} = (D_M)_{P_M}$.
- If $P_M \in v_M\text{-max}(D_M)$, then $P^* \in \widetilde{v}_M\text{-max}(\widetilde{D}_M)$.
- If $J \subset D_M$ and $J^* \subset \widetilde{D}_M$ are ideals such that $J^* \cap D_M = J \subset P_M$, then $J^* \subset P^*$.

Now we set $\widetilde{P} = P^* \cap \widetilde{D}$. Then $\widetilde{P} \cap D = P^* \cap D_M \cap D = P_M \cap D = P$, and by the Theorems 2.6.6.2(c) and 1.3.6.2. it follows that $\widetilde{P} \in \widetilde{v}\text{-spec}(\widetilde{D})$ and $P^* = \widetilde{P}_M$.

To prove the uniqueness of $\widetilde{P}$, suppose that $P' \in \widetilde{v}\text{-spec}(\widetilde{D})$ is such that $P' \cap D = P$. Then $P'_M \in \widetilde{v}_M\text{-spec}(\widetilde{D}_M))$ and $P'_M \cap D_M = (P' \cap D)_M = P_M$, hence $P'_M = P^*$ (by the uniqueness of P^*), and $P' = P'_M \cap \widetilde{D} = P^* \cap \widetilde{D} = \widetilde{P}$.

It remains to prove that $\widetilde{P}$ has the asserted properties. By Theorem 1.3.8 we obtain

$$D_P = (D_M)_{P_M} = (\widetilde{D}_M)_{P^*} = (\widetilde{D}_M)_{\widetilde{P}_M} = \widetilde{D}_{\widetilde{P}}\,.$$

Hence D_P is a dv-monoid if and only if $\widetilde{D}_{\widetilde{P}}$ is a dv-monoid, and therefore P is strong if and only if $\widetilde{P}$ is strong. If $P \in \mathcal{R}(D) \subset v\text{-max}(D)$, then $P_M \in v_M\text{-max}(D_M)$, hence $P^* = \widetilde{P}_M \in \widetilde{v}_M\text{-max}(\widetilde{D}_M)$, and therefore $\widetilde{P} \in \widetilde{v}\text{-max}(\widetilde{D})$. Since P is not strong, it follows that $\widetilde{P} \in \mathcal{R}(\widetilde{D})$. Assume finally that $I \in \mathcal{I}_v(D)$, $\widetilde{I} \in \mathcal{I}_{\widetilde{v}}(\widetilde{D})$ and $\widetilde{I} \cap D = I \subset P$. Then $I_M \in D_M$, $\widetilde{I}_M \subset \widetilde{D}_M$ and $\widetilde{I}_M \cap \widetilde{D}_M = (\widetilde{I} \cap D)_M = I_M \subset P_M$. Hence it follows that $\widetilde{I}_M \subset P^* = \widetilde{P}_M$, and $\widetilde{I} \subset \widetilde{I}_M \cap \widetilde{D} \subset \widetilde{P}_M \cap \widetilde{D} = \widetilde{P}$. □

Theorem 5.3.5. *Let D be a Mori monoid, $I \in \mathcal{I}_v(D)^\bullet$, and suppose that there is no $P \in \mathcal{S}(D)$ such that $I \subset P$. Then there exists a unique $\widetilde{I} \in \mathcal{I}_{\widetilde{v}}(\widetilde{D})$ such that $\widetilde{I} \cap D = I$, and there is no $P^* \in \mathcal{S}(\widetilde{D})$ such that $P^* \supset \widetilde{I}$.*

Proof. By Theorem 3.2.7.2, $\{P \in v\text{-max}(D) \mid I \subset P\} = \{P_1, \ldots, P_n\}$ for some $n \in \mathbb{N}$. For $i \in [1, n]$ we have $P_i \in \mathcal{R}(D)$, and by Theorem 5.3.4 there exists some $\widetilde{P}_i \in \mathcal{R}(\widetilde{D})$ such that $\widetilde{P}_i \cap D = P_i$, $D_{P_i} = \widetilde{D}_{\widetilde{P}_i}$ and, if $I' \in \mathcal{I}_{\widetilde{v}}(\widetilde{D})$ is such that $I' \cap D = I$, then $I' \subset \widetilde{P}_i$ and $I'_{\widetilde{P}_i} = I_{P_i}$.

We set $\widetilde{I} = I_{P_1} \cap \ldots \cap I_{P_n} \cap \widetilde{D}$. For $i \in [1, n]$, $D_{P_i} = \widetilde{D}_{\widetilde{P}_i}$ is a dv-monoid, hence $I_{P_i} = ID_{P_i} = I\widetilde{D}_{\widetilde{P}_i}$ is a principal ideal, and therefore $I_{P_i} \cap \widetilde{D} \in \widetilde{v}\text{-spec}(\widetilde{D})$. Hence it follows that $\widetilde{I} \in \mathcal{I}_{\widetilde{v}}(\widetilde{D})$ and $\widetilde{I} \cap D = I$, since

$$I = \bigcap_{P \in v\text{-}\max(D)} I_P = I_{P_1} \cap \ldots \cap I_{P_n} \cap D$$

If $P^* \in \widetilde{v}\text{-max}(\widetilde{D})$ is such that $P^* \supset \widetilde{I}$, then $P^* \cap D \in v\text{-spec}(D)$ and $P^* \cap D \supset I$. Hence there exists some $i \in [1, n]$ such that $P^* \cap D \subset P_i$, and as $P_i \in \mathfrak{X}(D)$, we obtain $P^* \cap D = P_i$ and therefore $P^* = \widetilde{P}_i \in \mathcal{R}(\widetilde{D})$.

It remains to prove the uniqueness of $\widetilde{I}$. Let $I' \in \mathcal{I}_{\widetilde{v}}(\widetilde{D})$ be such that $I' \cap D = I$. Then $I'_{\widetilde{P}_i} = I_{P_i} = \widetilde{I}_{\widetilde{P}_i}$ for all $i \in [1, n]$, and it suffices to prove that $\{\widetilde{P}_1, \ldots, \widetilde{P}_n\} = \{P' \in \widetilde{v}\text{-max}(\widetilde{D}) \mid P' \supset I'\}$. Indeed, once this is done, we obtain

$$I' = \bigcap_{P' \in \widetilde{v}\text{-}\max(\widetilde{D})} I'_{P'} = I'_{\widetilde{P}_1} \cap \ldots \cap I'_{\widetilde{P}_n} \cap \widetilde{D} = I_{P_1} \cap \ldots \cap I_{P_n} \cap \widetilde{D} = \widetilde{I}\,.$$

For $i \in [1, n]$, we have $\widetilde{P}_i = (\widetilde{P}_i)_{\widetilde{P}_i} \cap \widetilde{D} \supset I'_{\widetilde{P}_i} \cap \widetilde{D} \supset I'$. Conversely, assume that $P' \in \widetilde{v}\text{-max}(\widetilde{D})$ is such that $P' \supset I'$. Then $P' \cap D \in v\text{-spec}(D)$, $P' \cap D \supset I' \cap D = I$, and therefore there exists some $i \in [1, n]$ such that $P' \cap D \subset P_i$. Since $P_i \in \mathfrak{X}(D)$, we obtain $P' \cap D = P_i$ and $P' = \widetilde{P}_i$. □

5.4. Seminormal Mori monoids

Theorem und Definition 5.4.1.

1. *The following assertions are equivalent*:
   (a) *If $x \in K$ and $\{x^2, x^3\} \subset D$, then $x \in D$.*
   (b) *If $x \in K$ and $x^n \in D$ for all sufficiently large $n \in \mathbb{N}$, then $x \in D$.*

If D satisfies these conditions, then it is called *seminormal.*

If D is root-closed, then D is seminormal.

2. *Let D be seminormal and $T \subset D^\bullet$ a multiplicatively closed subset. T hen $T^{-1}D$ is seminormal.*
3. *Let $(D_\lambda)_{\lambda\in\Lambda}$ be a family of seminormal monoids such that $D_\lambda \subset K$ for all $\lambda \in \Lambda$ and*
$$D = \bigcap_{\lambda\in\Lambda} D_\lambda\,.$$
Then D is seminormal.
4. *Let D be seminormal, $x, y \in D^\bullet$ and $k \in \mathbb{N}$ such that $x^k(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$. Then it follows that already $x(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$.*

PROOF. 1. (a) $\Rightarrow$ (b) Let $x \in K$, and let $m \in \mathbb{N}_0$ be minimal such that $x^n \in D$ for all $n > m$. We must prove that $m = 0$, and we assume to the contrary that $m \ge 1$. Then $x^m \notin D$, and since $3m > 2m > m$, we obtain $\{(x^m)^2, (x^m)^3\} \subset D$, a contradiction.

(b) $\Rightarrow$ (a) If $x \in K$ is such that $\{x^2, x^3\} \subset D$, then $x^k \in D$ for all $k \ge 2$, and thus also $x \in D$.

2. Let $x \in K$ be such that $\{x^2, x^3\} \subset T^{-1}D$. Then there exist $a, b \in D$ and $t \in T$ such that $x^2 = t^{-1}a$ and $x^3 = t^{-1}b$, and therefore $(tx)^2 = ta \in D$ and $(tx)^3 = t^2a \in D$. Since D is seminormal, it follows that $tx \in D$ and $x = t^{-1}(tx) \in T^{-1}D$.

3. Let $x \in K$ be such that $\{x^2, x^3\} \subset D$. For all $\lambda \in \Lambda$, this implies $\{x^2, x^3\} \subset D_\lambda$, hence $x \in D_\lambda$, and therefore we obtain $x \in D$.

4. If $n \in \mathbb{N}$, then it follows that $[\,x(xy^{-1})^n\,]^j = x^k(xy^{-1})^{nj}\,x^{j-k} \in D$ for all $j \ge k$, which implies $x(xy^{-1})^n \in D$. □

Theorem 5.4.2. *Let D be a seminormal Mori monoid.*
1. *If $x, y \in D^\bullet$, then $xy^{-1} \in \widehat{D}$ if and only if $x(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$.*
2. *$\widehat{D}$ is completely integrally closed.*

PROOF. 1. By definition, if $x, y \in D^\bullet$ and $x(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$, then $xy^{-1} \in \widehat{D}$.

Thus assume that $x, y \in D^\bullet$, $xy^{-1} \in \widehat{D}$, and let $c \in D^\bullet$ be such that $c(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, we consider the ideal
$$I_n = \bigcap_{i=0}^{n} \big((x^{-1}y)^i D \cap D\big)\,.$$
By definition, $I_n \in \mathcal{I}_v(D)$, $I_n \supset I_{n+1}$ and $c \in I_n$ for all $n \in \mathbb{N}$. As D is a Mori monoid, there exists some $k \in \mathbb{N}$ such that $I_k = I_{k+n}$ for all $n \in \mathbb{N}$, and since $y^k = (x^{-1}y)^k x^k \in I_k$, we obtain $y^k \in I_{k+n}$ for all $n \in \mathbb{N}$. Hence for every $n \in \mathbb{N}$ there exists some $b_n \in D$ such that $y^k = (x^{-1}y)^{k+n}b_n$ and therefore $x^k(xy^{-1})^n = x^k b_n \in D$. Consequently, $x(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$ holds by Theorem 5.4.1.4.

2. Suppose that $u = y^{-1}x \in \widehat{\widehat{D}}$, where $x, y \in D^\bullet$, and let $d \in \widehat{D}^\bullet$ be such that $du^n = dx^n(y^n)^{-1} \in \widehat{D}$ for all $n \in \mathbb{N}$. We may assume that $d \in D^\bullet$. By 1. it follows that $dx^n\,[\,dx^n y^{-n}\,]^m \in D$ for all $m, n \in \mathbb{N}$. For $m \in \mathbb{N}$ and $n \ge m+1$, this implies that $[\,dx(y^{-1}x)^m\,]^n = dx^n(dx^ny^{-n})^m d^{n-m-1} \in D$, hence $dx(y^{-1}x)^m \in D$, since D is seminormal and therefore $u = y^{-1}x \in \widehat{D}$. Hence $\widehat{D}$ is completely integrally closed. □

Theorem 5.4.3. *Let D be a seminormal Mori domain.*
1. *Let $I \subset D$ be a strong ideal and $C = (D\!:\!I) = (I\!:\!I)$. If I is a radical ideal of C, then C is seminormal.*
2. *$\widetilde{D}$ is seminormal, and if $P \in \mathcal{S}(D)$, then $P\widetilde{D}_P = P_P$ is a radical ideal of $\widetilde{D}_P$.*

3. *If $Q \in \mathcal{S}(\widetilde{D})$, then $Q \cap D \notin v\text{-max}(D)$. In particular, the assignment $Q \mapsto Q \cap D$ defines a bijective map*

$$\{Q \in \widetilde{v}\text{-spec}(\widetilde{D}) \mid Q \text{ strong}\} \;\to\; \{P \in v\text{-spec}(D)\,,\; P \text{ strong},\; P \notin \mathcal{S}(D)\,\}\,.$$

Proof. 1. By Theorem 5.1.3.1 C is a Mori monoid. If $v^* = v(C)$, then

$$C = \bigcap_{P \in v^*\text{-}\max(C)} C_P\,,$$

and therefore it suffices to prove that C_P is seminormal for all $P \in v^*\text{-max}(C)$. Suppose that $P \in v^*\text{-max}(C)$, and consider the following two cases.

CASE 1: $I \not\subset P$. Theorem 5.1.3 implies that $C_P = D_{D \cap P}$, and the latter monoid is seminormal by Theorem 5.4.1.2.

CASE 2: $I \subset P$. By Theorem 5.1.2 we obtain $(P : P) = C$, and since $P \in v^*\text{-max}(C)$, it follows that $(C : P) \supsetneq C$. Hence P is not strong, and by Theorem 5.1.1.2 C_P is a dv-monoid. Hence C_P is root-closed and therefore seminormal.

2. If $P \in \mathcal{R}(D)$, then D_P is a dv-monoid, hence it is root-closed and therefore seminormal.

Assume now that $P \in \mathcal{S}(D)$. Then $\widetilde{D}_P = (P_P : P_P)$ by Theorem 5.3.3, and therefore we get $P\widetilde{D}_P = P_P\widetilde{D}_P = P_P(P_P : P_P) = P_P$. We show that P_P is a radical ideal of $(P_P : P_P)$. Thus let $x \in (P_P : P_P)$ be in the radical of P_P. Then $x^n \in P_P \subset D_P$ for all sufficiently large $n \in \mathbb{N}$, and as D_P is seminormal, it follows that $x \in D_P$. Hence $x \in P_P$, since $P_P \subset D_P$ is a prime ideal. By 1. it follows that $\widetilde{D}_P$ is seminormal.

Now $\widetilde{D}$ is seminormal, since

$$\widetilde{D} = \bigcap_{P \in \mathcal{R}(D)} D_P \cap \bigcap_{P \in \mathcal{S}(D)} (D_P : P_P) = \bigcap_{P \in \mathcal{R}(D)} D_P \cap \bigcap_{P \in \mathcal{S}(D)} \widetilde{D}_P\,.$$

3. Suppose to the contrary that $Q \in \mathcal{S}(\widetilde{D})$ and $P = Q \cap D \in v\text{-max}(D)$. Then Theorem 5.3.3 yields $P \in \mathcal{S}(D)$ and $\widetilde{D}_P = (D_P : P_P) = (P_P : P_P)$. By 2., P_P is a radical ideal of $\widetilde{D}_P$, and since $P_P \subset Q_P$, Theorem 5.1.2 implies $(Q_P : Q_P) = \widetilde{D}_P$. On the other hand, Q_P is strong, hence $(D_P : Q_P) = (Q_P : D_P) = \widetilde{D}_P$ and $Q_P = (Q_P)_{\widetilde{v}_P} = \widetilde{D}_P$, a contradiction.

In particular, if $Q \in \widetilde{v}\text{-spec}(\widetilde{D})$ is strong, then the arguments above together with Theorem 5.3.3 show that $Q \cap P \in v\text{-spec}(D) \setminus \mathcal{S}(D)$ is strong. Conversely, if $P \in v\text{-spec}(D) \setminus \mathcal{S}(D)$, then Theorem 5.3.4 shows that there is a unique strong $Q \in \widetilde{v}\text{-spec}(\widetilde{D})$ such that $Q \cap D = P$. □

Theorem 5.4.4. *Let D be a seminormal Mori monoid, and let the sequence $(D_i)_{i \ge 0}$ of Mori monoids be recursively defined by $D_0 = D$ and $D_{i+1} = \widetilde{D_i}$ for all $i \ge 0$.*

If $k \in \mathbb{N}$ and $Q \in \mathcal{S}(D_k)$, then there exist strong prime ideals $P_0, \ldots, P_k \in v\text{-spec}(D)$ such that $P_0 = Q \cap D \subsetneq P_1 \subsetneq \ldots \subsetneq P_k$.

Proof. 1. We use induction on k.

$k = 1$: If $Q \in \mathcal{S}(\widetilde{D})$, then $P_0 = Q \cap D \in v\text{-spec}(D)$ is strong and $P_0 \notin \mathcal{S}(D)$ by Theorem 5.4.3.3. Hence there exists some $P_1 \in v\text{-spec}(D)$ such that $P_0 \subsetneq P_1$, and P_1 is strong, since $P_1 \notin \mathfrak{X}(D)$.

$k \ge 2$, $k-1 \to k$: Note that $D_1 = \widetilde{D}$. By the induction hypothesis, there exist strong prime ideals $P_0', \ldots, P_{k-1}' \in \widetilde{v}\text{-spec}(\widetilde{D})$ such that $P_0' = Q \cap \widetilde{D} \subsetneq P_1' \subsetneq \ldots \subsetneq P_{k-1}'$, and we set $P_i = P_i' \cap D$ for all $i \in [0, k-1]$. By Theorem 5.4.3.3 it follows that $P_0 = Q \cap D \subsetneq P_1 \subsetneq \ldots \subsetneq P_{k-1}$, and $P_i \in v\text{-spec}(D) \setminus \mathcal{S}(D)$ is strong for all $i \in [0, k-1]$. Hence there exists some $P_k \in v\text{-max}(D)$ such that $P_{k-1} \subsetneq P_k$, and clearly P_k is strong. □

Theorem 5.4.5. *Let D be a seminormal Mori monoids, let the sequence $(D_i)_{i\ge 0}$ of Mori monoids be recursively defined by $D_0 = D$ and $D_{i+1} = \widetilde{D_i}$ for all $i \ge 0$. Then*

$$\widehat{D} = \bigcup_{i\ge 0} D_i \quad \textit{is a Krull monoid.}$$

Proof. $(D_i)_{i\ge 0}$ is an ascending sequence of Mori monoids. Hence

$$D^* = \bigcup_{i\ge 0} D_i \subset K$$

is a monoid. We set $v^* = v(D^*)$, $t^* = t(D^*)$ and $v_i = v(D_i)$, $t_i = t(D_i)$, and we obtain $t \le t_i \le t_{i+1} \le t^*$ for all $i \ge 0$. In particular, if $J \in \mathcal{I}_{v^*}(D^*)$ or if $J \in \mathcal{I}_{v_{i+1}}(D_{i+1})$, then $J \cap D_i \in \mathcal{I}_{v_i}(D_i)$. It is now sufficient to prove the following three assertions.

I. $D^* \subset \widehat{D}$.

II. D^* is a Mori monoid.

III. $\mathcal{S}(D^*) = \emptyset$.

Indeed, by II and III it follows that D^* is a Mori monoid satisfying $v^*\text{-max}(D^*) = \mathcal{R}(D^*)$. Hence D^*_P is a dv-monoid for all $P \in v^*\text{-max}(D^*)$, and therefore D^* is a Krull monoid by Theorem 5.2.5. In particular, D^* is completely integrally closed, hence $\widehat{D} \subset \widehat{D^*} = D^*$, and therefore $D^* = \widehat{D}$ by I.

I. It clearly suffices to prove that $D_i \subset \widehat{D}$ for all $i \ge 0$, and we proceed by induction on i. For $i = 0$, there is nothing to do. Thus suppose that $i \ge 0$ and $D_i \subset \widehat{D}$. Since $\widehat{D}$ is completely integrally closed by Theorem 5.4.2, Theorem 5.3.3 implies that $D_{i+1} \subset \widehat{D_i} \subset \widehat{D}$.

II. Let $(I_n)_{n\ge 0}$ be an ascending chain in $\mathcal{I}_{v^*}(D^*)$. For $i,\, n \ge 0$, we set $I_{n,i} = I_n \cap D_j$. For every $i \ge 0$, $(I_{n,i})_{n\ge 0}$ is an ascending sequence in $\mathcal{I}_{v_i}(D_i)$, and it terminates since D_i is a Mori domain. Let $n_i \ge 0$ be minimal such that $I_{n,i} = I_{n+1,i}$ for all $n \ge n_i$. Then the sequence $(n_i)_{i\ge 0}$ is monotonically increasing, and since

$$I_n = \bigcup_{i\ge 0} I_{n,i} \quad \text{for all} \quad n \ge 0\,,$$

it suffices to prove that there exists some $k \ge 0$ such that $n_{i+1} = n_i$ for all $i \ge k$. Indeed, then it follows that $I_n = I_{n+1}$ for all $n \ge n_k$. Replacing the sequence $(I_n)_{n\ge 0}$ by a suitable end piece, we may assume that $I = I_{0,0} \ne \{0\}$ and $n_0 = 0$. Then it follows that $I_{n,i} \cap D = I$ for all $n,\, i \ge 0$.

Let $k \in \mathbb{N}$ be such that there is no chain $I \subset P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_k$, where $P_0, \ldots, P_k \in v\text{-spec}(D)$, and suppose that there is some $i \ge k$ such that $n_{i+1} > n_i$. Then there exists some $n \ge n_i$ such that $I_{n,i+1} \subsetneq I_{n+1,i+1}$, and since $I_{n,i+1} \cap D_i = I_{n+1,i+1} \cap D_i = I_{n,i}$, Theorem 5.3.5 implies that there is some $P \in \mathcal{S}(D_i)$ such that $I_{n,i} \subset P$. By Theorem 5.4.4 there exists a chain $P \cap D = P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_i$ in $v\text{-spec}(D)$, and since $I = I_{n,i} \cap D \subset P_0$ and $i \ge k$, this contradicts our choice of k.

III. Assume to the contrary that there is some $P^* \in \mathcal{S}(D^*)$. For $i \ge 0$, set $P_i = P^* \cap D_i \in v_i\text{-spec}(D_i)$. Then $(D_i)_{P_i} \subset D^*_P \subsetneq K$, $P_i^\bullet \ne \emptyset$, and D^*_P is not a dv-monoid. Hence $(D_i)_{P_i}$ is not a dv-monoid, and therefore P_i is strong. If $Q_i \in v_i\text{-max}(D_i)$ is such that $P_i \subset Q_i$, then $Q_i \in \mathcal{S}(D_i)$, $P_0 \subset Q_i \cap D$, and Theorem 5.4.4 implies that there is a chain $P_0 \subset P_1 \subsetneq \ldots \subsetneq P_i$ in $v\text{-spec}(D)$. As $i \ge 0$ is arbitrary, this contradicts Theorem 3.2.7.2. $\square$

CHAPTER 6

Ideal theory of polynomial rings

6.1. The content and the Dedekind-Mertens Lemma

Throughout this Section, let D be a ring, $D[X]$ a polynomial ring, $d = d(D)$ and $v = v(D)$.

Definition 6.1.1. Let $R \supset D$ be an overring. For D-submodules $M, N \subset R$ we write (as usual in ring theory) MN instead of ${}_D(MN)$.

For a polynomial $g = b_0 + b_1X + \ldots + b_mX^m \in R[X]$, the D-module

$$\mathsf{c}_D(g) = {}_D(b_0, \ldots, b_m) = \sum_{j=0}^{m} Db_j \subset R$$

is called the *D-content* of g. If $J \subset R$ is a D-submodule, then $g \in J[X]$ if and only if $\mathsf{c}_D(g) \subset J$.
Obviously, $\mathsf{c}_D(af) = a\mathsf{c}_D(f)$ and $\mathsf{c}_D(fg) \subset \mathsf{c}_D(f)\mathsf{c}_D(g)$ for all $a \in R$ and $f, g \in R[X]$, but equality need not hold [indeed, if $D = R = \mathbb{Z}[2\mathsf{i}]$ and $f = 2\mathsf{i} + 2X$, then $f^2 = -4 + 8\mathsf{i}X + 4X^2$, hence $\mathsf{c}(f) = (2\mathsf{i}, 2)$, $\mathsf{c}(f^2) = (4)$, and $\mathsf{c}(f)^2 = (4, 4\mathsf{i}) \ne \mathsf{c}(f^2)$].
The *Dedekind-Mertens number* of a non-zero polynomial $g \in R[X]$ with respect to D is defined by

$$\mu_D(g) = \inf\bigl\{k \in \mathbb{N} \mid \mathsf{c}_D(f)^k\mathsf{c}_D(g) = \mathsf{c}_D(f)^{k-1}\mathsf{c}_D(fg) \text{ for all } f \in R[X]\bigr\} \in \mathbb{N} \cup \{\infty\}\,.$$

If $f, g \in R[X]$, then $\mathsf{c}_D(fg) \le \mathsf{c}_D(f)\,\mathsf{c}_D(g)$ implies $\mathsf{c}_D(f)^{k-1}\mathsf{c}_D(fg) \le \mathsf{c}_D(f)^k\mathsf{c}_D(g)$ for all $k \in \mathbb{N}$, and therefore

$$\mu_D(g) = \inf\bigl\{k \in \mathbb{N} \mid \mathsf{c}_D(f)^k\mathsf{c}_D(g) \le \mathsf{c}_D(f)^{k-1}\mathsf{c}_D(fg) \text{ for all } f \in R[X]\bigr\} \in \mathbb{N} \cup \{\infty\}\,.$$

We shall see in Theorem 6.1.2 that $\mu_D(g)$ only depends on the D-module $\mathsf{c}_D(g)$ and not on the embedding ring R.
The classical *Dedekind-Mertens Lemma* asserts that $\mu_D(g) \le \deg_D(g) + 1$ for all $g \in D[X]^\bullet$. We shall prove a more general statement in Theorem 6.1.2.

Theorem 6.1.2. *Let $R \supset D$ be an overring, $g \in R[X]$ and $\delta(g)$ the number of non-zero coefficients of g. For $M \in \max(D)$, we denote by $\rho_M(g)$ the minimal number of generators of the D_M-module $\mathsf{c}_D(g)_M$, that is, $\rho_M(g) = \dim_{D/M}(\mathsf{c}_D(g)_M/M\mathsf{c}_D(g)_M)$. Then*

$$\mu_D(g) \le \max\{\rho_M(g) \mid M \in \max(D)\} \le \delta(g) \le \deg(g) + 1\,.$$

For the proof we need the following variant of Nakayama's Lemma.

Lemma 6.1.3. *Let D be local with maximal ideal M.*

1. *Let A, B be D-modules such that $A \subset B$ and B/A is finitely generated. If $B = A + MB$, then $B = A$.*
2. *Let L be a D-module and $A, B \subset L$ submodules. If A is finitely generated and $A \subset B + MA$, then $A \subset B$.*

PROOF. 1. This is the classical form of Nakayama's Lemma.

2. If $A \subset B + MA$, then $A+B \subset B+MA \subset B+M(A+B) \subset A+B$ implies $A+B = B+M(A+B)$, and by 1. we obtain $B = A+B \supset A$. $\square$

PROOF OF THEOREM 6.1.2. For $f \in R[X]$, we set $C_f = \mathsf{c}_D(f)$. If $f,\, g \in R[X]$, then we obviously have $C_fC_g \subset C_{fg}$ and therefore $C_f^kC_g \subset C_f^{k-1}C_{fg}$ for all $k \in \mathbb{N}$.

It suffices to prove the result if D is local with maximal ideal M. Indeed, suppose that this is done. Let $g \in R[X]$ and $k \in \mathbb{N}$ be such that $k \ge \rho_M(g)$ for all $M \in \max(D)$. We must prove that $C_f^kC_g = C_f^{k-1}C_{fg}$ for all $f \in R[X]$. For $f \in R[X]$ and $M \in \max(D)$, let $f_M \in R_M[X]$ be the image of f in $R_M[X]$. Then $\mathsf{c}_{D_M}(f_M) = (C_f)_M$, and the local result implies $\mathsf{c}_{D_M}(f_M)^k\mathsf{c}_{D_M}(g_M) = \mathsf{c}_{D_M}(f_M)^{k-1}\mathsf{c}_{D_M}(f_Mg_M)$, that is, $(C_f^kC_g)_M = C_{f_M}^kC_{g_M} = C_{f_M}^{k-1}C_{f_Mg_M} = (C_f^{k-1}C_{fg})_M$. Since this holds for all $M \in \max(D)$, the assertion follows.

Assume now that D is local, $M = D \setminus D^\times$, $R \supset D$ is an overring, and for $g \in R[X]$, we set $\rho(g) = \rho_M(g)$. We prove first:

A. If $g,\, g_1 \in R[X]$ and $C_{g-g_1} \subset MC_g$, then $C_g = C_{g_1}$ and $\mu_D(g) = \mu_D(g_1)$.

Proof of **A.** Since $g = g_1 + (g - g_1)$, we obtain $C_g \subset C_{g_1} + C_{g-g_1} \subset C_{g_1} + MC_g$ and therefore $C_g \subset C_{g_1}$ by Lemma 6.1.3. But $C_{g_1-g} = C_{g-g_1} \subset MC_g \subset MC_{g_1}$, hence we obtain also $C_{g_1} \subset C_g$ and therefore $C_g = C_{g_1}$.

By symmetry, it is now sufficient to prove that $\mu_D(g) \le \mu_D(g_1)$, and for this we may assume that $k = \mu_D(g_1) < \infty$. If $f \in R[X]$, then

$$\begin{aligned} C_f^kC_g = C_f^kC_{g_1} = C_f^{k-1}C_{fg_1} = C_f^{k-1}C_{fg+f(g_1-g)} \subset C_f^{k-1}(C_{fg} + C_{f(g_1-g)}) \\ \subset C_f^{k-1}(C_{fg} + C_fC_{g_1-g}) = C_f^{k-1}C_{fg} + MC_f^kC_g\,. \end{aligned}$$

By Lemma 6.1.3 we obtain $C_f^kC_g \subset C_f^{k-1}C_{fg}$. $\square$[**A.**]

We prove Theorem 6.1.2 by induction on $\rho(g)$. If $g = 0$, then $\mu_D(g) = 0$. Thus we may assume that

$$g = \sum_{j=0}^m b_jX^j\,, \quad \text{where} \quad m \in \mathbb{N}_0\,,\ b_0, \ldots, b_m \in R \quad \text{and} \quad b_m \ne 0\,.$$

$\boldsymbol{\rho(g) = 1}$: Then $C_g = Db$ for some $b \in R$. For $j \in [0, m]$, there exists some $d_j \in D$ such that $b_j = d_jb$, and we assert that there is some $l \in [0, m]$ such that $d_j \notin M$ (indeed, otherwise we have $C_g \subset MC_g$ and consequently $C_g = 0$ by Lemma 6.1.3). Let $l \in [0, m]$ be such that $d_l \notin M$ and $d_j \in M$ for all $j \in [0, l-1]$. We must prove that $C_fC_g \subset C_{fg}$ for all $f \in R[X]$. Thus suppose that

$$f = \sum_{i=0}^n a_iX^i\,, \quad \text{where} \quad n \in \mathbb{N}_0\,,\ a_0, \ldots, a_n \in R \quad \text{and} \quad c_k = \sum_{i=0}^k a_{k-i}d_ib\,.$$

Then $C_fC_g = C_fb = {}_D(a_0b, \ldots, a_nb)$. If $a_i = 0$ for all $i > n$ and $b_j = 0$ for all $j > m$, then

$$fg = \sum_{k=0}^{m+n} c_kX^k\,, \quad \text{where} \quad c_k = \sum_{i=0}^k a_{k-i}d_ib \quad \text{for all} \quad k \in [0, m+n]\,.$$

It suffices to prove that $a_ib \in C_{fg} + MC_fC_g$ for all $i \in [0, n]$. Indeed, once this is done, it follows that $C_fC_g \subset C_{fg} + MC_fC_g$ and therefore $C_fC_g \subset C_{fg}$ by Lemma 6.1.3.

We proceed by induction on i. Let $i \in [0, n]$ and suppose that $a_\nu b \in C_{fg} + MC_fC_g$ for all $\nu \in [0, i-1]$. Then

$$c_{i+l} = a_id_lb \;+\; \sum_{\nu=0}^{l-1} a_{i+l-\nu}d_\nu b \;+\; \sum_{\nu=l+1}^{l+i} a_{i+l-\nu}d_\nu b \in C_{fg}\,.$$

If $\nu \in [0, l-1]$, then $d_\nu \in M$ and $a_{i+l-\nu} d_\nu b \in MC_f C_g$. If $\nu \in [l+1, l+i]$, then $i+l-\nu \in [0, i-1]$ and $a_{i+l-\nu} d_\nu b \in D(C_{fg} + MC_f C_g) = C_{fg} + MC_f C_g$ by the induction hypothesis. Hence it follows that $a_i d_l b \in C_{fg} + MC_f C_g$, and since $d_l \in D \setminus M = D^\times$, we obtain $a_i b \in C_{fg} + MC_f C_g$.

$\boldsymbol{\rho(g) = k \geq 2\,,\ k-1 \to k}$: If

$$g_1 = \sum_{\substack{j=0 \\ b_j \notin MC_g}}^{m} b_j X^j\,, \quad \text{then} \quad C_{g-g_1} \subset MC_g\,, \quad \text{hence} \quad C_g = C_{g_1} \quad \text{and} \quad \mu_D(g) = \mu_D(g_1)\,.$$

Therefore we may assume that $g = g_1$. Since $b_m \notin MC_g$, there exists a subset $L \subset [0, m-1]$ such that $|L| = k-1$ and $\{b_m\} \cup \{b_\mu \mid \mu \in L\}$ is a minimal generating set of C_g. Then $C_g = Db + E$, where $E = {}_D(\{b_\mu \mid \mu \in L\})$, and for every $j \in [0, m]$, there is a representation

$$b_j = \lambda_j b_m + b_j'\,, \quad \text{where} \quad b_j' = \sum_{\mu \in L} \lambda_{j,\mu} b_\mu \in E\,,$$

such that $\lambda_j,\, \lambda_{j,\mu} \in D$ for all $j \in [0, m]$ and $\mu \in L$, $\lambda_m = 1$ and $\lambda_{m,\mu} = 0$ for all $\mu \in L$, and if $j \in L$, then $\lambda_{j,j} = 1$ and $\lambda_j = \lambda_{j,\nu} = 0$ for all $\nu \in L \setminus \{j\}$. We set

$$g_0 = \sum_{j=0}^{m} d_j b X^j = b_m X^m + \ldots \quad \text{and} \quad g_1 = \sum_{j=0}^{m-1} b_j' X^j\,.$$

Then $g = g_0 + g_1$, $C_{g_0} = b_m D$, $C_{g_1} = E$, $\rho(g_0) = 1$, and $\rho(g_1) = k-1$. By the induction hypothesis and since $\rho(g_0) = 1$, we have $C_f^{k-1} C_{g_1} = C_f^{k-2} C_{fg_1}$ and $C_{fg_0} = C_f C_{g_0} = b_m C_f$ for all $f \in R[X]$, and we must prove that $C_f^k C_g \subset C_f^{k-1} C_{fg}$ for all $f \in R[X]$. We proceed by induction on $\deg(f)$. We may assume that $f \neq 0$,

$$f = \sum_{i=0}^{n} a_i X^i = a_n X^n + f_1\,, \quad \text{where} \quad n \in \mathbb{N}_0\,,\ a_0, \ldots, a_n \in R\,,\ a_n \neq 0 \quad \text{and} \quad C_{f_1}^k C_g \subset C_{f_1}^{k-1} C_{f_1 g}\,.$$

Then it follows that $a_n b_m \in C_{fg}$. We use the induction hypothesis to prove the following assertion.

B. $C_{fg_1} \subset C_{fg} + b_m C_{f_1}$ and $C_{f_1 g} \subset C_{fg} + a_n C_{g_1}$

Proof of **B.** Since $C_{fg_0} = C_{a_n X^n g_0 + f_1 g_0} \subset C_{a_n X^n g_0} + C_{f_1 g_0} \subset a_n b_m D + C_{f_1} b_m \subset C_{fg} + C_{f_1} b_m$, we obtain $C_{fg_1} = C_{f(g-g_0)} \subset C_{fg} + C_{fg_0} \subset C_{fg} + b_m C_{f_1}$.

In the same way, $C_{a_n X^n g} = C_{X^n(a_n g_0 + a_n g_1)} \subset C_{a_n g_0} + C_{a_n g_1} = a_n b_m D + a_n C_{g_1} \subset C_{fg} + a_n C_{g_1}$, and therefore $C_{f_1 g} = C_{(f - a_n X^n) g} \subset C_{fg} + C_{a_n X^n g} \subset C_{fg} + a_n C_{g_1}$. $\square$[**B.**]

$C_f^k C_g$ is the D-module generated by the set A of all elements $\alpha = a_0^{v_0} \cdot \ldots \cdot a_{n-1}^{v_{n-1}} a^v b_j \in R$, where $v_0, \ldots, v_{n-1}, v \in \mathbb{N}_0$, $v_0 + \ldots + v_{n-1} + v = k$ and $j \in [0, m]$.

- If $v \neq 0$ and $j \in J$, then $\alpha = a_0^{v_0} \cdot \ldots \cdot a_{n-1}^{v_{n-1}} a^{v-1} a b d_j \in C_f^{k-1} C_{fg}$.
- If $v \neq 0$ and $j \notin J$, then $\alpha = a_0^{v_0} \cdot \ldots \cdot a_{n-1}^{v_{n-1}} a^{v-1} a b_j \in C_f^{k-1} a C_{g_1}$.
- If $v = 0$, then $\alpha = a_0^{v_0} \cdot \ldots \cdot a_{n-1}^{v_{n-1}} b_j \in C_{f_1}^k C_g \subset C_{f_1}^{k-1} C_{f_1 g} \subset C_f^{k-1} C_{f_1 g} \subset C_f^{k-1}(C_{fg} + a_n C_{g_1})$ (by the induction hypothesis, **B**, and since $C_{f_1} \subset C_f$).

Putting the three cases together, we get

$$C_f^k C_{fg} \subset C_f^{k-1} C_{fg} + C_f^{k-1} a_n C_{g_1} + C_f^{k-1}(C_{fg} + a_n C_{g_1}) = C_f^{k-1} C_{fg} + C_f^{k-1} a_n C_{g_1}\,.$$

Using **B** and the induction hypothesis, it follows that

$$C_f^{k-1} a_n C_{g_1} \subset a_n C_f^{k-2} C_{fg_1} \subset a_n C_f^{k-2}(C_{fg} + b_m C_f) \subset a_n C_f^{k-2} C_{fg} + C_f^{k-1} a_n b_m \subset C_f^{k-1} C_{fg}\,,$$

which completes the proof. $\square$

Corollary 6.1.4. *Let $R \supset D$ be an overring.*

1. *For every $g \in R[X]$ there exists some $m \in \mathbb{N}$ such that $\mathsf{c}_D(f)^m \mathsf{c}_D(g) = \mathsf{c}_D(f)^{m-1} \mathsf{c}_D(fg)$ for all $f \in R[X]$.*
2. *Let $f,\, g \in R[X]$, and suppose that $\mathsf{c}_D(f)$ is a finitely cancellative D-submodule of R (that means, $\mathsf{c}_D(f)M = \mathsf{c}_D(f)N$ implies $M = N$ for all finitely generated D-submodules $M,\, N \subset R$). Then $\mathsf{c}_D(fg) = \mathsf{c}_D(f)\mathsf{c}_D(g)$.*
3. *Let D be a domain, $K = \mathsf{q}(D)$ and r a module system on K such that $r \ge d$. If $f \in K[X]$ and $\mathsf{c}_D(f)_r$ is r-finitely r-cancellative, then $\mathsf{c}_D(fg)_r = [\,\mathsf{c}_D(f)\mathsf{c}_D(g)\,]_r$ for all $g \in K[X]$.*

Proof. Obvious by Theorem 6.1.2. □

Theorem 6.1.5. *Let D be a domain and $K = \mathsf{q}(D)$. Then the following assertions are equivalent:*

(a) *D is integrally closed.*

(b) *For all $f,\, g \in K[X]$ we have $\mathsf{c}_D(fg)_v = [\,\mathsf{c}_D(f)\mathsf{c}_D(g)\,]_v$.*

(c) *For all $f,\, g \in K[X]$ we have $\mathsf{c}_D(f)\mathsf{c}_D(g) \subset \mathsf{c}_D(fg)_v$.*

(d) *For all $f \in K[X]$ we have $fK[X] \cap D[X] = f\mathsf{c}_D(f)^{-1}[X]$.*

Proof. (a) $\Rightarrow$ (b) Since D is integrally closed, we have $D_{d_\mathsf{a}} = D$, and therefore d_a is a finitely cancellative ideal system on D. Hence $\mathsf{c}_D(fg)_{d_\mathsf{a}} = [\,\mathsf{c}_D(f)\mathsf{c}_D(g)\,]_{d_\mathsf{a}}$, and since $d_\mathsf{a} \le v$, the assertion follows.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (d) Let $f \in K[X]$. We must prove that, for all $g \in K[X]$, we have $fg \in D[X]$ if and only if $g \in \mathsf{c}_D(f)^{-1}[X]$. If $g \in K[X]$ and $fg \in D[X]$, then $\mathsf{c}_D(f)\mathsf{c}_D(g) \subset \mathsf{c}_D(fg)_v \subset D$, hence $\mathsf{c}_D(g) \subset \mathsf{c}_D(f)^{-1}$ and therefore $g \in \mathsf{c}_D(f)^{-1}[X]$. Conversely, if $g \in \mathsf{c}_D(f)^{-1}[X]$, then $\mathsf{c}_D(g) \subset \mathsf{c}_D(f)^{-1}$ and therefore $\mathsf{c}_D(fg) \subset \mathsf{c}_D(f)\mathsf{c}_D(g) \subset D$, which implies $fg \in D[X]$.

(d) $\Rightarrow$ (a) Let $u \in K$ be integral over D, and let $g \in D[X]$ be a monic polynomials such that $g(u) = 0$. Then $g = (X-u)h$, where $h \in K[X]$, and therefore $g \in (X-u)K[X] \cap D[X] = (X-u)\{1,u\}^{-1}[X]$. Hence $h \in \{1,u\}^{-1}[X]$, which implies that $uh \in D[X]$ and thus $u \in D$, since h is monic. □

Theorem 6.1.6. *Let D be a domain and $K = \mathsf{q}(D)$. Then the following assertions are equivalent:*

(a) *D is local and integrally closed.*

(b) *If $f \in D[X]$, $u \in K^\times$, $f(u) = 0$ and $\mathsf{c}_D(f)$ is invertible, then $u \in D$ or $u^{-1} \in D$.*

(c) *If $f \in D[X]$ be such that some coefficient of f lies in $D^\times$ and $u \in K^\times$ is such that $f(u) = 0$, then $u \in D$ or $u^{-1} \in D$.*

Proof. (a) $\Rightarrow$ (b) Let $f \in D[X]$ and $u = b^{-1}a \in K^\times$, where $a,\, b \in D^\bullet$, be such that $f(u) = 0$ and $\mathsf{c}_D(f)$ is invertible. Then $f = (bX - a)h$ for some $h \in K[X]$, and

$$\mathsf{c}_D(f) = \mathsf{c}_D(f)_v \supset \mathsf{c}_D(bX-a)\mathsf{c}_D(h) = (a,b)\,\mathsf{c}_D(h) \supset \mathsf{c}_D(f)\,.$$

Hence $\mathsf{c}_D(f) = (a,b)\,\mathsf{c}_D(h)$, and therefore (a,b) is invertible. Since D is local, Theorem 4.1.4 implies $(a,b) = (b)$ or $(a,b) = (a)$, and therefore $u \in D$ or $u^{-1} \in D$.

(b) $\Rightarrow$ (c) Let $f \in D[X]$ and some coefficient of f lies in $D^\times$, then $\mathsf{c}_D(f) = D$.

(c) $\Rightarrow$ (a) Let $u \in K^\times$ be integral over D, and let $f = X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0 \in D[X]$ be a monic polynomial of minimal degree such that $f(u) = 0$. If $u \notin D$, then $n \ge 2$, $u^{-1} \in D$ and $u^{n-1} + a_{n-1}u^{n-2} + \ldots + (a_1 + a_0u^{-1}) = 0$, which contradicts the minimality of n. Hence D is integrally closed.

In order to prove that D is local, we take some $M \in \max(D)$ and prove that $D \setminus M \subset D^\times$. If $u \in D \setminus M$, then $M + Du = D$, and there exist elements $a \in M$ and $b \in D^\bullet$ such that $a + bu = 1$. If $a = 0$, then $u \in D^\times$ and we are done. Thus suppose that $a \ne 0$. Then $u^{-1}a$ is a zero of the polynomial

$f = (uX - a)(X - b) = uX^2 - X + ab \in D[X]$, and therefore either $u^{-1}a \in D$ or $a^{-1}u \in D$. If $a^{-1}u \in D$, then $u \in aD \subset M$, a contradiction. If $u^{-1}a \in D$, then $a = ud$ for some $d \in D$, hence $1 = u(d + b)$ and $u \in D^\times$. □

6.2. Nagata rings

Remarks and Definition 6.2.1. Let D be a ring and $K = \mathsf{q}(D)$ its total quotient ring.

1. We denote by $\mathcal{F}(D) = \{c^{-1}J \mid c \in D^*,\ J \lhd D\}$ the set of all fractional ideals of D. If $I,\,J \in \mathcal{F}(D)$ and $a \in D$, then $aI,\ I+J,\ IJ \in \mathcal{F}(D)$. For $I \in \mathcal{F}(D)$, we define
$$I[X] = \Big\{\sum_{i=0}^n a_iX^i \,\Big|\, n \in \mathbb{N}_0\,,\ a_0,\ldots,a_n \in I\Big\} \subset K[X]\,.$$
2. Let $R \supset D$ be an overring such that $R^* \subset D^*$, and assume that $\mathsf{q}(D) \subset \mathsf{q}(R)$. For $I \in \mathcal{F}(D)$, we denote by
$$IR = {}_RI = \{x_1a_1 + \ldots + x_na_n \mid n \in \mathbb{N}\,,\ x_1,\ldots,x_n \in I,\ a_1,\ldots,a_n \in R\} \in \mathcal{F}(R)$$
the the R-submodule of $\mathsf{q}(R)$ generated by I. If $I,\,J \in \mathcal{F}(D)$, then $(IJ)R = (IR)(JR)$, and if $I = {}_D(a_1,\ldots,a_n) = Da_1 + \ldots + Da_n$, then $IR = {}_R(a_1,\ldots,a_n) = Ra_1 + \ldots + Ra_n$.
3. For a $D[X]$-submodule $J \subset K[X]$, we call
$$\mathsf{c}_D(J) = \sum_{f \in J} \mathsf{c}_D(f) \ \subset K$$
the *content* of J. By definition, $\mathsf{c}_D(J) \subset K$ is a D-submodule.
4. Let $I \lhd D$ be an ideal. We identify the rings $D[X]/I[X]$ and $(D/I)[X]$ by means of the canonical isomorphism. Explicitly, we set
$$\sum_{i\ge0} a_iX^i + I[X] = \sum_{i\ge0}(a_i + I)X^i \quad \text{for every polynomial} \quad f = \sum_{i\ge0} a_iX^i \in D[X]\,.$$
For a multiplicatively closed subset $T \subset D^\bullet$, we identify the rings $(T^{-1}D)[X]$ and $T^{-1}D[X]$ by means of the canonical isomorphism. Explicitly, we set
$$\sum_{i\ge0} \frac{a_i}{t}\,X^i = \sum_{i\ge0} a_iX^i\Big/t \quad \text{for every polynomial} \quad f = \sum_{i\ge0} a_iX^i \in D[X] \quad \text{and} \quad t \in T\,.$$

Theorem 6.2.2. *Let D be a ring, $K = \mathsf{q}(D)$ and $I,\,J \in \mathcal{F}(D)$.*

1. *$ID[X] = I[X] = \{f \in K[X] \mid \mathsf{c}_D(f) \subset I\} \in \mathcal{F}(D[X])$, $\mathsf{c}_D(I[X]) = I$, $I[X] \cap K = I$, and $(IJ)[X] = I[X]J[X]$.*
2. *I is finitely generated [principal] if and only if $I[X]$ is finitely generated [principal]. More precisely, if $I = {}_D(a_1,\ldots,a_n)$ for some $a_1,\ldots,a_n \in K$, then $I[X] = {}_{D[X]}(a_1,\ldots,a_n)$, and if $I[X] = {}_{D[X]}(f_1,\ldots,f_n)$ for some $f_1,\ldots,f_n \in K[X]$, then $I = {}_D(f_1(0),\ldots,f_n(0))$.*
3. *Let D be a domain and $J^\bullet \ne \emptyset$. Then $(I:J)[X] = (I[X]:J[X])$. In particular (for $I = D$), $J^{-1}[X] = J[X]^{-1}$.*

Proof. 1. By definition, $I[X] = \{f \in K[X] \mid \mathsf{c}_D(f) \subset I\} \subset ID[X]$, $\mathsf{c}_D(I[X]) \subset I$, and $I \subset I[X]$ implies $I = \mathsf{c}_D(I) \subset \mathsf{c}_D(I[X]$. Therefore $I[X] \cap K = \{a \in K \mid \mathsf{c}_D(a) = aD \subset I\} = I$.

If $f \in ID[X]$, then $f = a_1f_1 + \ldots + a_nf_n$, where $n \in \mathbb{N}$, $a_1,\ldots,a_n \in I$ and $f_1,\ldots,f_n \in D[X]$. For $i \in [1,n]$, we have $\mathsf{c}_D(a_if_i) = a_i\mathsf{c}_D(f_i) \subset a_iD \subset I$, hence $a_if_i \in I[X]$ and $f \in I[X]$. Consequently, $(IJ)[X] = (IJ)D[X] = (ID[X])(JD[X]) = I[X]J[X]$.

2. Obviously, $I = {}_D(a_1,\ldots,a_n)$ implies $I[X] = {}_{D[X]}(a_1,\ldots,a_n)$. Thus let $f_1,\ldots,f_n \in K[X]$ be such that $I[X] = {}_{D[X]}(f_1,\ldots,f_n)$. For all $i \in [1,n]$, $f_i \in I[X]$ implies $f_i(0) \in I$, and therefore ${}_D(f_1(0),\ldots,f_n(0)\,) \subset I$. If $a \in I \subset I[x]$, then $a = f_1g_1 + \ldots + f_ng_n$, for some $g_1,\ldots,g_n \in D[X]$, and therefore $a = (f_1g_1 + \ldots + f_ng_n)(0) = f_1(0)g_1(0) + \ldots + f_n(0)g_n(0) \in {}_D(f_1(0),\ldots f_n(0)\,)$.

3. Since $(I\!:\!J)[X]\,J[X] = ((I\!:\!J)J)[X] \subset I[X]$, we obtain $(I\!:\!J)[X] \subset (I[X]\!:\!J[X])$. Suppose now that $c \in I^\bullet$ and $F \in (I[X]\!:\!J[X]) \subset K(X)$. Then $Fc \in I[X] \subset K[X]$ and therefore $F \subset K[X]$. If $b \in J$, then $bF \in I[X]$ implies $I \supset \mathsf{c}_D(bF) = b\mathsf{c}_D(F)$, hence $J\mathsf{c}_D(F) \subset I$, $\mathsf{c}_D(F) \subset (I\!:\!J)$ and consequently $F \in (I\!:\!J)[X]$. □

Theorem und Definition 6.2.3. *Let D be a ring, $K = \mathsf{q}(D[X])$ the total quotient ring of the polynomial ring $D[X]$ and $N = \{f \in D[X] \mid \mathsf{c}_D(f) = D\}$.*

1. *$N \subset D[X]^*$ is a multiplicatively closed subset.*
   The ring $D(X) = N^{-1}D[X] \subset K$ is called the *Nagata ring* of D. If D is a field, then $N = D[X]^\bullet$, and $D(X)$ is just the field of rational functions (thus the terminology is consistent).
2. *Let $J \subsetneq D$ be an ideal, and let $\pi\colon D[X] \to D/J[X]$ be the canonical epimorphism. Then $JD(X) = N^{-1}J[X] \lhd D(X)$, $JD(X) \cap D = J[X] \cap D = J$, and there is an isomorphism*
   $$\Phi\colon D(X)/JD(X) \;\to\; (D/J)(X)\,, \quad \textit{given by} \quad \Phi\Big(\frac{f}{g} + JD(X)\Big) = \frac{\pi(f)}{\pi(g)}\,.$$
3. *If $P \in \operatorname{spec}(D)$, then $P[X] \in \operatorname{spec} D[X]$, $PD(X) \in \operatorname{spec} D(X)$, and the natural embedding $j_P\colon D[X] \to D_P[X] = (D \setminus P)^{-1}D[X]$ induces an isomorphism $\iota_P\colon D[X]_{P[X]} \overset{\sim}{\to} D_P(X)$.*
4. $\max D(X) = \{PD(X) \mid P \in \max(D)\}$.

Proof. 1. If $f \in N$ and $g \in D[X]^\bullet$, then $\mathsf{c}_D(fg) = \mathsf{c}_D(f)\mathsf{c}_D(g) = \mathsf{c}_D(g) \neq \{0\}$ by Corollary 6.1.4. Hence $fg \neq 0$, which implies $f \in D[X]^*$. If $f,\,g \in N$, then $\mathsf{c}_D(fg) = \mathsf{c}_D(f)\mathsf{c}_D(g) = D$, hence $fg \in N$, and N is multiplicatively closed.

2. Clearly, $JD(X) = JN^{-1}D[X] = N^{-1}JD[X] = N^{-1}J[X] \lhd D(X)$. If $a \in JD(X) \cap D$, then there is some $f \in N$ such that $af \in J[X]$, and therefore $\mathsf{c}_D(af) = a\mathsf{c}_C(f) = aD \subset J$, which implies $a \in J$. Hence $JD(X) \cap D \subset J \subset J[X] \cap D \subset JD(X) \cap D$, and thus equality holds.
There is an isomorphism
$$\Phi\colon D(X)/JD(X) = N^{-1}D[X]/N^{-1}J[X] \;\overset{\sim}{\to}\; N^{-1}(D[X]/J[X]) = N^{-1}(D/J)[X] = \pi(N)^{-1}(D/J)[X]\,,$$
given by
$$\Phi\Big(\frac{f}{g} + JD(X)\Big) = \frac{\pi(f)}{\pi(g)} \quad \text{for all} \quad f \in D[X] \;\text{ and }\; g \in \mathbb{N}\,.$$
Therefore it suffices to prove that $\pi(N) = \{\pi(f) \mid f \in D[X]\,,\ \mathsf{c}_{D/J}(\pi(f)) = D/J\}$. If $f \in D[X]$, then $\mathsf{c}_{D/J}(\pi(f)) = \mathsf{c}_D(f) + J/J$, and therefore $f \in N$ implies $\mathsf{c}_{D/J}(\pi(f)) = D/J$. To prove the converse, let $f \in D[X]$ be such that $\mathsf{c}_{D/J}(\pi(f)) = D/J$. Then $\mathsf{c}_D(f) + J = D$, and there exists some $u \in J$ such that $\mathsf{c}_D(f) + uD = D$. If $n \in \mathbb{N}$ and $n > \deg(f)$, then $\mathsf{c}_D(f + uX^n) = \mathsf{c}_D(f) + uD = D$, hence $f + uX^n \in N$ and $\pi(f) = \pi(f + uX^n) \in \pi(N)$.

3. Let $P \in \operatorname{spec}(D)$. Then $D[X]/P[X] = (D/P)[X]$ is a domain. Hence $P[X] \in \operatorname{spec} D[X]$, and since $P[X] \cap N = \emptyset$, it follows that $PD(X) = N^{-1}P[X] \in \operatorname{spec} D(X)$. By definition,
$$D_P(X) = N_P^{-1}D_P[X]\,, \quad \text{where} \quad N_P = \{F \in D_P[X] \mid \mathsf{c}_{D_P}(F) = D_P\}\,.$$
If $f \in D[X] \setminus P[X]$, then $\mathsf{c}_D(f) \not\subset P$, hence $\mathsf{c}_{D_P}(j_P(f)) = \mathsf{c}_D(f)_P = D_P$ and therefore $j_P(f) \in N_P$. Hence it follows that $j_P(D[X] \setminus P[X]) \subset N_P$, and therefore j_P induces a ring homomorphism
$$\iota_P\colon D[X]_{P[X]} \;\to\; D_P(X)\,, \quad \text{given by} \quad \iota_P\Big(\frac{g}{f}\Big) = \frac{g/1}{f/1} \quad \text{for all } \; g \in D[X] \;\text{ and }\; f \in D[X] \setminus P[X]\,.$$

ι_P is surjective: If $z \in D_P(X)$, then there exist $g \in D[X]$, $f \in D[X] \setminus P[X]$ and $s, t \in D \setminus P$ such that

$$z = \frac{g/s}{f/t} = \frac{tg/1}{sf/1} = \iota_P\Big(\frac{tg}{sf}\Big) \quad \text{(note that } sf \in D[X] \setminus P[X]\,).$$

ι_P is injective: If $z \in \mathrm{Ker}(\iota_P) \subset D[X]_{P[X]}$, then

$$z = \frac{g}{f}\,, \quad \text{where } g \in D[X]\,,\ f \in D[X] \setminus P[X] \text{ and } \frac{g/1}{f/1} = 0 \in D_P(X)\,, \text{ hence } \frac{g}{1} = \frac{0}{1} \in D_P[X]\,.$$

Therefore there exists some $s \in D \setminus P$ such that $sg = 0$, and as $s \in D[X] \setminus P[X]$, this implies $z = 0$.

4. If $P \in \max(D)$, then $D(X)/PD(X) \simeq (P/D)(X)$ is a field, and therefore $PD(X) \in \max D(X)$. Thus assume that $M \in \max D(X)$. Then $M = N^{-1}Q$, where $Q \in \operatorname{spec} D[X]$ is maximal such that $Q \cap N = \emptyset$. It is now sufficient to prove that

$$J = \sum_{f \in Q} \mathsf{c}_D(f) \ \neq\ D\,.$$

Indeed, then there exists some $P \in \max(D)$ such that $J \subset P$, hence $Q \subset P[X]$, and it follows that $M = N^{-1}Q \subset N^{-1}P[X] = PD(X)$, and therefore $M = PD(X)$.

Assume to the contrary that $J = D$. Then there exist $f_1, \ldots, f_m \in Q$ such that $1 \in \mathsf{c}(f_1) + \ldots + \mathsf{c}(f_m)$. Let $k_2, \ldots, k_m \in \mathbb{N}$ be such that $k_j > \deg(f_1 + X^{k_2} f_2 + \ldots + X^{k_{j-1}} f_{j-1})$ for all $j \in [2, m]$, and consider the polynomial $f = f_1 + X^{k_2} f_2 + \ldots + X^{k_m} f_m$. Then $\mathsf{c}_D(f) = \mathsf{c}_D(f_1) + \ldots + \mathsf{c}_D(f_m)$, hence $1 \in \mathsf{c}_D(f)$ and $f \in Q$, a contradiction. □

Theorem 6.2.4. *Let K be a field, v be valuation of K and v^* the trivial extension of v to $K(X)$. Then $\mathcal{O}_{v^*} = \mathcal{O}_v(X)$.*

Proof. By definition, $\mathcal{O}_v(X) = N^{-1}\mathcal{O}_v[X]$, where

$$N = \{f \in \mathcal{O}_v[X] \mid \mathsf{c}_{\mathcal{O}_v}(f) = \mathcal{O}_v\} = \Big\{\sum_{i \ge 0} a_i X^i \in \mathcal{O}_v[X] \ \Big|\ v(a_i) = 0 \text{ for some } i \ge 0\Big\}\,,$$

and therefore $N = \{f \in \mathcal{O}_v[X] \mid v^*(f) = 0\}$. If $f \in \mathcal{O}_v[X]^\bullet$, then $f = af_0$, where $a \in \mathcal{O}_v^\bullet$, $f_0 \in N$ and $v(a) = v^*(f_0)$. Therefore we obtain

$$\mathcal{O}_{v^*} = \Big\{\frac{af_0}{g_0} \ \Big|\ a \in K\,,\ v(a) \ge 0\,,\ f_0, g_0 \in N\Big\} = \Big\{\frac{f}{g_0} \ \Big|\ f \in \mathcal{O}_v[X]\,,\ g_0 \in N\Big\} = \mathcal{O}_v(X)\,. \qquad \square$$

Theorem und Definition 6.2.5. *Let D be a domain, $K = \mathsf{q}(D)$ and r be a finitary module system on K such that $r \ge d = d(D)$ (then $\{1\}_d = D$ implies $\{1\}_r = D_r \supset D$).*

1. $N_r = \{f \in D[X] \mid \mathsf{c}_D(f)_r = D_r\} \subset D[X]$ *is a multiplicatively closed subset.*
   The domain $\mathsf{N}_r(D) = N_r^{-1}D[X] \subset K[X]$ is called the *r-Nagata domain* of D. Note that $D(X) = \mathsf{N}_d(D)$.
2. *Let $J \in \mathcal{F}(D)$ be a fractional ideal of D. Then $J\mathsf{N}_r(D) = N_r^{-1}J[X]$ is a fractional ideal of $\mathsf{N}_r(D)$, and $J \subset J\mathsf{N}_r(D) \cap K \subset J_r$.*
3. *If $I, J \in \mathcal{F}(D)$, $J^\bullet \ne \emptyset$ and $I_r = I$, then $(I : J)\mathsf{N}_r(D) = (I\mathsf{N}_r(D) : J\mathsf{N}_r(D))$. In particular, $(J\mathsf{N}_r(D))^{-1} = J^{-1}\mathsf{N}_r(D)$.*
4. $\max \mathsf{N}_r(D) = \{P\mathsf{N}_r(D) \mid P \in r_D\text{-}\max(D)\}$. *If $P \in r_D$-$\max(D)$ and $M = P\mathsf{N}_r(D)$, then $\mathsf{N}_r(D)_M = D[X]_{P[X]} = D_P(X)$.*
5. *If $J \in \mathcal{F}(D)$, then*

$$J\mathsf{N}_r(D) \cap K = \bigcap_{P \in r_D\text{-}\max(D)} J_P\,.$$

6. *If* $J \in \mathcal{F}(D)$*, then* $(JJ^{-1})_r = D_r$ *if and only if* $J\mathsf{N}_r(D)$ *is an invertible fractional ideal of* $\mathsf{N}_r(D)$*. In particular, if* r *is an ideal system on* D *and* $J \in \mathcal{F}_r(D)$*, then* J *is* r*-invertible if and only if* $J\mathsf{N}_r(D)$ *is an invertible fractional ideal of* $\mathsf{N}_r(D)$*.*
7. $\operatorname{Pic}\mathsf{N}_r(D) = \mathbf{0}$*. Every invertible fractional ideal of* $\mathsf{N}_r(D)$ *is principal.*

Proof. 1. Since $D_r = \{1\}_r$ is r-cancellative, we may apply Corollary 6.1.4. If $f, g \in N_r$, then $\mathsf{c}_D(fg)_r = [\,\mathsf{c}_D(f)\mathsf{c}_D(g)\,]_r = \mathsf{c}_D(f)_r \cdot_r \mathsf{c}_D(g)_r = D_r$ and thus $fg \in N_r$.

2. Clearly, $N_r^{-1}J[X]$ is an $\mathsf{N}_r(D)$-submodule of $K(X) = \mathsf{q}(\mathsf{N}_r(D))$, and if $a \in D^\bullet$ and $aJ \subset D$, then $aN_r^{-1}J[X] \subset \mathsf{N}_r(D)$. Hence $N_r^{-1}J[X]$ is a fractional ideal of $\mathsf{N}_r(D)$, and $J \subset J[X] \subset N_r^{-1}J[X] \cap K$. If $a \in N_r^{-1}J[X] \cap K$, then there exists some $g \in N_r$ such that $ag \in J[X]$, hence $\mathsf{c}_D(ag) \subset J$ and $\mathsf{c}_D(ag)_r = a\mathsf{c}_D(g)_r = aD_r \subset J_r$, and therefore $a \in J_r$.

3. $N_r^{-1}(IJ)[X] = N_r^{-1}(I[X]\,J[X]) = (N_r^{-1}I[X])(N_r^{-1}J[X])$. Hence it follows that

$$\big(N_r^{-1}(I\!:\!J)[X]\big)(N_r^{-1}J[X]) = N_r^{-1}\big((I\!:\!J)J\big)[X] \subset N_r^{-1}I[X]\,,$$

and therefore $N_r^{-1}(I\!:\!J)[X] \subset (N_r^{-1}I[X]\!:\!N_r^{-1}J[X])$. If $J = \{0\}$, then equality holds.

Assume now that $I = I_r$, $b \in J^\bullet$ and $F \in (N_r^{-1}I[X]\!:\!N_r^{-1}J[X])$. Since $J \subset N_r^{-1}J[X]$, we obtain $bF \in N_r^{-1}I[X]$, and therefore there exist some $f \in b^{-1}I[X] \subset K[X]$ and $g \in N_r$ such that $gF = f$. If $a \in J$, then $af = aFg \in N_r^{-1}I[X]$, and there exists some $h \in N_r$ such that $afh \in I[X]$. Hence it follows that $\mathsf{c}_D(afh) \subset I$, and $a\mathsf{c}_D(f) = \mathsf{c}_D(af) \subset \mathsf{c}_D(af)_r = \mathsf{c}_D(af)_r \cdot_r \mathsf{c}_D(h)_r = \mathsf{c}_D(afh)_r \subset I_r = I$. Since $a \in J$ was arbitrary, we obtain $J\mathsf{c}_D(f) \subset I$, hence $\mathsf{c}_D(f) \subset (I\!:\!J)$, and $F \in N_r^{-1}(I\!:\!J)[X]$.

4. For the proof of $\max \mathsf{N}_r(D) = \{P\mathsf{N}_r(D) \mid P \in r_D\text{-max}(D)\}$ we proceed in three steps:

- If $P \in r_D$-spec(D), then $N_r^{-1}P[X] \in \operatorname{spec}\mathsf{N}_r(D)$.

If $P \in r_D$-spec(D) and $f \in P[X]$, then $\mathsf{c}_D(f) \subset P = P_r \cap D$. Hence it follows that $\mathsf{c}_D(f)_r \subset P_r \subsetneq D_r$, $P[X] \cap N_r = \emptyset$, and $N_r^{-1}P[X] \in \operatorname{spec}\mathsf{N}_r(D)$.

- If $M \in \max \mathsf{N}_r(D)$, then there exists some $P \in r_D$-max(D) such that $M = P\mathsf{N}_r(D)$.

Suppose that $M \in \max \mathsf{N}_r(D)$, say $M = N_r^{-1}Q$ for some $Q \in \operatorname{spec} D[X]$ such that $Q \cap N_r = \emptyset$. We set

$$J = \sum_{f \in Q} \mathsf{c}_D(f) \subset D\,, \quad \text{and we assert that} \quad J_r = \Big(\bigcup_{f\in Q} \mathsf{c}_D(f)\Big)_r \neq D_r\,.$$

Assume the contrary. Since r is finitary, there exist $f_1, \ldots, f_m \in Q$ such that $1 \in [\,\mathsf{c}_D(f_1) \cup \ldots \cup \mathsf{c}_D(f_m)\,]_r$. Let $k_2, \ldots, k_m \in \mathbb{N}$ be such that $k_j > \deg(f_1 + X^{k_2}f_2 + \ldots + X^{k_{j-1}}f_{j-1})$ for all $j \in [2, m]$. Then we obtain $f = f_1 + X^{k_2}f_2 + \ldots + X^{k_m}f_m \in Q$, and $\mathsf{c}_D(f)_r = [\,\mathsf{c}_D(f_1) + \ldots + \mathsf{c}_D(f_m)\,]_r = [\,\mathsf{c}_D(f_1) \cup \ldots \cup \mathsf{c}_D(f_m)\,]_r$, hence $\mathsf{c}_D(f)_r = D_r$ and $f \in N_r$, a contradiction.

As $J_r \neq D_r$, we obtain $J \subset J_{r_D} = J_r \cap D \subsetneq D$, and there exists some $P \in r_D$-max(D) such that $J \subset P$. If $f \in Q$, then $\mathsf{c}_D(f) \subset J \subset P$, hence $f \in P[X]$, and therefore $Q \subset P[X]$. Hence it follows that $M = N_r^{-1}Q \subset N_r^{-1}P[X] = P\mathsf{N}_r(D)$, and therefore $M = P\mathsf{N}_r(D)$.

- If $P \in r_D$-max(D), then $P\mathsf{N}_r(D) \in \max \mathsf{N}_r(D)$.

If $P \in r_D$-max(D), then $P\mathsf{N}_r(D) \in \operatorname{spec}\mathsf{N}_r(D)$, and there exists some $M \in \max \mathsf{N}_r(D)$ such that $P\mathsf{N}_r(D) \subset M$. As we have just proved, $M = P'\mathsf{N}_r(D)$ for some $P' \in r_D$-max(D), and we obtain $P \subset P\mathsf{N}_r(D) \cap D \subset P'\mathsf{N}_r(D) \cap D \subset P'_r = P'$, hence $P = P'$ and $P\mathsf{N}_r(D) = M$.

If $P \in r_D$-max(D), then $\mathsf{N}_r(D)_{N_r^{-1}P[X]} = N_r^{-1}D[X]_{N_r^{-1}P[X]} = D[X]_{P[X]} = D_P(X)$ by Theorem 6.2.3.3 (note that in our case all rings are subrings of $K(X)$ the isomorphism ι_P given there is the identity map).

5. If $J \in \mathcal{F}(D)$, then Theorem 3.2.2 implies

$$J\mathsf{N}_r(D) \cap K = \bigcap_{M \in \max \mathsf{N}_r(D)} J\mathsf{N}_r(D)_M \cap K = \bigcap_{P \in r_D\text{-}\max(D)} JD_P(X) \cap K\,.$$

Hence it suffices to prove that $JD_P(X) \cap K = J_P$ for all $P \in r_D\text{-max}(D)$. If $P \in r_D\text{-max}(D)$, then clearly $J_P \subset JD_P(X) \cap K$. Thus suppose that $a \in JD_P(X) \cap K$. Since $JD_P(X) = JN_P^{-1}D_P[X] = N_P^{-1}J_P[X]$, where $N_P = \{g \in D_P[X] \mid \mathsf{c}_{D_P}(g) = D_P\}$, there exists some $g \in \mathbb{N}_P$ such that $ag \in J_P[X]$. Hence $\mathsf{c}_{D_P}(ag) = a\mathsf{c}_{D_P}(g) \subset J_P$, and if $s \in \mathsf{c}_{D_P}(g) \setminus P_P = D_P^\times$, then $as \in J_P$ and therefore $a \in J_P$.

6. Suppose that $(JJ^{-1})_r \neq D_r$. Then $(JJ^{-1})_{r_D} = (JJ^{-1})_r \cap D \neq D$, and therefore there exists some $P \in r_D\text{-max}(D)$ such that $JJ^{-1} \subset P$. Hence $J\mathsf{N}_r(D)(J\mathsf{N}_r(D))^{-1} = (JJ^{-1})\mathsf{N}_r(D) \subset P\mathsf{N}_r(D) \subsetneq \mathsf{N}_r(D)$ by 3., and therefore $J\mathsf{N}_r(D)$ is not invertible.

Conversely, assume that $J\mathsf{N}_r(D)$ is not invertible. Then there exists some $M \in \max \mathsf{N}_r(D)$ such that $J\mathsf{N}_r(D)(J\mathsf{N}_r(D))^{-1} \subset M$. By 4. there exists some $P \in r_D\text{-max}(D)$ such that $M = P\mathsf{N}_r(D)$, and then $JJ^{-1} \subset (JJ^{-1})\mathsf{N}_r(D) \cap D = J\mathsf{N}_r(D)(J\mathsf{N}_r(D))^{-1} \cap D \subset P\mathsf{N}_r(D) \cap D \subset P_r$, which implies that $(JJ^{-1})_r \subset P_r \subsetneq D_r$.

6. Let $J \subset \mathsf{N}_r(D) = N_r^{-1}D[X]$ be an invertible ideal. Then $J = (f_1, \ldots, f_m)$ for some $m \in \mathbb{N}$ and $f_1, \ldots, f_m \in D[X]^\bullet$. Let $k_2, \ldots, k_m \in \mathbb{N}$ be such that $k_j > \deg(f_1 + X^{k_2}f_2 + \ldots + X^{k_{j-1}}f_{j-1})$ for all $j \in [2, m]$. If $f = f_1 + X^{k_2}f_2 + \ldots + X^{k_m}f_m \in J$, then $\mathsf{c}_D(f) = \mathsf{c}_D(f_1) + \ldots + \mathsf{c}_D(f_m)$, and we assert that $J = f\mathsf{N}_r(D)$. By Theorem 3.2.2 it suffices to prove that $J_M = f\mathsf{N}_r(D)_M$ for all $M \in \max \mathsf{N}_r(D)$. Let $M \in \max \mathsf{N}_r(D)$ and $P \in r_D\text{-max}(D)$ such that $M = P\mathsf{N}_r(D)$. Then $\mathsf{N}_r(D)_M = D[X]_{P[X]}$, and by Theorem 4.1.4 there exists some $j \in [1, m]$ such that $J_M = f_j\mathsf{N}_r(D)_M = f_jD[X]_{P[X]}$. Since $f \in J_M$, there exists some $h \in D[X] \setminus P[X]$ and some $g \in D[X]$ such that $fh = f_jg$, and it suffice to prove that $g \notin P[X]$, for then $g, h \in (D[X]_{P[X]})^\times = \mathsf{N}_r(D)_M{}^\times$ and $J_M = f_j\mathsf{N}_r(D)_M = f\mathsf{N}_r(D)_M$.

Assume to the contrary that $g \in P[X]$. Then $\mathsf{c}_D(fh) = \mathsf{c}_D(f_jg) \subset \mathsf{c}_D(f_j)\mathsf{c}_D(g) \subset \mathsf{c}_D(f_j)P$, and since $h \notin P[X]$, it follows that $\mathsf{c}_D(h) \not\subset P$ and $\mathsf{c}_{D_P}(h) = \mathsf{c}_D(h)_P = D_P$. Hence we obtain

$$\mathsf{c}_D(f_j)_P \subset \mathsf{c}_D(f)_P = \mathsf{c}_{D_P}(f) = \mathsf{c}_{D_P}(fh) = \mathsf{c}_D(fh)_P \subset \mathsf{c}_D(f_j)_P\, P_P$$

and therefore $\mathsf{c}_D(f_j)_M = \{0\}$ by Lemma 6.1.3. But this implies that $f_j = 0$, a contradiction. □

6.3. Kronecker domains

Definition 6.3.1. Let K be a field. A subring $R \subset K(X)$ is called a *Kronecker domain* if $X \in R^\times$ and $f(0) \in fR$ for all $f \in K[X]$.

Theorem 6.3.2. *Let K be a field and $R \subset K(X)$ a Kronecker domain.*

1. *If $f = a_0 + a_1X + \ldots + a_nX^n \in K[X]$, then $fR = Ra_0 + \ldots + Ra_n$.*
2. *R is a Bezout domain, and $K(X) = \mathsf{q}(R)$. In particular, R is a* GCD*-domain, $t(R) = d(R)$, $\mathrm{Pic}(R) = \mathcal{C}(R) = \mathbf{0}$, and a domain Y such that $R \subset Y \subset K(X)$ is a valuation domain if and only if Y is a $t(R)$-valuation monoid.*
3. *Let $R \subset Y \subset K(X)$ be a valuation domain. Then $V = Y \cap K$ is a valuation domain of K, and $Y = V(X)$.*

Proof. 1. Clearly, $X \in R$ implies $fR \subset a_0R + \ldots + a_nR$. For the reverse inclusion we prove that $a_i \in fR$ for all $i \in [0, n]$ by induction on i.

$i = 0$: $a_0 = f(0) \in fR$.

$i \in [1, n]$, $i - 1 \to i$: If $a_0, \ldots, a_{i-1} \in fR$, then $f' = X^{-i}[f - (a_0 + a_1X + \ldots + a_{i-1}X^{i-1})] \in fR$, and therefore $a_i = f'(0) \in f'R \subset fR$.

2. We prove that every ideal of R generated by two elements is a principal ideal. Thus let $\alpha, \beta \in R$ and $f, g, h \in K[X]^\bullet$ such that $\alpha = \frac{f}{h}$ and $\beta = \frac{g}{h}$. If $n > \deg(f)$, then $fR + gR = fR + X^ngR = (f + X^ng)R$ by 1., and therefore $\alpha R + \beta R = (\alpha + X^n\beta)R$.

In order to prove that $K(X) = \mathsf{q}(R)$, it suffices to prove that $K[X] \subset \mathsf{q}(R)$. If $f \in K[X]$, then $h = (1 + Xf)^{-1} \in R$, and therefore $f = X^{-1}(h^{-1} - 1) \in \mathsf{q}(R)$.

3. If $x \in K \setminus V$, then $x \in K(X) \setminus Y$, hence $x^{-1} \in K \cap Y = V$, and therefore V is a valuation domain of K. Let $y\colon K(X) \to \Gamma \cup \{\infty\}$ be a valuation such that $\mathcal{O}_y = Y$. Then $y(X) = 0$, and if $f = a_0 + a_1X + \ldots + a_nX^n \in K[X]$, then $a_i \in fR \subset fY$ and therefore $y(a_i) \ge y(f)$ for all $i \in [0, n]$. On the other hand, $y(f) \ge \min\{y(a_iX^i) \mid i \in [0, n]\} = \min\{y(a_i) \mid i \in [0, n]\} \ge y(f)$. Hence equality holds, and we obtain $y(K) = y(K[X])$. Since $\Gamma = y(K(X)^\times) = \mathsf{q}(y(K[X]^\bullet)) = \mathsf{q}(y(K^\times)) = y(K^\times)$, it follows that $v = y\,|\,K\colon K \to \Gamma \cup \{\infty\}$ is a valuation such that $\mathcal{O}_v = V$, and $y = v^*$, the trivial extension of v to $K(X)$. Hence $Y = V(X)$ by Theorem 6.2.4. □

Definition 6.3.3. Let D be a domain, $K = \mathsf{q}(D)$ and r a finitary module system on K such that $r \ge d(D)$. Then

$$\mathsf{K}_r(D) = \Big\{ \frac{f}{g} \;\Big|\; f \in D[X]\,,\; g \in D[X]^\bullet\,,\; \mathsf{c}_D(f) \subset \mathsf{c}_D(g)_{r_\mathsf{a}} \Big\} \subset K(X)$$

is called the *r-Kronecker function domain* of D.

Theorem 6.3.4. *Let D be a domain, $K = \mathsf{q}(D)$ and r a finitary module system on K such that $r \ge d(D)$.*

1. $\mathsf{K}_r(D)$ *is a Kronecker domain of* $K(X)$, *and if* $f \in K[X]$ *and* $g \in K[X]^\bullet$, *then* $\frac{f}{g} \in \mathsf{K}_r(D)$ *if and only if* $\mathsf{c}_D(f) \subset \mathsf{c}_D(g)_{r_\mathsf{a}}$.
2. *There is a surjective monoid homomorphism*

$$\varepsilon\colon K(X) \to \Lambda_r(K)\,, \quad \textit{given by} \;\; \varepsilon\Big(\frac{f}{g}\Big) = \mathsf{c}_D(g)_{r_\mathsf{a}}^{[-1]}\,\mathsf{c}_D(f)_{r_\mathsf{a}} \;\; \textit{for all} \;\in K[X] \;\textit{and}\; g \in K[X]^\bullet\,.$$

$\varepsilon^{-1}(\Lambda_r^+(K)) = \mathsf{K}_r(D)$, $\varepsilon^{-1}(1) = \mathsf{K}_r(D)^\times$, *and ε induces monoid isomorphisms*

$$K(X)/\mathsf{K}_r(D)^\times \xrightarrow{\sim} \Lambda_r(K) \quad \textit{and} \quad \mathsf{K}_r(D)/\mathsf{K}_r(D)^\times \xrightarrow{\sim} \Lambda_r^+(K)\,.$$

$\varepsilon\,|\,K = \tau_r\colon K \to \Lambda_r(K)$ *is the Lorenzen r-homomorphism.*

3. *Let $t = t(\Lambda_r^+(K))$. Denote by $\mathcal{W}$ the set of all t-valuation monoids of $\Lambda_r(K)$, by $\mathcal{Y}$ set of all valuation domains Y such that $\mathsf{K}_r(D) \subset Y \subset K(X)$ and by $\mathcal{V}$ the set of all valuation domains $V \subset K$ such that $V_r = V$. Then there are bijective maps*

$$\widetilde{\tau}_r\colon \begin{cases} \mathcal{W} \to \mathcal{V} \\ W \mapsto \tau_r^{-1}(W)\,, \end{cases} \quad \widetilde{\varepsilon}\colon \begin{cases} \mathcal{Y} \to \mathcal{W} \\ Y \mapsto \varepsilon(Y)\,, \end{cases} \quad \widetilde{\eta}\colon \begin{cases} \mathcal{Y} \to \mathcal{V} \\ Y \mapsto Y \cap K\,, \end{cases} \quad \widetilde{\theta}\colon \begin{cases} \mathcal{V} \to \mathcal{Y} \\ V \mapsto V(X)\,, \end{cases}$$

where $\widetilde{\eta} = \widetilde{\tau}_r \circ \widetilde{\varepsilon}$ *and* $\widetilde{\theta} = \widetilde{\eta}^{-1}$.

Proof. 1. Let $f, g \in K[X]$, $g \ne 0$ and $a \in D^\bullet$ such that $af, ag \in D[X]$. If $\mathsf{c}_D(f) \subset \mathsf{c}_D(g)_{r_\mathsf{a}}$, then $\mathsf{c}_D(af) = a\mathsf{c}_D(f) \subset a\mathsf{c}_D(g)_{r_\mathsf{a}} = \mathsf{c}_D(af)_{r_\mathsf{a}}$ and therefore $\frac{f}{g} = \frac{af}{ag} \in \mathsf{K}_r(D)$ by definition. Conversely, assume that $\frac{f}{g} \in \mathsf{K}_r(D)$, and let $f_1, g_1 \in D[X]$ be such that $g_1 \ne 0$, $\mathsf{c}_D(f_1) \subset \mathsf{c}_D(g_1)_{r_\mathsf{a}}$ and $\frac{f}{g} = \frac{f_1}{g_1}$. Then $\mathsf{c}_D(f_1)_{r_\mathsf{a}} \subset \mathsf{c}_D(g_1)_{r_\mathsf{a}}$, $fg_1 = f_1g$, and since r_a is finitely cancellative, we obtain

$$[\,\mathsf{c}_D(f)\,\mathsf{c}_D(g_1)\,]_{r_\mathsf{a}} = \mathsf{c}_D(fg_1)_{r_\mathsf{a}} = \mathsf{c}_D(f_1g)_{r_\mathsf{a}} = [\,\mathsf{c}_D(f_1)\,\mathsf{c}_D(g)\,]_{r_\mathsf{a}} \subset [\,\mathsf{c}_D(g_1)\,\mathsf{c}_D(g)\,]_{r_\mathsf{a}}$$

and therefore $\mathsf{c}_D(f) \subset \mathsf{c}_D(f)_{r_\mathsf{a}} \subset \mathsf{c}_D(g)r_\mathsf{a}$.

Next we prove that $\mathsf{K}_r(D) \subset K(X)$ is a subring. Suppose that $\alpha, \beta \in \mathsf{K}_r(D)$, say $\alpha = \frac{f}{h}$ and $\beta = \frac{g}{h}$, where $f, g, h \in K[X]$, $h \ne 0$ and $\mathsf{c}_D(f) \cup \mathsf{c}_D(g) \subset \mathsf{c}_D(h)_{r_\mathsf{a}}$. Then $\alpha + \beta = \frac{f+g}{h}$, $\alpha\beta = \frac{fg}{h^2}$, $\mathsf{c}_D(f+g) \subset \mathsf{c}_D(f) + \mathsf{c}_D(g) \subset \mathsf{c}_D(h)_{r_\mathsf{a}}$ and $\mathsf{c}_D(fg) \subset \mathsf{c}_D(f)\mathsf{c}_D(g) \subset \mathsf{c}_D(h)_{r_\mathsf{a}}^2 = \mathsf{c}_D(h^2)_{r_\mathsf{a}}$, which implies $\alpha + \beta \in \mathsf{K}_r(D)$ and $\alpha\beta \in \mathsf{K}_r(D)$.

Clearly, $X \in \mathsf{K}_r(D)$, $X^{-1} \in \mathsf{K}_r(D)$, and if $f \in K[X]$, then $\mathsf{c}_D(f(0)) = Df(0) \subset \mathsf{c}_D(f) \subset \mathsf{c}_D(f)_{r_\mathsf{a}}$, hence $\frac{f(0)}{f} \in \mathsf{K}_r(D)$ and therefore $f(0) \in f\mathsf{K}_r(D)$. Hence $\mathsf{K}_r(D)$ is a Kronecker domain.

2. If $f,\, f_1 \in K[X]$ and $g,\, g_1 \in K[X]^\bullet$ are such that $\frac{f}{g} = \frac{f_1}{g_1}$, then $fg_1 = f_1 g$, and as $r_{\mathsf a}$ is finitely cancellative, we obtain $[\,\mathsf{c}_D(f)\,\mathsf{c}_D(g_1)\,]_{r_{\mathsf a}} = \mathsf{c}_D(fg_1)_{r_{\mathsf a}} = \mathsf{c}_D(f_1 g)_{r_{\mathsf a}} = [\,\mathsf{c}_D(f_1)\,\mathsf{c}_D(g)\,]_{r_{\mathsf a}}$, and therefore $\mathsf{c}_D(g)_{r_{\mathsf a}}^{[-1]}\,\mathsf{c}_D(f)_{r_{\mathsf a}} = \mathsf{c}_D(g_1)_{r_{\mathsf a}}^{[-1]}\,\mathsf{c}_D(f_1)_{r_{\mathsf a}}$. Hence there is a map $\varepsilon\colon K(X) \to \Lambda_r(K)$ as announced, and it obviously is a homomorphism. If $E = \{a_0, \ldots, a_n\} \in \mathbb{P}_{\mathsf f}(K)$, then $E_{r_{\mathsf a}} = \mathsf{c}_D(a_0 + a_1 X + \ldots + a_n X^n)_{r_{\mathsf a}}$, and since $\Lambda_r(K) = \{E_{r_{\mathsf a}}'^{[-1]} E_{r_{\mathsf a}} \mid E,\, E' \in \mathbb{P}_{\mathsf f}(K)\,,\ E'^\bullet \neq \emptyset\,\}$, it follows that ε is surjective.

If $f \in D[X]$ and $g \in D[X]^\bullet$, then $\varepsilon\big(\frac{f}{g}\big) = \mathsf{c}_D(g)_{r_{\mathsf a}}^{[-1]}\mathsf{c}_D(f)_{r_{\mathsf a}} \in \Lambda_r^+(K)$ if and only if $\mathsf{c}_D(f)_{r_{\mathsf a}} \subset \mathsf{c}_D(g)_{r_{\mathsf a}}$, which is equivalent to $\frac{f}{g} \in \mathsf{K}_r(D)$, and $\varepsilon\big(\frac{f}{g}\big) = 1$ if and only if $\frac{f}{g} \in \mathsf{K}_r(D)^\times$. Hence $\varepsilon^{-1}(\Lambda_r^+(K)) = \mathsf{K}_r(D)$, $\varepsilon^{-1}(1) = \mathsf{K}_r(D)^\times$, and ε induces an isomorphism ε^* as asserted.

If $a \in K$, then $\varepsilon(a) = \mathsf{c}_D(a)_{r_{\mathsf a}} = \{a\}_{r_{\mathsf a}} = \tau_r(a)$, and therefore $\varepsilon \,|\, K = \tau_r$.

3. By Theorem 4.4.3.2 (b) $\widetilde{\tau}_r$ is bijective. By 2., ε induces a commutative diagram

$$\begin{array}{ccccc} \tau_r\colon K & \longrightarrow & K(X) & \xrightarrow{\ \varepsilon\ } & \Lambda_r(K) \\ \mathcal{V}\big\uparrow & & \mathcal{Y}\big\uparrow & & \mathcal{W}\big\uparrow \\ D & \longrightarrow & \mathsf{K}_r(D) & \xrightarrow{\ \varepsilon\ } & \Lambda_r^+(K)\,, \end{array}$$

where the upwards arrows are inclusions. If $t^* = t(\mathsf{K}_r(D))$, then $t^* = \varepsilon^* t$ by Theorem 2.6.2, and by Theorem 6.3.2.2, $\mathcal{Y}$ is the set of all t-valuation monoids Y such that $\mathsf{K}_r(D) \subset Y \subset K(X)$, and by Theorem 3.4.10 the assignment $Y \mapsto \varepsilon(Y)$ defines a bijective map $\widetilde{\varepsilon}\colon \mathcal{Y} \to \mathcal{W}$. Hence $\widetilde{\eta} = \widetilde{\tau}_r \circ \widetilde{\varepsilon}\colon \mathcal{Y} \to \mathcal{V}$ is bijective. If $Y \in \mathcal{Y}$, then $\mathsf{K}_r(D)^\times = \varepsilon^{-1}(1) \subset Y$, and $\widetilde{\eta}(Y) = \tau_r^{-1} \circ \varepsilon(Y) = (\varepsilon \,|\, K)^{-1} \circ \varepsilon(Y) = Y \cap K$. If $V \in \mathcal{V}$, then $Y = \widetilde{\eta}^{-1}(V) \in \mathcal{Y}$, $V = Y \cap K$, and therefore $Y = V(X) = \widetilde{\theta}(V)$ by Theorem 6.3.2.3. $\square$

6.4. v-ideals and t-ideals in polynomial domains

Throughout this section, let D be a domain and $K = \mathsf{q}(D)$.
We use t and v for the corresponding operations both for D and $D[X]$.

Definition 6.4.1.

1. An ideal $J \lhd D[X]$ is called *almost principal* if there exist $f \in J \setminus D$ and $r \in D^\bullet$ such that $J \subset r^{-1} f D[X]$.
2. For a $D[X]$-submodule $J \subset K[X]$, we call

$$\mathsf{c}_D(J) = \sum_{f \in J} \mathsf{c}_D(f)$$

the *content* of J. By definition, $\mathsf{c}_D(J) \subset K$ is a D-submodule, and $J \subset \mathsf{c}_D(J)[X]$.

Theorem 6.4.2.

1. *Let $J \lhd D[X]$ be an ideal. Then $JK = JK[X] = fK[X]$ for some $f \in J$, and the following assertions are equivalent:*
   (a) $f \in D^\bullet$.
   (b) $J \cap D^\bullet \neq \emptyset$.
   (c) $JK = K[X]$.

   In particular, if J is almost principal and $f \in J \setminus D$ and $r \in D^\bullet$ are such that $J \subset r^{-1} f D[X]$, then $JK = fK[X] \neq K[X]$, and $J \cap D^\bullet = \emptyset$.
2. *Let q be an ideal system on $D[X]$ such that $q \geq d(D[X])$, $S \subset D[X]^\bullet$ a set of polynomials of bounded degree and $J = S_q \lhd D[X]$. If $JK \neq K[X]$, then J is almost principal.*

3. *If* $f \in D[X]$, *then* $(fK[X] \cap D[X])K = fK[X]$.
4. *If* $\{0\} \neq J \subset D[X]$, *then* J *is a prime ideal such that* $J \cap D^\bullet = \emptyset$ *if and only if* $J = fK[X] \cap D[X]$ *for some irreducible polynomial* $f \in K[X]$.
5. *The following assertions are equivalent*:
   (a) *For every fractional ideal* $F \in \mathcal{F}(D[X])$ *such that* $F \subset K[X]$ *there exists some* $s \in D^\bullet$ *such that* $sF \subset D[X]$.
   (b) *Every fractional ideal* $F \in \mathcal{F}(D[X])$ *is of the form* $F = hB$, *where* $h \in K(X)$ *and* $B \lhd D[X]$ *is an ideal satisfying* $B \cap D^\bullet \neq \emptyset$.
   (c) *For every* $f \in D[X]^\bullet$ *we have* $fK[X] \cap D[X] = r^{-1}fB$, *where* $r \in D^\bullet$ *and* $B \lhd D[X]$ *is an ideal satisfying* $B \cap D^\bullet \neq \emptyset$.
   (d) *Every non-zero ideal* $J \lhd D[X]$ *such that* $JK \neq K[X]$ *is almost principal.*
6. *The equivalent conditions in* 5. *are fulfilled in the following cases*:
   - D *is noetherian or* $D[X]$ *is* q*-noetherian for some ideal system* $q \geq d(D[X])$.
   - *If* $\overline{D}$ *denotes the integral closure of* D, *then there exists some* $c \in D^\bullet$ *such that* $c\overline{D} \subset D$.

PROOF. 1. Clearly, $JK = \{cg \mid c \in K\,,\ g \in J\} = JKD[X] = JK[X] = f'K[X]$ for some $f' \in JK$. If $f' = cf$, where $f \in J$ and $c \in K^\times$, then $JK = f'K[X] = fK[X]$.

(a) $\Rightarrow$ (b) $f \in J \cap D^\bullet$.

(b) $\Rightarrow$ (c) If $c \in J \cap D^\bullet$, then $1 = cc^{-1} \in JK$, and therefore $JK = K[x]$.

(c) $\Rightarrow$ (a) If $JK = K[X] = fK[X]$, then $f \in K[X]^\times \cap J \subset K^\times \cap D[X] = D^\bullet$.

In particular, if $f \in D \setminus J$ and $r \in D^\bullet$ are such that $J \subset r^{-1}fD[X]$, then $fD[X] \subset J \subset r^{-1}fD[X]$ implies $JK = fK[X]$, and by the above we obtain $JK \neq K[X]$ and $J \cap D^\bullet = \emptyset$.

2. Since $JK \neq K[X]$, there exists some polynomial $f \in J \setminus D$ such that $JK = fK[X]$. We set $f = X^t(a_nX^n + \ldots + a_1X + a_0)$, where $t, n \in \mathbb{N}_0$, $t+n = \deg(f) > 0$, $a_0, \ldots, a_n \in D$ and $a_0a_n \neq 0$. Let $m \in \mathbb{N}_0$ be such that $\deg(h) \leq m + \deg(f)$ for all $h \in S$. It suffices to prove that $a_0^{m+1}h \subset fD[X]$ for all $h \in S$. Indeed, if this is done, then it follows that $a_0^{m+1}S \subset fD[X]$ and $J = S_q \subset (a_0^{m+1})^{-1}fD[X]$.

Thus let $h \in S \subset J \subset fK[X]$, say $h = fg$, where $g \in K[X]$. Then $\deg(g) = \deg(h) - \deg(f) \leq m$, and we set $g = b_mX^m + b_{m-1}X^{m-1} + \ldots + b_0$, where $b_0, \ldots, b_m \in K$. Then

$$h = fg = X^t \sum_{i=0}^{n+m} c_iX^i \in D[X]\,, \quad \text{where} \quad c_l = \sum_{i=0}^{l} a_{l-i}b_i \quad \text{for all} \quad l \in [0, m] \quad (\text{with } a_i = 0 \text{ for } i > n)\,.$$

We use induction on l to prove that $a_0^{l+1}b_l \in D$ for all $l \in [0, m]$. Clearly, $a_0b_0 = c_0 \in D$. Thus let $l \in [1, m]$, and suppose that $a_0^{j+1}b_j \in D$ for all $j \in [0, l-1]$. Then

$$a_0^l c_l = a_0^{l+1}b_l + \sum_{i=0}^{l-1} a_{l-i}a_0^{l-1-i}(a_0^{i+1}b_i) \in D\,, \quad \text{and therefore} \quad a_0^{l+1}b_l \in D\,.$$

Hence it follows that $a_0^{m+1}g \in D[X]$ and $a_0^{m+1}h \in fD[X]$.

3. If $f \in D[X]$, then $fK[X] = fD[X]K \subset (fK[X] \cap D[X])K \subset fK[X]$.

4. Suppose that $\{0\} \neq J \subset D[X]$. If $J = fK[X] \cap D[X]$ for some irreducible polynomial $f \in K[X]$, then $J \cap D^\bullet = \emptyset$ by 1., and J is a prime ideal of $D[X]$, since $fK[X]$ is a prime ideal of $K[X]$.

To prove the converse, let J be a prime ideal such that $J \cap D^\bullet = \emptyset$. Then $JK = fK[X]$ for some $f \in J$ by 1., and since $JK = D^{\bullet -1}J$ and $J \cap D^\bullet = \emptyset$, it follows that JK is a prime ideal of $K[X]$, and $J = JK \cap D[X] = fK[X] \cap D[X]$.

5. (a) $\Rightarrow$ (b) Let $F \in \mathcal{F}(D[X])$ be a fractional ideal and $v \in D[X]^\bullet$ such that $C = vF \subset D[X]$. If $C = \{0\}$, then $J = \{0\}$ and the assertion follows with $h = 0$ and $B = D[X]$. If $C \cap D^\bullet \neq \emptyset$, then the assertion follows with $h = v^{-1}$ and $B = C$.

We may now assume that $C \neq \{0\}$ and $C \cap D^\bullet = \emptyset$. Then $CK \subsetneq K[X]$ is a non-zero ideal, and thus $CK = fK[X]$ for some $f \in D[X] \setminus D$. Consequently, $E = f^{-1}C \subset K[X]$ is a fractional ideal, and by (a) there exists some $s \in D^\bullet$ such that $B = sE \lhd D[X]$. Since $fK[X] = CK = fs^{-1}BK = fBK$, we obtain $BK = K[X]$ and therefore $B \cap D^\bullet \neq \emptyset$. As $F = v^{-1}C = v^{-1}fs^{-1}B$, the assertion follows with $h = v^{-1}fs^{-1} \in K(X)$.

(b) $\Rightarrow$ (c) Let $f \in D[X]^\bullet$. By assumption, $fK[X] \cap D[X] = hB'$, where $h \in K(X)$, $B' \lhd D[X]$ and $B' \cap D^\bullet \neq \emptyset$. Hence $B'K = K[X]$, and $fK[X] = (fK[X] \cap D[X])K = hB'K = hK[X]$ (by 3.). Therefore we obtain $h = r^{-1}af$ for some $a, r \in D^\bullet$, and $fK[X] \cap D[X] = r^{-1}afB' = r^{-1}fB$, where $B = aB' \lhd D[X]$, and $B \cap D^\bullet \supset a(B' \cap D^\bullet) \neq \emptyset$.

(c) $\Rightarrow$ (d) Let $\{0\} \neq J \lhd D[X]$ be such that $JK \neq K[X]$. By 1. there exists some $f \in J \setminus D$ such that $JK = fK[X]$. By (c) there exist $r \in D^\bullet$ and $B \lhd D[X]$ such that $fK[X] \cap D[X] \subset r^{-1}fB$, and therefore $J \subset fK[X] \cap D[X] \subset r^{-1}fB \subset r^{-1}fD[X]$.

(d) $\Rightarrow$ (a) Let $F \in \mathcal{F}(D[X])$ be a fractional ideal such that $F \subset K[X]$, and let $f \in D[X]^\bullet$ be such that $J = fF \subset D[X]$. If $f \in D$, we are done. Thus suppose that $f \notin D$. Then $J \subset J' = fK[X] \cap D[X]$, and $J'K = fK[X] \neq K[X]$. By (d) there exists some $f' \in J' \setminus D$ and some $r \in D^\bullet$ such that $J' \subset r^{-1}f'D[X]$, and therefore $f'K[X] = J'K = fK[X]$ by 3. Hence $f' = b^{-1}af$ for some $a, b \in D^\bullet$, and if $s = br \in D^\bullet$, then $sF = brF = brf^{-1}J \subset bf^{-1}rJ' \subset bf^{-1}f'D[X] = aD[X] \subset D[X]$.

6. If D is noetherian, then $D[X]$ is noetherian, and if $D[X]$ is q-noetherian for some ideal system $q \geq d(D[X])$, then (d) follows by 2.

If D is integrally closed, we verify (c). Let $f \in D[X]^\bullet$. If $f \in D$, then $fK[X] \cap D[X] = D[X]$ and (c) holds with $r = f$ and $B = D[X]$. If $f \notin D$, then $fK[X] \cap D[X] = f\mathsf{c}_D(f)^{-1}[X]$ by Theorem 6.1.5. If $0 \neq r \in \mathsf{c}_D(f)$, then (c) holds with $B = r\mathsf{c}_D(f)^{-1}[X]$.

Assume finally that $\overline{D}$ is the integral closure of D and there is some $c \in D^\bullet$ such that $c\overline{D} \subset D$. Then (d) holds for $\overline{D}$, and we verify it for D. Let $J \lhd D[X]$ be a non-zero ideal such that $JK \neq K[X]$. Then $\overline{J} = J\overline{D}[X]$ is a non-zero ideal of $\overline{D}[X]$ and $\overline{J}K = JK[X] \neq K[X]$. Hence there exist some $\overline{f} \in \overline{J} \setminus \overline{D}$ and $\overline{r} \in \overline{D}^\bullet$ such that $\overline{r}\overline{J} \subset \overline{f}\,\overline{D}[X]$. Then $f = c\overline{f} \in J \setminus D$, $r = c^2\overline{r} \in D^\bullet$, and $rJ \subset c^2\overline{r}\overline{J} \subset (c\overline{f})c\overline{D}[X] \subset fD[X]$. $\square$

Theorem 6.4.3.

1. *The assignment* $I \mapsto I[X]$ *defines injective monoid homomorphisms* $j\colon \mathcal{F}(D) \to \mathcal{F}(D[X])$,
$$j_t = j \,|\, \mathcal{F}_t(D)\colon \mathcal{F}_t(D) \to \mathcal{F}_t(D[X])\,, \quad j_v = j \,|\, \mathcal{F}_v(D)\colon \mathcal{F}_v(D) \to \mathcal{F}_v(D[X])\,,$$
and it induces group monomorphisms $j'_v = j_v \,|\, \mathcal{F}_v(D)^\times\colon \mathcal{F}_v(D)^\times \to \mathcal{F}_v(D[X])^\times$,
$$j' = j'_v \,|\, \mathcal{F}(D)^\times\colon \mathcal{F}(D)^\times \to \mathcal{F}(D[X])^\times \quad \text{and} \quad j'_t = j'_v \,|\, \mathcal{F}_t(D)^\times\colon \mathcal{F}_t(D)^\times \to \mathcal{F}_t(D[X])^\times\,.$$
2. *Let* $I \in \mathcal{F}(D)^\bullet$ *be a non-zero fractional ideal.*
   (a) *I is invertible [finitely generated, a principal ideal] if and only if $I[X]$ is invertible [finitely generated, a principal ideal].*
   (b) *$I[X]_v = I_v[X]$, and if $I \in \mathcal{F}_v(D)$, then I is v-invertible [v-finitely generated] if and only if $I[X]$ is v-invertible [v-finitely generated].*
   (c) *$I[X]_t = I_t[X]$, and if $I \in \mathcal{F}_t(D)$, then I is t-invertible [t-finitely generated] if and only if $I[X]$ is t-invertible [t-finitely generated].*

   In particular, j'_v *induces a group monomorphism* $j^*\colon \mathcal{C}_v(D) \to \mathcal{C}_v(D[X])$, *mapping* $\mathrm{Pic}(D)$ *into* $\mathrm{Pic}(D[X])$ *and* $\mathcal{C}(D)$ *into* $\mathcal{C}(D[X])$.

Proof. By Theorem 6.2.2, the assignment $I \mapsto I[X]$ defines an injective monoid homomorphism $j\colon \mathcal{F}(D) \to \mathcal{F}(D[X])$. If $I \in \mathcal{F}(D)$, then $I^{-1}[X] = I[X]^{-1}$, and I is finitely generated [a principal ideal] if and only if $I[X]$ is finitely generated [a principal ideal]. Hence $j' = j \,|\, \mathcal{F}(D)^\times \colon \mathcal{F}(D)^\times \to \mathcal{F}(D[X])^\times$ is a group monomorphism.

If $I \in \mathcal{F}(D)$, then $I[X]_v = (I[X]^{-1})^{-1} = (I^{-1})^{-1}[X] = I_v[X]$. To prove the corresponding result for the t-operation, let $\mathcal{F}(I)$ denote the set of all finitely generated fractional ideals $J \in \mathcal{F}(D)$ such that $J \subset I$. If $J \in \mathcal{F}(I)$, then $J[X] \in \mathcal{F}(D[X])$ is also finitely generated, hence $J_t = J_v$, $J[X]_t = J[X]_v$, and we obtain

$$I_t[X] = \bigcup_{J\in\mathcal{F}(I)} J_t[X] = \bigcup_{J\in\mathcal{F}(I)} J_v[X] = \bigcup_{J\in\mathcal{F}(I)} J[X]_v = \bigcup_{J\in\mathcal{F}(I)} J[X]_t = \Bigl(\bigcup_{J\in\mathcal{F}(I)} J[X]\Bigr)_t = I[X]_t$$

(note that the union is taken over a directed family).

Next we prove that a fractional t-ideal $I \in \mathcal{F}_t(D)$ is t-finitely generated if and only if $I[X]$ is t-finitely generated (note that a fractional v-ideal is v-finitely generated if and only if it is t-finitely generated). If $I \in \mathcal{F}_{t,\mathsf{f}}(D)$, then $I = J_t$ for some $J \in \mathcal{F}(I)$, and therefore $I[X] = J_t[X] = J[X]_t \in \mathcal{F}_{t,\mathsf{f}}(D[X])$. Conversely, assume that $I[X] \in \mathcal{F}_{t,\mathsf{f}}(D[X])$. Then $I[X] = E_t$ for some finite set $E \subset I[X]$. Since

$$I[X] = \bigcup_{J\in\mathcal{F}(I)} J_t[X] \quad \text{(directed union)},$$

there exists some $J \in \mathcal{F}(I)$ such that $E \subset J_t[X]$, which implies $I[X] = E_t = J_t[X]$, and therefore $I = I[X] \cap K = J_t[X] \cap K = J_t \in \mathcal{F}_{t,\mathsf{f}}(D)$.

We have proved that $j(\mathcal{F}_v(D)) \subset \mathcal{F}_v(D[X])$ and $j(\mathcal{F}_t(D)) \subset \mathcal{F}_t(D[X])$, and we assert that the injective maps $j_v = j \,|\, \mathcal{F}_v(D) \colon \mathcal{F}_v(D) \to \mathcal{F}_v(D[X])$ and $j_t = j \,|\, \mathcal{F}_t(D) \colon \mathcal{F}_t(D) \to \mathcal{F}_t(D[X])$ are monoid homomorphisms. Indeed, if $I_1,\, I_2 \in \mathcal{F}_v(D)$, then $(I_1 \cdot_v I_2)[X] = (I_1I_2)_v[X] = (I_1I_2)[X]_v = I_1[X]_v \cdot_v I_2[X]_v$, and the same argument holds for t instead of v. Hence j_v and j_t induce group monomorphisms $j_v' \colon \mathcal{F}_v(D)^\times \to \mathcal{F}_v(D[X])^\times$ and $j_t' \colon \mathcal{F}_t(D)^\times \to \mathcal{F}_t(D[X])^\times$. Since $\mathcal{F}(D)^\times \subset \mathcal{F}_t(D)^\times \subset \mathcal{F}_v(D)^\times$ are subgroups, we obtain $j' = j_v' \,|\, \mathcal{F}(D)^\times$ and $j_t' = j_v' \,|\, \mathcal{F}_t(D)^\times$ by definition. In particular, if $I \in \mathcal{F}(D)^\bullet$ is invertible [I_v is v-invertible, I_t is t-invertible], then $I[X]$ is invertible [$I[X]_v$ is v-invertible, $I[X]_t$ is t-invertible].

If $I \in \mathcal{F}(D)^\bullet$ and $I[X]$ is invertible, then $D[X] = I[X]I[X]^{-1} = I[X]I^{-1}[X] = (II^{-1})[X]$, and therefore $D = (II^{-1})[X] \cap D = II^{-1}$. Hence I is invertible.

If $I_v[X]$ is v-invertible, then $D[X] = (I_v[X]I_v[X]^{-1})_v = (I_vI_v^{-1})[X]_v = (I_vI_v^{-1})_v[X]$, and therefore $D = (I_vI_v^{-1})_v[X] \cap D = (I_vI_v^{-1})_v$. Hence I_v is v-invertible. The same argument holds for t instead of v.

If $I \in \mathcal{F}_v(D)^\times$, then $I[X]$ is principal if and only if $I[X]$ is principal. Hence j_v' induces a group monomorphism $j^* \colon \mathcal{C}_v(D) \to \mathcal{C}_v(D[X])$. For $I \in \mathcal{F}_v(D)^\times$, we denote by $[I] \in \mathcal{C}_v(D)$ the class of I, and for $J \in \mathcal{F}_v(D[X])^\times$ we denote by $[\![J]\!] \in \mathcal{C}_v(D[X])$ the class of J. If $\mathfrak{c} = [I] \in \mathcal{F}_v(D)^\times$, then $j^*(\mathfrak{c}) = [\![I[X]]\!]$. If $\mathfrak{c} \in \mathrm{Pic}(D)$, then $I \in \mathcal{F}(D)^\times$, hence $I[X] \in \mathcal{F}(D[X])^\times$ and $j^*(\mathfrak{c}) = [\![I[X]]\!] \in \mathrm{Pic(D[X])}$. If $\mathfrak{c} \in \mathcal{C}_t(D)$, then $I \in \mathcal{F}_t(D)^\times$, hence $I[X] \in \mathcal{F}_t(D[X])^\times$ and $j^*(\mathfrak{c}) = [\![I[X]]\!] \in \mathcal{C}(D[X])$. □

Theorem 6.4.4. *The following assertions are equivalent*:

(a) *D is integrally closed.*

(b) *If $J \lhd D[X]$ and $J \cap D^\bullet \ne \emptyset$, then $J_v = \mathsf{c}_D(J)_v[X]$.*

(c) *If $J \in \mathcal{I}_v(D[X])$ and $J \cap D^\bullet \ne \emptyset$, then $J \cap D \in \mathcal{I}_v(D)$, and $J = (J \cap D)[X]$.*

(d) *If $J \in \mathcal{I}_t(D[X])$ and $J \cap D^\bullet \ne \emptyset$, then $J \cap D \in \mathcal{I}_t(D)$, and $J = (J \cap D)[X]$.*

(e) *If $J \lhd D[X]$ and $J \cap D^\bullet \ne \emptyset$, then $J_t = \mathsf{c}_D(J)_t[X]$.*

(f) *If $f,\, g \in D[X]^\bullet$ and $a \in D^\bullet$ are such that $\mathsf{c}_D(fg) \subset aD$, then $\mathsf{c}_D(f)\mathsf{c}_D(g) \subset aD$.*

Proof. (a) ⇒ (b) Suppose that $J \lhd D[X]$ and $J \cap D^\bullet \ne \emptyset$. Then $J \subset \mathsf{c}_D(J)[X]$ and therefore $J_v \subset \mathsf{c}_D(J)[X]_v = \mathsf{c}_D(J)_v[X]$. For the proof of the reverse inclusion, observe that J_v is the intersection of all fractional principal ideals containing J. Hence it suffices to prove:

If $h \in K(X)^\bullet$ and $J \subset hD[X]$, then $\mathsf{c}_D(J)[X]_v \subset hD[X]$.

Let $h = g^{-1}b \in K(X)$, where $g,\, b \in D[X]^\bullet$ are coprime in $K[X]$, and suppose that $J \subset hD[X]$. Then it clearly suffices to prove that $\mathsf{c}_D(J)[X] \subset hD[X]$. We have $gJ \subset bD[X]$, and if $c \in J \cap D^\bullet$, then $cg = bq$ for some $q \in D[X]$, and as b and g are coprime in $K[X]$, we obtain $b \in D^\bullet$. For all $q \in J$, we obtain $\mathsf{c}_D(gq) \subset bD$, and therefore, by Theorem 6.1.5, $\mathsf{c}_D(g)\mathsf{c}_D(q) \subset [\,\mathsf{c}_D(g)\mathsf{c}_D(q)\,]_v = \mathsf{c}_D(gq)_v \subset bD$, hence $g\mathsf{c}_D(q) \subset bD[X]$. Consequently, we obtain $g\mathsf{c}_D(J) \subset bD[X]$ and $\mathsf{c}_D(J) \subset g^{-1}bD[X] = hD[X]$.

(b) $\Rightarrow$ (c) If $J \in \mathcal{I}_v(D[X])$ and $J \cap D^\bullet \ne \emptyset$, then $J = \mathsf{c}_D(J)_v[X]$ by (b), and thus it follows that $\mathsf{c}_D(J)_v = J \cap D \in \mathcal{I}_v(D)$.

(c) $\Rightarrow$ (d) Let $J \in \mathcal{I}_t(D[X])$ be such that $J \cap D^\bullet \ne \emptyset$, and denote by $\mathcal{F}(J)$ the set of all finitely generated ideals $B \subset J$ such that $B_v \cap D^\bullet \ne \emptyset$. Then

$$J = J_t = \bigcup_{B \in \mathcal{F}(J)} B_v \quad \text{implies} \quad J \cap D = \bigcup_{B \in \mathcal{F}'(J)} B_v \cap D \quad \text{and} \quad (J \cap D)[X] = \bigcup_{B \in \mathcal{F}'(J)} (B_v \cap D)[X]\,.$$

If $B \in \mathcal{F}(J)$, then $B_v \cap D \in \mathcal{I}_v(D)$, and $B_v = (B_v \cap D)[X]$. Since all unions are directed, it follows that $J \cap D \in \mathcal{I}_t(D)$ and $J = (J \cap D)[X]$.

(d) $\Rightarrow$ (e) Suppose that $J \lhd D[X]$ and $J \cap D^\bullet \ne \emptyset$. By (d) we have $J_t = (J_t \cap D)[X]$, and $\mathsf{c}_D(J_t) = J_t \cap D \in \mathcal{I}_t(D)$. As $\mathsf{c}_D(J) \subset \mathsf{c}_D(J_t)$, it follows that $\mathsf{c}_D(J)_t \subset \mathsf{c}_D(J_t)$, and therefore

$$J_t \subset \mathsf{c}_D(J)[X]_t = \mathsf{c}_D(J)_t[X] \subset \mathsf{c}_D(J_t)[X] = (J_t \cap D)[X] = J_t\,.$$

(e) $\Rightarrow$ (f) Let $f,\, g \in D[X]^\bullet$ and $a \in D^\bullet$ be such that $\mathsf{c}_D(fg) \subset aD$, and set $J = {}_{D[X]}(a, g) \lhd D[X]$. Then $J \cap D^\bullet \ne \emptyset$, and therefore $J_t = \mathsf{c}_D(J)_t[X]$ by (e). Since $fJ = {}_{D[X]}(fa, fg) \subset aD[X]$, we obtain $f\mathsf{c}_D(g)[X] \subset f\mathsf{c}_D(J)_t[X] = fJ_t \subset aD[X]$, and therefore $\mathsf{c}_D(f)\mathsf{c}_D(g) \subset aD$.

(f) $\Rightarrow$ (a) Let $u \in K$ be integral over D and $f \in D[X]$ a monic polynomial such that $f(u) = 0$. Then $f = (X - u)g$ for some monic polynomial $g \in K[X]$. Let $t \in D^\bullet$ be such that $tu \in D$ and $tg \in D[X]$. Then $h = t^2 f = t(X-u)(tg) \in t^2D[X]$, hence $\mathsf{c}_D(h) \subset t^2D$, and therefore $\mathsf{c}_D(t(X-u))\mathsf{c}_D(tg) \subset t^2D$. Since $tu \in \mathsf{c}_D(t(X-u))$ and $t \in \mathsf{c}_D(tg)$, we obtain $t^2u \in t^2D$ and therefore $u \in D$. □

Theorem 6.4.5. *Let D be integrally closed. Then the group monomorphism $j^*\colon \mathcal{C}_v(D) \to \mathcal{C}_v(D[X])$* (see Theorem 6.4.3) *is an isomorphism, $j^*(\mathrm{Pic}(D)) = \mathrm{Pic}(D[X])$ and $j^*(\mathcal{C}(D)) = \mathcal{C}(D[X])$.*

Proof. By Theorem 6.4.3 it suffices to prove that $\mathcal{C}_v(D[X]) \subset j^*(\mathcal{C}_v(D))$, $\mathrm{Pic}(D[X]) \subset j^*(\mathrm{Pic(D)})$ and $\mathcal{C}_t(D[X]) \subset j^*(\mathcal{C}_t(D))$. For $I \in \mathcal{F}_v(D)^\times$ we denote by $[I] \in \mathcal{C}_v(D)$ the class of I, for $J \in \mathcal{F}_v(D[X])^\times$ we denote by $[\![J]\!] \in \mathcal{C}_v(D[X])$ the class of J.

Let $\mathfrak{c} = [\![F]\!] \in \mathcal{C}_v(D[X])$, where $F \in \mathcal{F}_v(D[X])^\times$. By Theorem 6.4.2 it follows that $F = hB$ for some ideal $B \lhd D[X]$ such that $B \cap D^\bullet \ne \emptyset$. Then $B \in \mathcal{C}_v(D[X])^\times$ and $\mathfrak{c} = [\![B]\!]$. By the Theorems 6.4.4 and 6.4.3 it follows that $B \cap D \in \mathcal{F}_v(D)^\times$ and $B = (B \cap D)[X]$. Hence we obtain $[B \cap D] \in \mathcal{C}_v(D)$ and $\mathfrak{c} = j^*([B \cap D])$.

If $\mathfrak{c} \in \mathcal{C}(D[X])$, then $F \in \mathcal{F}_t(D[X])^\times$, $B = (B \cap D)[X] \in \mathcal{F}_t(D[X])^\times$, hence $B \cap D \in \mathcal{F}_t(D)^\times$, $[B \cap D] \in \mathcal{C}(D)$ and $\mathfrak{c} = j^*([B \cap D]) \in j^*(\mathcal{C}(D))$.

If $\mathfrak{c} \in \mathrm{Pic}(D[X])$, then $F \in \mathcal{F}(D[X])^\times$, $B = (B \cap D)[X] \in \mathcal{F}(D[X])^\times$, hence $B \cap D \in \mathcal{F}(D)^\times$, $[B \cap D] \in \mathrm{Pic}(D)$ and $\mathfrak{c} = j^*([B \cap D]) \in j^*(\mathrm{Pic}(D))$. □

Theorem 6.4.6. *Each of the following assertions hold for $R = D$ if and only if it holds for $R = D[X]$.*

1. *R is integrally closed.*
2. *R is completely integrally closed* (*equivalently, every non-zero v-ideal is v-invertible*).
3. *R is a v-domain* (*equivalently, every v-finitely generated non-zero v-ideal is v-invertible*).
4. *R is a Krull domain* (*equivalently, every non-zero t-ideal is t-invertible*).
5. *R is a* PVMD (*equivalently, every t-finitely generated non-zero t-ideal is t-invertible*).

6. *R is factorial (equivalently, R is a Krull domain and $\mathcal{C}(R) = \mathbf{0}$).*
7. *R is a* GCD*-domain (equivalently, R is a* PVMD *and $\mathcal{C}(R) = \mathbf{0}$).*

Proof. **A.** We prove first: If $D[X]$ is integrally closed, then D is integrally closed.

Let $D[X]$ be integrally closed and $x \in K$ integral over D. Then $x \in K(X)$ is integral over $D[X]$, hence $x \in D[X] \cap K = D$. □[**A.**]

B. Let $r \in \{v, t\}$. To prove 2. 3. 4. and 5., it suffices to show the equivalence of the following two assertions:

(a) Every (r-finitely generated) non-zero r-ideal of D is r-invertible.

(b) Every (r-finitely generated) non-zero r-ideal of $D[X]$ is r-invertible.

Proof. If every r-finitely generated r-ideal of D is r-invertible, then r is finitely cancellative, hence D is r-closed and thus integrally closed by Theorem 4.3.2. In the same was, if every r-finitely generated r-ideal of $D[X]$ is r-invertible, then $D[X]$ is integrally closed, and therefore D is integrally closed by **A.**. Hence for the proof of **B** we may assume that D is integrally closed.

(a) $\Rightarrow$ (b) Let $F \subset D[X]$ be an (r-finitely generated) non-zero r-ideal. By Theorem 6.4.2 $F = hB$ for some $h \in K(X)^\times$ and $B \lhd D[X]$ such that $B \cap D^\bullet \neq \emptyset$. Then B is an (r-finitely generated) non-zero r-ideal. By Theorem 6.4.4 $B \cap D$ is an r-ideal, and $B = (B \cap D)[X]$. If B is r-finitely generated, then $B \cap D$ is also r-finitely generated by Theorem 6.4.3. By assumption, $B \cap D$ is r-invertible, hence B is r-invertible by Theorem 6.4.3, and therefore F is r-invertible.

(b) $\Rightarrow$ (a) Let $I \subset D$ be an (r-finitely generated) non-zero r-ideal. By Theorem 6.4.3, $I[X]$ is an (r-finitely generated) non-zero r-ideal and as $I[X]$ is r-invertible by assumption, it follows that I is r-invertible.

C. The assertions 6. and 7. follow by **B** and Theorem 6.4.5.

D. Finally we prove: If D is integrally closed, then $D[X]$ is integrally closed.

Proof. Let D be integrally closed. By Corollary 4.4.5

$$D = \bigcap_{V \in \mathcal{V}} V \quad \text{and therefore} \quad D[X] = \bigcap_{V \in \mathcal{V}} V[X]\,,$$

where $\mathcal{V}$ is the set of all valuation domains V such that $D \subset V \subset K$. Therefore it suffices to prove that $V[X]$ is integrally closed for all $V \in \mathcal{V}$.

If $V \in \mathcal{V}$, then every t-ideal of V is principal, hence V is a PVMD, and by **B**, $V[X]$ is a PVMD. Hence $V[X]$ is integrally closed. □

Theorem 6.4.7. *Let D be a Mori domain, and suppose that either D integrally closed or D contains an uncountable subfield. Then $D[X]$ is a Mori domain.*

Proof. CASE 1: D is integrally closed.

We prove that every $J \in \mathcal{I}_t(D[X])^\bullet$ is t-finitely generated. If $J \in \mathcal{I}_t(D[X])^\bullet$, then Theorem 6.4.2 implies that $J = hB$ for some $h \in K(X)^\times$ and $B \lhd J[X]$ such that $B \cap D^\bullet \neq \emptyset$. By Theorem 6.4.4 we obtain $B \cap D \in \mathcal{I}_t(D)$ and $B = (B \cap D)[X] \in \mathcal{I}_{t,\mathfrak{f}}(D)$, since D is a Mori domain. By Theorem 6.4.3 it follows that B and therefore also J is t-finitely generated.

CASE 2: D contains an uncountable field Δ.

Assume to the contrary that $D[X]$ is not t-noetherian. Then there exists a sequence $(g_n)_{n \geq 0}$ in $D[X]$ such that $\{g_0, \ldots, a_{n-1}\}_v \subsetneq \{g_0, \ldots, a_n\}_v$ for all $n \geq 1$, and therefore

$$(D[X]\colon\{g_0, \ldots, a_n\}) \subsetneq (D[X]\colon\{g_0, \ldots, a_{n-1}\})\,.$$

For $n \in \mathbb{N}$, let $h_n \in K(X)$ be such that $h_n g_i \in D[X]$ for all $i \in [0, n-1]$ and $h_n g_n \notin D[X]$. Since $K[X]$ is noetherian, there exists some $m \in \mathbb{N}$ such that, for all $n \geq m$,

$$(K[X]\colon\{g_0, \ldots, a_n\}) = (K[X]\colon\{g_0, \ldots, a_{n-1}\})\,.$$

For $n \geq m$ we have $h_n \in (K[X]\colon\{g_0, \ldots, a_{n-1}\})$, hence $h_n g_n \in K[X] \setminus D[X]$, and by the subsequent Lemma 6.4.8 the set $C_n = \{c \in \Delta \mid (h_n g_n)(c) \in D\}$ is finite. Hence there exists some $c \in \Delta$ such that for all $n > m$ we have $h_n(c) g_n(c) \notin D$, and $h_n(c) g_i(c) \in D$ for all $i \in [m, n-1]$. Consequently,

$$(D\colon\{g_0(c), \ldots, g_n(c)\}) \subsetneq (D\colon\{g_0(c), \ldots, g_{n-1}(c)\})$$

and $\big(\{g_0(c), \ldots, g_n(c)\}_v\big)_{n \geq m}$ is a properly ascending sequence of v-ideals of D, a contradiction. □

Lemma 6.4.8. *Let D be a domain, $K = \mathsf{q}(D)$, $\Delta \subset D$ a subfield, $g \in K[X]$ a polynomial such that $\deg(g) = d \in \mathbb{N}$. If $c_0, \ldots, c_d \in \Delta$ are distinct such that $g(c_i) \in D$ for all $i \in [0, d]$, then $g \in D[X]$.*

Proof. If $g = a_0 + a_1 X + \ldots + a_d X^d$, then $(a_0, \ldots, a_d)$ is a solution of the system of equations

$$\begin{pmatrix} 1 & c_0 & c_0^2 & \ldots & c_0^d \\ 1 & c_1 & c_1^2 & \ldots & c_1^d \\ . & . & . & \ldots & . \\ 1 & c_d & c_d^2 & \ldots & c_d^d \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ . \\ a_d \end{pmatrix} = \begin{pmatrix} g(c_0) \\ g(c_1) \\ . \\ g(c_d) \end{pmatrix} \in D^{d+1}$$

with a determinant in $\Delta^\times \subset D^\times$. Hence $a_0, \ldots, a_d \in D$. □