Idealtheorie kommutativer Ringe und Monoide

Franz Halter-Koch
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CHAPTER 1

Generalities on Monoids

For a set $X$, we denote by $\mathcal{P}(X)$ the power set and by $\mathcal{P}_f(X)$ the set of all finite subsets of $X$. If $A$, $B$ are sets, then $A \subset B$ or $B \supset A$ means that $A$ is a subset of $B$ which may be equal to $B$. If $A$ is a proper subset of $B$, we write $A \varsubsetneq B$ or $B \varsupsetneq A$.

As usual, we denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of integers, rational numbers, real numbers and complex numbers. We denote by $\mathbb{N} = \{x \in \mathbb{Z} \mid x > 0\}$ the set of positive integers, and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $x, y \in \mathbb{Z}$ and $x \leq y$, we set $[x, y] = \{z \in \mathbb{Z} \mid x \leq z \leq y\}$. For a set $X$, we denote by $|X| \in \mathbb{N}_0 \cup \{\infty\}$ its cardinality.

Let $X$ be a set. A subset $\Sigma \subset \mathcal{P}(X)$ is called
• directed if, for any $A, B \in \Sigma$, there is some $C \in \Sigma$ such that $A \cup B \subset C$;
• a chain if, for any $A, B \in \Sigma$, we have $A \subset B$ or $B \subset A$.

A family $(A_\lambda)_{\lambda \in \Lambda}$ of subsets of $X$ is called directed or a chain if the set $\{A_\lambda \mid \lambda \in \Lambda\}$ has this property. If $(A_\lambda)_{\lambda \in \Lambda}$ is directed and $E$ is a finite set, then
$$E \subset \bigcup_{\lambda \in \Lambda} A_\lambda \quad \text{implies} \quad E \subset A_\lambda \quad \text{for some } \lambda \in \Lambda.$$

We shall frequently use Zorn’s Lemma in the following form:

Let $X$ be a set, $\emptyset \neq \Sigma \subset \mathcal{P}(X)$, and suppose that the union of every chain in $\Sigma$ belongs to $\Sigma$. Then $\Sigma$ contains maximal elements (with respect to the inclusion).

A partial ordering on a set $X$ is a binary relation $\leq$ such that the following assertions hold for all $x, y \in X$:
• $x \leq x$;
• $x \leq y$ and $y \leq x$ implies $x = y$.
• $x \leq y$ and $y \leq z$ implies $x \leq z$.

If $\leq$ is a partial ordering on $X$, we call $(X, \leq)$ a partially ordered set. We call $\leq$ a total ordering and $(X, \leq)$ a totally ordered set if, for all $x, y \in X$ we have either $x \leq y$ or $y \leq x$.

Let $(X, \leq)$ be a partially ordered set. Then every subset of $X$ is again a partially ordered set with the induced order. A totally ordered subset of $X$ is called a chain. Sometimes we will use the abstract form of Zorn’s Lemma as follows:

Let $(X, \leq)$ be a non-empty partially ordered set, and assume that every non-empty chain in $X$ has an upper bound. Then $X$ contains maximal elements.

For a partially ordered set $(X, \leq)$, the following assertions are equivalent:
• For every sequence $(a_n)_{n \geq 0}$ in $X$ satisfying $a_n \leq a_{n+1}$ for all $n \geq 0$, there exists some $m \geq 0$ such that $a_n = a_m$ for all $n \geq m$ [in other words, every ascending sequence in $X$ becomes ultimately stationary].
• Every non-empty subset of $X$ contains a maximal element.
If these conditions are fulfilled, then \((X, \leq)\) is said to be **noetherian** or to satisfy the ACC (the *ascending chain condition*).

### 1.1. Preliminaries on Monoids

Let \(K\) be a multiplicative semigroup. An element \(n \in K\) is called a **zero element** if \(na = n\) for all \(a \in K\). An element \(e \in K\) is called a **unit element** if \(ea = a\) for all \(a \in K\). Plainly, \(K\) possesses at most one zero element, denoted by \(0 = 0_K\) and at most one unit element, denoted by \(1 = 1_K\). For subsets \(X, Y \subset K\) and \(a \in K\), we define \(XY = \{xy \mid x \in X, y \in Y\}\) and \(aX = \{a\}X\). For \(n \in \mathbb{N}\), we define \(X^n\) recursively by \(X^1 = 1\) and \(X^{n+1} = X^nX\), and we set \(X^{(n)} = \{x^n \mid x \in X\}\).

By a **monoid** we mean a multiplicative semigroup \(K\) containing a zero element \(0 = 0_K\) and a unit element \(1 = 1_K\). Clearly, \(0_K = 1_K\) if and only if \(|K| = 1\), and in this case \(K\) is called a **trivial monoid**.

A **monoid without zero** is a multiplicative semigroup \(K\) which is either trivial or does not contain a zero element. Thus the trivial monoid is both a monoid and a monoid without zero. A subset \(S \subset K\) is called **multiplicatively closed** if \(1 \in S\) and \(SS \subset S\).

Let \(K\) be a monoid. An element \(a \in K\) is called **cancellative** if \(ab = ac\) implies \(b = c\) for all \(b, c \in K\). For a subset \(X \subset K\), we set \(X^* = X \setminus \{0\}\), and we denote by \(X^\ast\) the set of all cancellative elements of \(X\). If \(K\) is non-trivial, then \(K^\ast \subset K^\ast\). \(K\) is called **cancellative** if \(K^\ast \subset K^\ast\). Hence \(K\) is cancellative if and only if either \(K\) is trivial or \(K^\ast = K^\ast\).

An element \(u \in K\) is called **invertible** if there exists some \(u' \in K\) such that \(uu' = 1\). In this case, \(u'\) is uniquely determined by \(u\), it is called the **inverse** of \(u\) and denoted by \(u^{-1}\). We denote by \(K^\times\) the set of all invertible elements of \(K\). Endowed with the induced multiplication, \(K^\times\) is a group, and \(K^\times \subset K^\ast\).

The monoid \(K\) is called
- **reduced** if \(K^\times = \{1\}\);
- **divisible** if \(K^\ast \subset K^\times\).

By definition, \(K\) is divisible if and only if either \(K\) is trivial or \(K^\ast = K^\times\). If \(K\) is divisible, then \(K\) is cancellative.

The most important example of a monoid is the multiplicative monoid \(D = (D, \cdot)\) of a ring \(D\) (throughout this volume, rings are assumed to be commutative and unitary, and modules and ring homomorphisms are assumed to be unitary). Note that \(D\) is a trivial monoid if and only if \(D\) is a zero ring, and \(D \setminus D^\ast\) is the set of zero divisors of \(D\). If \(D\) is non-trivial, then \(D\) is cancellative if and only if \(D\) is a domain, and \(D\) is divisible if and only if \(D\) is a field.

Let \(D\) be a monoid. A subset \(Q \subset D\) is called
- **multiplicatively closed** if \(1 \in Q\) and \(QQ \subset Q\) (then \(QQ = Q\));
- a **submonoid** if it is multiplicatively closed and \(0 \in Q\);
- a *(semigroup)* ideal of \(D\) if \(0 \in Q\) and \(DQ \subset Q\) (then \(DQ = Q\));
- a **principal ideal** of \(D\) if \(Q = Da\) for some \(a \in D\).
- a **prime ideal** of \(D\) if \(Q\) is an ideal and \(D \setminus Q\) is multiplicatively closed.

By definition, \(\{0, 1\}\) is the smallest submonoid of \(D\), \(\{0\} = D0\) and \(D = D1\) are principal ideals of \(D\), and \(D \setminus D^\times\) is a prime ideal of \(D\).

If \(D\) is cancellative (reduced), then every submonoid of \(D\) is also cancellative (reduced).

For \(a, b \in D\) we define \(a \mid b\) if \(bD \subset aD\). If \(b = au\) for some \(u \in D^\times\), then \(aD = bD\). Conversely, if \(D\) is cancellative and \(aD = bD\), then \(b = au\) for some \(u \in D^\times\).

**Lemma 1.1.1.** Let \(D\) be a monoid.

1. If \(J \subset D\) is an ideal, then \(J = D\) if and only if \(J \cap D^\times \neq \emptyset\).
2. $D$ is divisible if and only if $\{0\}$ and $D$ are the only ideals of $D$.

3. If $D$ is cancellative and not trivial, then $D^*$ is a multiplicatively closed subset and $\{0\}$ is a prime ideal of $D$.

4. Let $(Q_\lambda)_{\lambda \in \Lambda}$ be a family of subsets of $D$,

$$Q^* = \bigcup_{\lambda \in \Lambda} Q_\lambda \text{ and } Q_* = \bigcap_{\lambda \in \Lambda} Q_\lambda.$$

(a) If $(Q_\lambda)_{\lambda \in \Lambda}$ is a family of ideals of $D$, then $Q^*$ and $Q_*$ are ideals of $D$.

(b) If $(Q_\lambda)_{\lambda \in \Lambda}$ is a family of prime ideals of $D$, then $Q^*$ is a prime ideal of $D$, and if $(Q_\lambda)_{\lambda \in \Lambda}$ is a chain, then $Q_*$ is also a prime ideal of $D$.

(c) If $(Q_\lambda)_{\lambda \in \Lambda}$ is a family of submonoids of $D$, then $Q_*$ is a submonoid of $D$, and if $(Q_\lambda)_{\lambda \in \Lambda}$ is directed, then $Q^*$ is also a submonoid of $D$.

**Proof.** 1. Let $J \subset D$ be an ideal. If $J = D$, then $J \cap D^* = D^* \neq \emptyset$. If $u \in J \cap D^*$ and $a \in D$, then $a = (au^{-1})u \in J$ and therefore $J = D$.

2. Let $D$ be divisible and $J \subset D$ an ideal of $D$. If $a \in J^*$, then $1 = a^{-1}a \in J$ and therefore $J = D$.

3. If $D$ is cancellative and not trivial, then $D^* = D^*$ is multiplicatively closed, and therefore $\{0\}$ is a prime ideal of $D$.

4. (a) If $a \in D$ and $x \in Q^*$, then $x \in Q_\lambda$ for some $\lambda \in \Lambda$ and therefore $ax \in Q_\lambda \subset Q^*$. If $a \in D$ and $x \in Q_\lambda$, then $x \in Q_\lambda$ and thus $ax \in Q_\lambda$ for all $\lambda \in \Lambda$, and therefore $ax \in Q_*$.

(b) If $a, b \in D \setminus Q^*$, then $a, b \in D \setminus Q_\lambda$ and therefore $ab \in D \setminus Q_\lambda$ for all $\lambda \in \Lambda$. Hence it follows that $ab \in D \setminus Q^*$, and therefore $Q^*$ is a prime ideal of $D$.

Let now $(Q_\lambda)_{\lambda \in \Lambda}$ be a chain and $a, b \in D \setminus Q_*$. Then there exist $\lambda, \mu \in \Lambda$ such that $a \notin Q_\lambda$ and $b \notin Q_\mu$, and we may assume that $Q_\lambda \subset Q_\mu$. Then it follows that $a, b \notin Q_\lambda$, hence $ab \notin Q_\lambda$ and therefore $ab \notin Q_*$. Hence $Q_*$ is a prime ideal of $D$.

(c) Let $(Q_\lambda)_{\lambda \in \Lambda}$ be a family of submonoids of $D$. Then $0 \in Q_* \subset Q^*$. If $a, b \in Q_\lambda$, then $a, b \in Q_\lambda$ and therefore $ab \in Q_\lambda$, and we define a submonoid of $D$.

Let now $(Q_\lambda)_{\lambda \in \Lambda}$ be directed and $a, b \in Q^*$. Then there exists some $\lambda \in \Lambda$ such that $a, b \in Q_\lambda$. Hence $ab \in Q_\lambda \subset Q^*$, and therefore $Q^*$ is a submonoid of $D$.

Let $K$ and $L$ be monoids. A map $f: K \to L$ is called a (monoid) homomorphism if

$$f(1_K) = 1_L, \quad f(0_K) = 0_L, \quad \text{and } f(xy) = f(x)f(y) \text{ for all } x, y \in K.$$

As usual, a homomorphism is called a monomorphism [an epimorphism, an isomorphism] if it is injective [surjective, bijective]. The monoids $K$ and $L$ are called isomorphic if there exists an isomorphism $f: K \to L$, and in this case we write $f: K \simeq L$.

Let $f: K \to L$ be a monoid homomorphism. Then $f(K^\times) \subset L^\times$, and $f^* : K^\times : K^\times \to L^\times$ is a group homomorphism. If $J \subset L$ is an ideal, then $f^{-1}(J) \subset K$ is also an ideal [indeed, if $x \in f^{-1}(J)$ and $a \in K$, then $f(ax) = f(a)f(x) \in LJ = J$ and therefore $ax \in f^{-1}(J)$].

Let $K$ be a monoid and $G \subset K^\times$ a subgroup. Then we set $K/G = \{aG \mid a \in K\}$, and we define a multiplication on $K/G$ by means of $(aG)(bG) = abG$ for all $a, b \in K$. This definition does not depend on the representatives, it makes $K/G$ into a monoid, and $\pi: K \to K/G$, defined by $\pi(a) = aG$ for all $a \in K$, is a monoid epimorphism, called canonical. By definition, $(K/G)^\times = \{G \mid a \in K^\times\}$, $(K/G)^* = \{aG \mid a \in K^\times\}$, and $(K/G)^* = K^\times/G$ (the factor group). Consequently, $K/G$ is cancellative [divisible] if and only if $K$ is cancellative [divisible].

If $G \subset K^\times$ is a subgroup, then the canonical epimorphism $\pi: K \to K/G$ is an isomorphism if and only if $G = \{1\}$, and in this case we identify $K$ with $K/\{1\}$ by means of $\pi$ and set $K = K/\{1\}$. The monoid $K/K^\times$ is reduced. It is called the associated reduced monoid of $K$. 
1. GENERALITIES ON MONOIDS

Let \( f : K \to L \) be a monoid homomorphism, and let \( G \subseteq K^\times \) and \( H \subseteq L^\times \) be subgroups such that \( f(G) \subseteq H \). Then there is a unique homomorphism \( f^* : K/G \to L/H \) such that \( f^*(aG) = f(a)H \) for all \( a \in K \). We say that \( f^* \) is \textit{induced by} \( f \).

Let \( K \) and \( L \) be divisible monoids. A map \( f : K \to L \) is a monoid homomorphism if and only if \( f(0_K) = 0_L \), and \( f / K^\times : K^\times \to L^\times \) is a group homomorphism. In this case, \( f^{-1}(1) = \text{Ker}(f / K^\times) \) is a subgroup of \( K^\times \), and \( f \) induces a monomorphism \( f^* : K / f^{-1}(1) \to L \).

Let \( K \) be a monoid. For subsets \( X, Y \subseteq K \) and \( y \in K \), we define
\[
(X : Y) = (X :KY) = \{ z \in K \mid zY \subseteq X \} \quad \text{and} \quad (X : y) = (X : \{ y \}).
\]

\textbf{Lemma 1.1.2.} Let \( K \) be a monoid and \( X, X', Y, Y' \subseteq K \).

1. If \( X \subseteq X' \) and \( Y \subseteq Y' \), then \( (X : Y') \subseteq (X' : Y) \).
2. \( (X : YY') = ((X : Y) : Y') \).
3. \( (X : X) \) is a submonoid of \( K \).
4. If \( a \in K^\times \), then \( (aX : Y) = a(X : Y) \) and \( (X : aY) = a^{-1}(X : Y) \).
5. If \( (Y_\lambda)_{\lambda \in \Lambda} \) is a family of subsets of \( K \), then
\[
\left( X : \bigcup_{\lambda \in \Lambda} Y_\lambda \right) = \bigcap_{\lambda \in \Lambda} (X : Y_\lambda), \quad \text{and if} \quad Y \subseteq K^\times, \quad \text{then} \quad (X : Y) = \bigcap_{y \in Y} y^{-1}X.
\]

\textbf{Proof.} 1. If \( z \in (X : Y') \), then \( zY \subseteq zY' \subseteq X \subseteq X' \), and therefore \( z \in (X' : Y) \).

2. If \( z \in K \), then
\[
z \in (X : Y') \iff zY' = (zY')Y \subseteq X \iff zY' \subseteq (X : Y) \iff z \in ((X : Y) : Y').
\]

3. Clearly, \( 0 \in (X : X) \), and if \( x, y \in (X : X) \), then \( xyX = x(yX) \subseteq xX \subseteq X \), and therefore \( xy \in (X : Y) \).

4. Let \( a \in K^\times \) and \( z \in K \). Then
\[
z \in (aX : Y) \iff zY \subseteq aX \iff a^{-1}zY \subseteq X \iff a^{-1}z \in (X : Y) \iff z \in a(X : Y)
\]
and
\[
z \in (X : aY) \iff zaY \subseteq X \iff za \in (X : Y) \iff z \in a^{-1}(X : Y).
\]

5. Let \( (Y_\lambda)_{\lambda \in \Lambda} \) be a family of subsets of \( K \) and \( z \in K \). Then
\[
z \in \left( \bigcup_{\lambda \in \Lambda} Y_\lambda \right) \iff zY_\lambda \subseteq X \quad \text{for all} \quad \lambda \in \Lambda \iff z \in (X : Y_\lambda) \quad \text{for all} \quad \lambda \in \Lambda \iff z \in \bigcap_{\lambda \in \Lambda} (X : Y_\lambda).
\]

If \( Y \subseteq K^\times \), then
\[
(X : Y) = \left( X : \bigcup_{y \in Y} \{ y \} \right) = \bigcap_{y \in Y} (X : \{ y \}) = \bigcap_{y \in Y} y^{-1}(X : \{ 1 \}) = \bigcap_{y \in Y} y^{-1}X.
\]
1.2. Quotient Monoids

Remarks and Definition 1.2.1. Let $K$ be a monoid and $T \subset K$ a multiplicatively closed subset.
For $(x, t), (x', t') \in K \times T$, we define

$$(x, t) \sim (x', t') \quad \text{if} \quad st'x = stx' \quad \text{for some} \quad s \in T.$$ 

Then $\sim$ is an equivalence relation on $K \times T$.

Proof. Obviously, $\sim$ is reflexive and symmetric. To prove transitivity, let $(x, t), (x', t'), (x'', t'') \in K \times T$ be such that $(x, t) \sim (x', t')$ and $(x', t') \sim (x'', t'')$. Then there exist $s, s', t \in T$ such that $st'x = stx'$ and $s't''x' = s't'x''$. Then it follows that $s's't'x = s't''x' = s't''x''$, hence $(x, t) \sim (x'', t'')$. □

We define the quotient monoid $T^{-1}K$ of $K$ with respect to $T$ by $T^{-1}K = K \times T / \sim$. For $(x, t) \in K \times T$, we denote by

$$\frac{x}{t} \in T^{-1}X \quad \text{the equivalence class of} \quad (x, t), \quad \text{and we define} \quad j_T : K \to T^{-1}K \quad \text{by} \quad j_T(x) = \frac{x}{1}.$$ 

The map $j_T$ is called the natural embedding (although it need not be injective). By definition, if $(x, t), (x', t') \in K \times T$, then

$$\frac{x}{t} = \frac{x'}{t'} \quad \text{if and only if} \quad st'x = stx' \quad \text{for some} \quad s \in T,$$

and if $T \subset K^*$, then

$$\frac{x}{t} = \frac{x'}{t'} \quad \text{if and only if} \quad t'x = tx'.$$

If $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in T^{-1}K$, then $z_1, \ldots, z_n$ have a common denominator, that is, there exist $x_1, \ldots, x_n \in K$ and $t \in T$ such that

$$z_i = \frac{x_i}{t} \quad \text{for all} \quad i \in [1, n].$$

For $x, x' \in K$ and $t, t' \in T$, we define

$$\frac{x}{t} \cdot \frac{x'}{t'} = \frac{xx'}{tt'}.$$

This definition does not depend on the choice of the representatives. Endowed with this multiplication, $T^{-1}K$ becomes a monoid with unit element $\frac{1}{1}$ and zero element $\frac{0}{1}$, and $j_T$ is a monoid homomorphism. If $0 \in T$, then $T^{-1}K$ is a trivial monoid.

Proof. Suppose that $(x, t), (x_1, t_1), (x', t'), (x''_1, t''_1) \in K \times T$, $(x, t) \sim (x_1, t_1)$ and $(x', t') \sim (x''_1, t''_1)$. We must prove that $(xx', tt') \sim (x_1x''_1, t_1t''_1)$. Let $s, s' \in T$ be such that $st_1x = stx_1$ and $st''_1x' = st't'x_1$. Then it follows that $s's't_1x = s't't_1x$, which implies $(xx', tt') \sim (x_1x''_1, t_1t''_1)$. Now it is obvious that this multiplication is associative and commutative, $\frac{1}{1}$ is a unit element and $\frac{0}{1}$ is a zero element. If $x, y \in K$, then

$$j_T(xy) = \frac{xy}{1} = \frac{x}{1} \cdot \frac{y}{1} = j_T(x)j_T(y), \quad j_T(0) = \frac{0}{1} \quad \text{and} \quad j_T(1) = \frac{1}{1}.$$ 

Hence $j_T$ is a monoid homomorphism. If $0 \in T$, then $(x, t) \sim (x', t')$ for all $(x, t), (x', t') \in K \times T$, and therefore $|T^{-1}K| = 1$. □

For every subset $X \subset K$, we set

$$T^{-1}X = \left\{ \frac{x}{t} \ \bigg| \ x \in X, \ t \in T \right\} \subset T^{-1}K.$$ 

If $X' \subset X \subset K$, then $T^{-1}X' \subset T^{-1}X \subset T^{-1}K$. Hence it follows that $T^{-1}(X \cap Y) \subset T^{-1}X \cap T^{-1}Y$ for any subsets $X, Y \subset K$. 

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**Theorem 1.2.2.** Let $K$ be a monoid, $T \subset K$ a multiplicatively closed subset and $j_T : K \to T^{-1}K$ the natural embedding.

1. If $X, Y \subset K$, then $T^{-1}(XY) = (T^{-1}X)(T^{-1}Y)$, and if additionally $TX = X$ and $TY = Y$, then $T^{-1}(X \cap Y) = T^{-1}X \cap T^{-1}Y$.

2. If $(X_\lambda)_{\lambda \in \Lambda}$ is a family of subsets of $K$, then
$$
T^{-1}\left(\bigcup_{\lambda \in \Lambda} X_\lambda\right) = \bigcup_{\lambda \in \Lambda} T^{-1}X_\lambda.
$$

3. If $J$ is an ideal of $K$, then $T^{-1}J$ is an ideal of $T^{-1}K$, $J \subset j_T^{-1}(T^{-1}J)$, and $T^{-1}J = T^{-1}K$ if and only if $J \cap T \neq \emptyset$.

4. If $V$ is an ideal of $T^{-1}J$, then $J = j_T^{-1}(V)$ is an ideal of $K$, and $V = T^{-1}J$.

**Proof.** Let $X, Y \subset K$. If $z \in T^{-1}(XY)$, then $z = \frac{x}{t}$ for some $x \in X$, $y \in Y$ and $t \in T$, and therefore $z = \frac{x}{t} = \frac{\frac{y}{s}}{\frac{t}{s}} \in (T^{-1}X)(T^{-1}Y)$. Conversely, if $z \in (T^{-1}X)(T^{-1}Y)$, then $z = \frac{\frac{y}{s}}{\frac{t}{s}} = \frac{y}{t}$ for some $x \in X$, $y \in Y$ and $s, t \in T$. Hence $z = \frac{yw}{st} \in T^{-1}(XY)$.

Assume now that $TX = X$ and $TY = Y$. Clearly, $T^{-1}(X \cap Y) \subset T^{-1}X \cap T^{-1}Y$. If $z \in T^{-1}X \cap T^{-1}Y$, then $z = \frac{x}{t} = \frac{y}{s}$, where $x \in X$, $y \in Y$ and $s, t \in T$. Then there is some $w \in T$ such that $wst = xty$. Since $wsx = wty \in TX \cap TY = X \cap Y$ it follows that $z = \frac{wsx}{wst} \in T^{-1}X \cap T^{-1}Y$.

2. If $\alpha \in \Lambda$, then
$$
X_\alpha \subset \bigcup_{\lambda \in \Lambda} X_\lambda \quad \text{implies} \quad T^{-1}X_\alpha \subset T^{-1}\left(\bigcup_{\lambda \in \Lambda} X_\lambda\right), \quad \text{and therefore} \quad \bigcup_{\lambda \in \Lambda} T^{-1}X_\lambda \subset T^{-1}\left(\bigcup_{\lambda \in \Lambda} X_\lambda\right).
$$

Conversely, if $z \in T^{-1}\left(\bigcup_{\lambda \in \Lambda} X_\lambda\right)$, then $z = \frac{x}{t}$, where $t \in T$ and $x \in X_\alpha$ for some $\alpha \in \Lambda$ and therefore $z \in T^{-1}X_\alpha \subset \bigcup_{\lambda \in \Lambda} T^{-1}X_\lambda$.

3. Obviously, $T^{-1}J$ is an ideal of $T^{-1}K$, and $J \subset j_T^{-1}(T^{-1}J)$.

Hence $\frac{1}{t} = \frac{a}{t}$ for some $a \in J$ and $t \in T$, and there exists some $s \in T$ such that $st = sa \in T \cap J$.

Conversely, if $s \in T \cap J$, then $\frac{1}{t} = \frac{s}{t} \in T^{-1}J$, which implies $T^{-1}J = T^{-1}K$.

4. Since $j_T$ is a monoid homomorphism, it follows that $J = j_T^{-1}(V)$ is an ideal of $K$. If $a \in J$ and $t \in T$, then $\frac{a}{t} \in V$ and therefore $\frac{a}{t} = \frac{1}{t} \frac{a}{t} \in V$. Hence $T^{-1}J \subset V$. To prove the converse, let $\frac{a}{t} \in V$, where $a \in K$ and $t \in T$. Then $\frac{a}{t} = \frac{a}{t} \frac{t}{t} \in V$, hence $a \in J$ and $\frac{a}{t} \in T^{-1}J$. $\Box$

**Theorem 1.2.3.** Let $K$ and $L$ be monoids, $T \subset K$ a multiplicatively closed subset and $\varphi : K \to L$ be a homomorphism such that $\varphi(T) \subset L^\times$. Then there exists a unique homomorphism $\Phi : T^{-1}K \to L$ such that $\Phi \circ j_T = \varphi$. It is given by
$$
\Phi\left(\frac{a}{t}\right) = \varphi(t)^{-1}\varphi(a) \quad \text{for all} \quad a \in K \quad \text{and} \quad t \in T.
$$

**Proof.** Let $\Phi : T^{-1}K \to L$ be a homomorphism satisfying $\Phi \circ j_T = \varphi$. For $a \in K$ and $t \in T$ we have $\varphi(t) \in L^\times$,
$$
\varphi(t)\Phi\left(\frac{a}{t}\right) = \Phi\left(\frac{t}{1}\right)\Phi\left(\frac{a}{t}\right) = \Phi\left(\frac{at}{t}\right) = \Phi\left(\frac{a}{1}\right) = \varphi(a), \quad \text{and therefore} \quad \Phi\left(\frac{a}{t}\right) = \varphi(t)^{-1}\varphi(a).
$$
This proves uniqueness and the formula for $\Phi$. To prove existence we define $\Phi$ by the formula above and prove that this definition does not depend on the choice of representatives.

If $a, a' \in K$ and $t, t' \in T$ are such that $\frac{a}{t} = \frac{a'}{t'}$, then there is some $s \in T$ such that $st'a = sta'$, hence $\varphi(s)\varphi(t')\varphi(a) = \varphi(s)\varphi(t)\varphi(a')$ and therefore $\varphi(t)^{-1}\varphi(a) = \varphi(t')^{-1}\varphi(a')$.

By the very definition, $\Phi_J(T) = \varphi$, $\Phi(J) = \varphi(1)^{-1}\varphi(0) = 1$ and $\Phi(\frac{1}{t}) = \varphi(1)^{-1}\varphi(1) = 1$. If $a, a' \in K$ and $t, t' \in T$, then

$$\Phi\left(\frac{a}{t} \frac{a'}{t'}\right) = \Phi\left(\frac{aa'}{tt'}\right) = \varphi(tt')^{-1}\varphi(aa') = \varphi(t)^{-1}\varphi(a)\varphi(t')^{-1}\varphi(a') = \Phi\left(\frac{a}{t}\right) \Phi\left(\frac{a'}{t'}\right).$$

Hence $\Phi$ is a homomorphism. \qed

**Theorem and Definition 1.2.4.** Let $K$ be a monoid, $T \subset K$ a multiplicatively closed subset, and $\overline{T} = \{s \in K \mid sK \cap T \neq \emptyset\}$. $\overline{T}$ is called the divisor-closure of $T$, and $T$ is called divisor-closed if $T = \overline{T}$.

1. Let $J_T$ be the set of all ideals $J \subset K$ such that $J \cap T = \emptyset$ and

   $$P = \bigcup_{J \in J_T} J.$$

   Then $T = K \setminus P$ is multiplicatively closed, $T \subset \overline{T} = \overline{T}$, and if $T \neq K$, then $P$ is a prime ideal, and it is the greatest ideal of $K$ such that $P \cap T = \emptyset$.

2. $(T^{-1}K)\times = T^{-1}T$, and there is an isomorphism

   $$\iota: T^{-1}K \cong \overline{T}^{-1}K,$$

   given by $\iota\left(\frac{x}{t}\right) = \frac{x}{t}$ for all $x \in K$ and $t \in T$.

   Note that $\iota$ is not the identity map, since the two fractions appearing in its description denote different equivalence classes. However, we shall identify them: $T^{-1}K = \overline{T}^{-1}K$.

3. Let $S \subset T$ be a multiplicatively closed subset. Then $ST \subset K$ and $T^{-1}S \subset T^{-1}K$ are multiplicatively closed subsets, and there is an isomorphism

   $$\Phi: (T^{-1}S)^{-1}(T^{-1}K) \cong (ST)^{-1}K,$$

   for all $K$, $t, t' \in T$ and $s \in S$.

4. If $X, Y \subset K$, then $(T^{-1}X : T^{-1}K)^{-1} = (T^{-1}X : T^{-1}Y) = (T^{-1}X : j_T(Y)) \cap T^{-1}(X : Y)$, and equality holds if $TX = X$ and $Y$ is finite.

**Proof.** 1. Suppose that $s \in S$, and let $J \in J_T$. If $a \in K$ is such that $sa \in S$, then $sa \notin J$ and thus $s \notin J$. Hence $\overline{T} \supset K \setminus P$. Conversely, if $s \in K \setminus P$, then $sK \notin J_T$, hence $sK \cap T \neq \emptyset$ and $s \in \overline{T}$.

   Clearly $T \subset \overline{T} \subset T$. If $s \in \overline{T}$, then $ts \in T$ for some $t \in K$, hence $t's \in T$ for some $t' \in K$, and therefore $s \in T$. Hence $T = \overline{T}$. If $s_1, s_2 \in T$, there exist $t_1, t_2 \in K$ such that $s_1t_1, s_2t_2 \in T$, which implies $s_1s_2t_1t_2 \in T$ and thus $s_1s_2 \in T$. Hence $T$ is multiplicatively closed. If $T \neq K$, then $P$ is an ideal of $K$ by Lemma 1.1.1. By definition, $P$ is the greatest ideal of $K$ such that $P \cap T = \emptyset$, and it is a prime ideal since $T$ is multiplicatively closed.

2. Let $x \in K$ and $t \in T$. We shall prove that $\frac{x}{t} \in (T^{-1}K)^\times$ if and only if $t \in \overline{T}$.

   If $\frac{x}{t} \in (T^{-1}K)^\times$, then there exist $x' \in K$ and $t' \in T$ such that $\frac{x}{t} \frac{x'}{t'} = 1$. Hence there is some $w \in T$ such that $wx' = wtt'$, and $wtt' \in T$ implies $x \in T$. Conversely, if $x \in T$ and $t \in T$, let $w \in K$ be such that $xw \in T$. Then $\frac{x}{w} \frac{x}{w} = \frac{x}{w} \frac{x}{w} = 1$, and therefore $\frac{x}{t} \in (T^{-1}K)^\times$.

   Let $j_T: K \to T^{-1}K$ be the natural embedding. Since $j_T(T) \subset T^{-1}T \subset (T^{-1}K)^\times$, Theorem 1.2.3 implies the existence of some homomorphism $\iota: T^{-1}K \to T^{-1}K$ satisfying $\iota\left(\frac{x}{t}\right) = \frac{x}{t}$ for all $x \in K$ and $t \in T$. 

is injective: Let \( x, x' \in K \) and \( t, t' \in T \) be such that \( \frac{x}{t} = \frac{x'}{t'} \) in \( T^{-1}K \). Then there exists some \( s \in T \) such that \( st'x = stx' \). If \( w \in K \) is such that \( ws \in T \), then \( (ws)t'x = (ws)tx' \) and therefore \( \frac{w}{s} = \frac{x}{t} \) in \( T^{-1}K \).

\( \iota \) is surjective: Let \( z \in T^{-1}K \), say \( z = \frac{x}{s} \), where \( x \in K \) and \( s \in T \). If \( t \in K \) is such that \( st \in T \), then \( y = \frac{sx}{st} \in T^{-1}K \), and \( \iota(y) = z \).

3. Clearly, \( ST \subset K \) and \( T^{-1}S \subset T^{-1}K \) are multiplicatively closed, and the homomorphism \( j_{ST}: K \rightarrow (ST)^{-1}K \) satisfies \( j_{ST}(T) \subset (ST)^{-1}T \subset ((ST)^{-1}K)^{\times} \). Hence Theorem 1.2.3 implies the existence of some homomorphism \( \varphi: T^{-1}K \rightarrow (ST)^{-1}K \) satisfying \( \varphi(x) = (\frac{t}{1})^{-1}x = \frac{x}{t} \) for all \( x \in K \) and \( t \in T \). Since \( \varphi(T^{-1}S) \subset (ST)^{-1}S \subset ((ST)^{-1}K)^{\times} \), again Theorem 1.2.3 implies the existence of a homomorphism \( \Phi: (T^{-1}S)^{-1}(T^{-1}K) \rightarrow (ST)^{-1}K \) satisfying
\[
\Phi\left( \frac{x}{t} \right) = \varphi\left( \frac{s}{t} \right)^{-1}\frac{x}{t} = \frac{t'x}{st} \text{ for all } x \in K, s \in S \text{ and } t, t' \in T.
\]

\( \Phi \) is injective: Let \( x, x_1 \in K, s, s_1 \in S \) and \( t, t_1, t', t'_1 \in T \) be such that
\[
\Phi\left( \frac{x}{t} \right) = \Phi\left( \frac{x_1}{t_1} \right) \in (ST)^{-1}K, \text{ that is, } \frac{t'x}{st} = \frac{t_1x_1}{s_1t_1}.
\]

Then there exist some \( v \in V \) and \( w \in T \) such that \( vws_1t_1x = vwst'_1x_1 \). Hence
\[
\frac{v}{w} \frac{x}{t} \frac{s_1}{t_1} = \frac{v}{w} \frac{x_1}{t_1} \frac{s}{t} \in T^{-1}K, \text{ and therefore } \frac{x}{t} = \frac{x_1}{t_1} \in (T^{-1}S)^{-1}(T^{-1}K).
\]

\( \Phi \) is surjective: Let \( z = \frac{x}{y} \in (ST)^{-1}K \), where \( s \in S \), \( t \in T \) and \( x \in K \). Then
\[
y = \frac{y}{z} \in (T^{-1}S)^{-1}(T^{-1}K) \text{ and } \Phi(y) = z.
\]

4. We may assume that \( Y \neq \emptyset \). Since \( [T^{-1}(X:Y)](T^{-1}Y) = T^{-1}[(X:Y)Y] \subset T^{-1}X \) and \( j_{TR}(Y) \subset T^{-1}Y \), we obtain
\[
T^{-1}(X:Y) \subset (T^{-1}X:T^{-1}Y) \subset (T^{-1}X:j_{TR}(Y)).
\]

For the proof of \( (T^{-1}X:j_{TR}(Y) \subset (T^{-1}X:T^{-1}Y) \), let \( z \in K \) and \( s \in T \) be such that \( z \in (T^{-1}X:j_{TR}(Y)) \). If \( \frac{z}{x} \in T^{-1}Y \) (where \( y \in Y \) and \( t \in T \)), then \( \frac{z}{x} \in j_{TR}(Y) \), and therefore \( \frac{y}{x} \frac{z}{x} = \frac{w}{x} \in T^{-1}X \), say \( \frac{w}{x} = \frac{z}{x} \) for some \( w \in X \) and \( x \in T \), which implies that \( \frac{z}{x} = \frac{z}{x} \in T^{-1}X \).

Assume now that \( TX = X \) and \( Y = \{y_1, \ldots, y_m\} \) for some \( m \in \mathbb{N} \), and let \( z \in (T^{-1}X:T^{-1}Y) \), where \( z \in K \) and \( t \in T \). For \( j \in [1, m] \), it follows that \( \frac{y_j}{x} \in T^{-1}X \), and thus there exist \( x_1, \ldots, x_m \in X \) and \( s \in T \) such that, for all \( j \in [1, m] \), we have \( \frac{w_j}{x_j} = \frac{w}{x} \) and therefore \( w_jz_y = w_jx_j \) for some \( w_j \in T \). Then \( w = w_1 \cdots w_m \in T \) and \( wsy_j = wtx_j \in TX = X \) for all \( j \in [1, m] \). Hence we obtain \( ws \in (X:Y) \), and \( z = \frac{ws}{w} = \frac{wz}{w} \in T^{-1}(X:Y) \).

\[\boxed{\text{Theorem and Definition 1.2.5. Let } K \text{ and } L \text{ be monoids, } T \subset K \text{ a multiplicatively closed subset and } \varphi: K \rightarrow L \text{ be a homomorphism. Then } \varphi(T) \subset L \text{ is a multiplicatively closed subset, and there exists a unique homomorphism } T^{-1}\varphi: T^{-1}K \rightarrow \varphi(T)^{-1}L \text{ such that } (T^{-1}\varphi) \circ j_{T} = j_{\varphi(T)} \circ \varphi. \text{ It is given by } (T^{-1}\varphi)\left( \frac{x}{t} \right) = \frac{\varphi(x)}{\varphi(t)} \text{ for all } x \in K \text{ and } t \in T.}
\]
PROOF. Clearly, $1 = \varphi(1) \in \varphi(T)$, and $\varphi(T)\varphi(T) = \varphi(TT) = \varphi(T)$, and therefore $\varphi(T) \subset L$ is multiplicatively closed.

By Theorem 1.2.4.2 we have $j_{\varphi(T)}(\varphi(T)) \subset (\varphi(T)^{-1}L)^*$, and by Theorem 1.2.3 there exists a monoid homomorphism $T^{-1}\varphi: T^{-1}K \to \varphi(T)^{-1}L$ such that $(T^{-1}\varphi) \circ j_T = j_{\varphi(T)} \circ \varphi$.

It remains to prove uniqueness and the formula. Thus let $\Phi: T^{-1}K \to \varphi(T)^{-1}L$ be a homomorphism such that $\Phi \circ j_T = j_{\varphi(T)} \circ \varphi$. If $x \in K$ and $t \in T$, then

$$\Phi\left(\frac{x}{t}\right) = \Phi\left(j_T(t)^{-1}j_T(x)\right) = \Phi \circ j_T(t)^{-1} \Phi \circ j_T(x) = \left(\frac{\varphi(t)}{1}\right)^{-1} \left(\frac{\varphi(x)}{\varphi(t)}\right) = \frac{\varphi(x)}{\varphi(t)}.$$  

\[\square\]

**Theorem 1.2.6.** Let $K$ be a monoid and $T \subset K^*$ a multiplicatively closed subset.

1. The natural embedding $j_T: K \to T^{-1}K$ is a monomorphism, and $(T^{-1}K)^* = T^{-1}K^*$.

2. If $a \in K$ and $s \in T$, then $\frac{a}{t} \in (T^{-1}K)^*$ if and only if $a \in K^*$. In particular, $(T^{-1}K)^* = T^{-1}K^*$, and $T^{-1}K$ is cancellative if and only if $K$ is cancellative.

**Proof.** 1. If $x, y \in K$ are such that $j_T(x) = j_T(y)$, then $sx = sy$ for some $s \in T$ and consequently $x = y$. In particular, if $j_T(x) = \frac{a}{t}$, then $x = 0$, and therefore $(T^{-1}K)^* = T^{-1}K^*$.

2. Let $a \in K$ and $s \in T$. If $a \in K^*$ and

$$\frac{a}{s} \cdot \frac{x}{t} = \frac{a}{s} \cdot \frac{x'}{t'}$$

for some $x, x' \in K$ and $t, t' \in T$, then $s't'x = stx'$, hence $t'x = tx'$ and $\frac{x}{t} = \frac{x'}{t'}$, since $sa \in K^*$. If $a \notin K^*$, then there exist $x, x' \in K$ such that $x \neq x'$ and $ax = ax'$. But then it follows that

$$\frac{a}{s} = \frac{a}{s} \cdot \frac{x}{t} = \frac{a}{s} \cdot \frac{x'}{t'}$$

and $\frac{x}{t} \neq \frac{x'}{t'}$, hence $\frac{a}{s} \notin (T^{-1}K)^*$. Hence it follows that $(T^{-1}K)^* = T^{-1}K^*$.

If $K$ is cancellative, then $K^* \subset K^*$, hence $(T^{-1}K)^* = T^{-1}K^* \subset T^{-1}K^* = (T^{-1}K)^*$, and thus $T^{-1}K$ is cancellative. If $K$ is not cancellative, then there is some $a \in K \setminus K^*$. Since $\frac{a}{t} \notin (T^{-1}K)^* \setminus (T^{-1}K)^*$, it follows that also $T^{-1}K$ is not cancellative. \[\square\]

**Remarks and Definition 1.2.7.** Let $K$ be a monoid and $T \subset K^*$ a multiplicatively closed subset. Then we identify $K$ with $j_T(K) \subset T^{-1}K$ by means of $j_T$. Hence

$$K \subset T^{-1}K, \quad a = \frac{a}{t} \text{ for all } a \in K, \quad T \subset (T^{-1}K)^*, \quad \text{and } \frac{a}{t} = t^{-1}a \text{ for all } a \in K \text{ and } t \in T.$$ In particular, it follows that $T^{-1}K = K$ if and only if $T \subset K^*$.

Let $K \subset K_1$ be a submonoid and $T \subset K \cap K_1^*$ a multiplicatively closed subset. Then $T \subset K^*$ and $T^{-1}K \subset T^{-1}K_1 = K_1$. Hence we obtain $K \subset T^{-1}K = \{t^{-1}x \mid x \in K, \ t \in T\} \subset K_1$.

The monoid $q(K) = K^{*-1}K$ is called the total quotient monoid of $K$. By Theorem 1.2.6 it follows that

$$q(K)^* = K^{*-1}K^* \quad \text{and} \quad q(K)^* = q(K)^* = K^{*-1}K^*.$$ In particular, $K^* \subset q(K)^*$, and therefore $K \subset T^{-1}K \subset q(K)$ for every multiplicatively closed subset $T \subset K^*$.

If $\varphi: K \to L$ is a monoid homomorphism satisfying $\varphi(K^*) \subset L^*$, then $q(\varphi) = K^{*-1}\varphi: q(K) \to q(L)$ is called the quotient homomorphism of $\varphi$.

**Theorem 1.2.8.** Let $D$ be a monoid and $K = q(D)$.

1. $K$ is divisible if and only if $D$ is cancellative.

2. If $G \subset D^*$ is a subgroup, then $K/G = q(D/G)$. In particular, $K/D^* = q(D/D^*)$. 

3. The following assertions are equivalent:
   (a) \( D = K \).
   (b) \( zD = K \) for some \( z \in K \).
   (c) \( D^* \cap \bigcap_{a \in D^*} aD \neq \emptyset \).

**Proof.** 1. If \( D \) is cancellative, then then \( K^* = D^{* - 1}D^* \subset D^{* - 1}D^* = K^* \), and therefore \( K \) is divisible. The converse is obvious, since \( D \subset K \) is a submonoid.

2. By definition, \( D/G \subset K/G \), and we assert that \( (D/G)^* \subset (K/G)^* \). Indeed, if \( aG \in (D/G)^* \) for some \( a \in D \), then \( a \in D^* \subset K^* \) and \( aG \in (K/G)^* \). Consequently, \( q(D/G) \subset K/G \), and if \( z \in K/G \), say \( z = a^{-1}bG \), where \( a \in D^* \) and \( b \in D \), then \( z = (aG)^{-1}(bG) \in q(D/G) \). Hence \( q(D/G) = K/G \).

3. (a) \( \Rightarrow \) (b) Obvious.
   (b) \( \Rightarrow \) (c) Let \( z \in K \) be such that \( zD = K \). Then \( z \in K^* \), say \( z = b^{-1}c \), where \( b, c \in D^* \), and \( b^{-1}D = c^{-1}K = K \). We assert that \( b \in aD \) for all \( a \in D^* \). Indeed, if \( a \in D^* \), then \( a^{-1} = b^{-1}u \) for some \( u \in D \) and therefore \( b = au \in aD \).
   (c) \( \Rightarrow \) (a) Let \( b \in D^* \) be such that \( b \in aD \) for all \( a \in D^* \). If \( x = a^{-1}c \in K \), where \( a \in D^* \) and \( c \in D \), then \( x = b^{-1}c(a^{-1}b) \in b^{-1}D \). Hence \( K = b^{-1}D \), and therefore \( D = bK = K \). \( \Box \)

**Remark 1.2.9.** Let \( K \) be a ring and \( T \subset K \) a multiplicatively closed subset.

For \( z, z' \in T^{-1}K \), let \( x, x' \in K \) and \( t \in T \) be such that
\[
z = \frac{x}{t}, \quad z' = \frac{x'}{t}, \quad \text{and define} \quad z + z' = \frac{x + x'}{t}.
\]
This definition does not depend on the choice of representatives. Endowed with this addition, \( T^{-1}K \) is the usual quotient ring of commutative ring theory. In particular, \( q(K) \) is the total quotient ring, and if \( K \) is a domain, then \( q(K) \) is the quotient field of \( K \).

### 1.3. Prime and primary ideals

*Throughout this section, let \( D \) be a monoid, and for \( X, Y \subset D \), we set \( (X:Y) = (X :_D Y) \).*

**Lemma 1.3.1.** Let \( Q \subset D \) be an ideal.

1. If \( Q \neq D \), then \( Q \) is a prime ideal if and only if, for all \( A, B \subset D \), \( AB \subset Q \) implies \( A \subset Q \) or \( B \subset Q \).

2. Let \( Q \) be a prime ideal, \( n \in \mathbb{N} \), and let \( J_1, \ldots, J_n \subset D \) be ideals such that either \( J_1, \ldots, J_n \subset Q \) or \( J_1 \cap \ldots \cap J_n \subset Q \). Then there exists some \( i \in [1, n] \) such that \( J_i \subset Q \).

**Proof.** 1. Let \( Q \neq D \) be a prime ideal, \( A, B \subset D \), \( AB \subset Q \) and \( A \nsubseteq Q \). If \( a \in A \setminus Q \) and \( b \in B \), then \( ab \in AB \subset Q \) and therefore \( b \in Q \). Hence it follows that \( B \subset Q \).

2. Since \( J_1, \ldots, J_n \subset J_1 \cap \ldots \cap J_n \), it suffices to prove the assertion for the product. But this follows from 1. by induction on \( n \). \( \Box \)

**Theorem und Definition 1.3.2.** Let \( J \subset D \) be an ideal. We call
\[
\sqrt{J} = D\sqrt{J} = \{x \in D \mid x^n \in J \text{ for some } n \in \mathbb{N}\}
\]
the **radical** of \( J \) (in \( D \)), and we call \( J \) a **radical ideal** of \( D \) if \( J = \sqrt{J} \). We denote by \( \Sigma(J) = \Sigma_D(J) \) the set of all prime ideals \( P \subset D \) such that \( J \subset P \), and we denote by \( \mathcal{P}(J) = \mathcal{P}_D(J) \) the set of minimal elements of \( \Sigma(J) \). The elements of \( \mathcal{P}(J) \) are called **prime divisors** of \( J \).

1. Let \( I \subset D \) be another ideal of \( D \).
1.3. Prime and Primary Ideals

(a) \( I \subset \sqrt{I} = \sqrt{I}, \) and \( I \subset J \) implies \( \sqrt{I} \subset \sqrt{J}. \)

(b) \( \sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}. \)

2. If \( J \neq D, \) then \( \mathcal{P}(J) \neq \emptyset, \) and for every \( P \in \Sigma(J) \) there exists some \( P_0 \in \mathcal{P}(J) \) such that \( P_0 \subset P. \)

3. If \( J \neq D, \) then \( \sqrt{J} \neq D, \)

\[ \sqrt{J} = \bigcap_{P \in \mathcal{P}(J)} P, \]

and \( \sqrt{J} \) is a prime ideal if and only if it is the only prime divisor of \( J. \)

**Proof.** (a) Clearly, \( I \subset \sqrt{I}, \) and \( I \subset J \) implies \( \sqrt{I} \subset \sqrt{J}. \) If \( x \in \sqrt{I}, \) then \( x^n \in \sqrt{I} \) for some \( n \in \mathbb{N}, \) hence \( x^{nm} = (x^n)^m \in I \) for some \( m \in \mathbb{N}, \) and therefore \( x \in \sqrt{I}. \)

(b) Since \( IJ \subset I \cap J \subset I, J, \) we obtain \( \sqrt{IJ} \subset \sqrt{I} \cap J \subset \sqrt{I} \cap \sqrt{J}. \) If \( a \in \sqrt{I} \cap \sqrt{J}, \) then there exist \( m, n \in \mathbb{N} \) such that \( a^m \in I \) and \( a^n \in J. \) Hence \( a^{m+n} = a^m a^n \in IJ, \) and \( a \in \sqrt{IJ}. \)

2. If \( J \neq P, \) then \( D \setminus D^{\times} \subset \Sigma(J). \) For \( P \in \Sigma(J), \) let \( \Omega_P = \{ P' \in \Sigma(J) \mid P' \subset P \}. \) The intersection of every family in \( \Omega_P \) belongs to \( \Omega_P, \) and by Zorn’s Lemma, applied for the partially ordered set \( (\Omega_P, \supseteq), \) it follows that \( \Omega_P \) has a minimal element \( P_0 \) with respect to the inclusion. Then \( P_0 \in \mathcal{P}(J) \) and \( P_0 \subset P. \)

3. If \( \sqrt{J} = D, \) then \( 1 \in \sqrt{J} \) implies \( 1 \in J \) and thus \( J = D. \) Clearly, \( \sqrt{J} \subset P \) for all \( P \in \mathcal{P}(J). \)

We prove that for every \( a \in D \setminus \sqrt{J} \) there exists some \( P_0 \in \mathcal{P}(J) \) such that \( a \notin P_0. \) Thus suppose that \( a \in D \setminus \sqrt{J}. \) Then \( T = \{ a^n \mid n \in \mathbb{N}_0 \} \) is a multiplicatively closed subset of \( D \) satisfying \( T \cap J = \emptyset. \) If \( T \) denotes the divisor-closure of \( T, \) then Theorem 1.2.4 implies \( P = D \setminus T \) is a prime ideal, and it is the greatest ideal of \( D \) such that \( P \cap T = \emptyset. \) Hence \( J \subset P, \) and by 2. there exists some \( P_0 \in \mathcal{P}(J) \) such that \( P_0 \subset P \) and therefore \( a \notin P_0. \)

**Theorem und Definition 1.3.3.** An ideal \( Q \subset D \) is called **primary** if \( Q \neq D \) and, for all \( a, b \in D, \)

if \( ab \in Q \) and \( a \notin Q, \) then \( b \in \sqrt{Q}. \)

1. Let \( Q \subset D \) be an ideal.

   (a) \( Q \) is a prime ideal if and only if \( Q \) is a primary ideal, and \( \sqrt{Q} = Q. \)

   (b) If \( Q \) is a primary ideal, then \( \sqrt{Q} \) is the only prime divisor of \( Q. \)

   If \( Q \) is a primary ideal and \( P = \sqrt{Q}, \) then \( Q \) is called \( P \)-primary.

2. For ideals \( Q, P \subset D \) the following assertions are equivalent:

   (a) \( Q \) is \( P \)-primary.

   (b) \( Q \subset P \subset \sqrt{Q}, \) and for all \( a, b \in D, \) if \( ab \in Q \) and \( a \notin Q, \) then \( b \in P. \)

   (c) \( Q \subset P \subset \sqrt{Q}, \) and for all \( A, B \subset D, \) if \( AB \subset Q \) and \( A \notin Q, \) then \( B \subset P. \)

3. Let \( P \subset D \) be a prime ideal.

   (a) If \( Q \) and \( Q' \) are \( P \)-primary ideals, then \( Q \cap Q' \) is also \( P \)-primary.

   (b) If \( Q \) is a \( P \)-primary ideal and \( B \subset D \) is any subset such that \( B \notin Q, \) then \( (Q : D)B \) is also \( P \)-primary.

4. Let \( \varphi : D \to D' \) be a monoid homomorphism and \( Q' \subset D' \) an ideal. Then \( \varphi^{-1}(Q') \subset D \) is an ideal, \( \sqrt{\varphi^{-1}(Q')} = \varphi^{-1}(\sqrt{Q'}). \) If \( Q' \) is primary \([ \text{a prime ideal} \text{]}, \) then so is \( \varphi^{-1}(Q'). \)

**Proof.** 1. Suppose that \( a, b \in D, \) \( \ ab \in \sqrt{Q} \) and \( a \notin \sqrt{Q}. \) Then there is some \( n \in \mathbb{N} \) such that \((ab)^n = a^n b^n \in Q \) and \( a^n \notin Q. \) Since \( Q \) is primary, we obtain \( b^n \in \sqrt{Q} \) and therefore \( b \in \sqrt{\sqrt{Q}} = \sqrt{Q}. \)

Hence \( \sqrt{Q} \) is a prime ideal, and we must prove that \( \sqrt{\sqrt{Q}} \) is the smallest prime ideal containing \( Q. \) Indeed, if \( P \subset D \) is a prime ideal and \( Q \subset P, \) then \( \sqrt{Q} \subset \sqrt{P} = P. \)

2. (a) \( \Rightarrow \) (b) and (c) \( \Rightarrow \) (b) Obvious.
(b) ⇒ (c) Suppose that \( Q \subset P \subset \sqrt{Q} \), \( A, B \subset D \), \( AB \subset Q \) and \( A \not\subset Q \). Let \( a \in A \setminus Q \). For all \( b \in B \), we have \( ab \in AB \subset Q \) and therefore \( b \in P \). Hence \( B \subset P \).

(b) ⇒ (a) If \( Q \subset P \subset \sqrt{Q} \), then \( P = \sqrt{Q} \) by 1. Hence \( Q \) is \( P \)-primary.

3.(a) If \( \sqrt{Q} = \sqrt{Q} = P \), then \( \sqrt{Q} \) for all \( a, b \in D \), \( ab \in Q \cap Q' \) and \( a \not\in Q \cap Q' \). Then it follows that \( b \in P \), and thus \( Q \cap Q' \) is \( P \)-primary.

(b) Note that \( Q \subset (Q : B) \subset D \), since \( B \not\subset Q \). Hence \( P = \sqrt{Q} \subset \sqrt{(Q : B)} \), and by 2. it suffices to prove that, for all \( a, b \in D \), if \( ab \in (Q : B) \) and \( a \not\in (Q : B) \), then \( b \in P \).

If \( a, b \in D \), \( ab \in (Q : B) \) and \( a \not\in (Q : B) \), then \( abB \subset Q \), \( aB \not\subset Q \) and hence \( b \in P \), again by 2.

4. Obviously, \( \varphi^{-1}(Q') \subset D \) is an ideal. If \( a \in D \), then \( a \in \sqrt{\varphi^{-1}(Q') \implies a^n \in \varphi^{-1}(Q') \text{ for some } n \in \mathbb{N} \implies \varphi(a)^n \in Q' \text{ for some } n \in \mathbb{N} \implies \varphi(a) \in \sqrt{Q'} \implies a \in \varphi^{-1}(\sqrt{Q'}) \). Hence \( \sqrt{\varphi^{-1}(Q')} = \varphi^{-1}(\sqrt{Q'}) \).

Now let \( Q' \) be primary, \( a, b \in D \), \( a \in \varphi^{-1}(Q') \) and \( a \not\in \varphi^{-1}(Q') \). Then \( \varphi(a) \varphi(b) \in Q' \) and \( \varphi(a) \not\in Q' \), hence \( \varphi(b) \in \sqrt{Q'} \) and therefore \( b \in \varphi^{-1}(\sqrt{Q'}) = \varphi^{-1}(\sqrt{Q'}) \). If \( Q' \) is a prime ideal, then it is primary and \( \sqrt{Q'} = Q' \). Hence the same holds for \( \varphi^{-1}(Q') \). \( \square \)

Definition 1.3.4. Let \( J \subset D \) be an ideal, \( n \in \mathbb{N}_0 \), \( Q_1, \ldots, Q_n \subset D \) distinct primary ideals, and \( \Omega = \{Q_1, \ldots, Q_n\} \).

1. \( \Omega \) is called a primary decomposition of \( J \) if \( J = Q_1 \cap \ldots \cap Q_n \).

2. \( \Omega \) is called reduced if \( \sqrt{Q_1}, \ldots, \sqrt{Q_n} \) are distinct, and \( Q_1 \cap \ldots \cap Q_{i-1} \cap Q_{i+1} \cap \ldots \cap Q_n \not\subset Q_i \) for all \( i \in \{1, n\} \).

Theorem 1.3.5. Let \( J \subset D \) be an ideal.

1. If \( \Omega \) is a primary decomposition of \( J \) for which \( |\Omega| \) is minimal, then \( \Omega \) is reduced. In particular, if \( J \) possesses a primary decomposition, then it also possesses a reduced one.

2. Let \( \Omega \) be a reduced primary decomposition of \( J \). For a prime ideal \( P \subset D \), the following conditions are equivalent:

(a) \( P = \sqrt{Q} \) for some \( Q \in \Omega \).

(b) There exists some \( z \in D \setminus J \) such that \( P = \sqrt{J : z} \).

3. Let \( \Omega \) and \( \Omega' \) be reduced primary decompositions of \( J \). Then there is a bijective map \( \sigma : \Omega \rightarrow \Omega' \) such that \( \sqrt{\sigma(Q)} = \sqrt{Q} \) for all \( Q \in \Omega \), and if \( \sqrt{Q_1} \) is minimal in \( \{\sqrt{Q} : Q \in \Omega\} \), then \( \sigma(Q_1) = Q_1 \).

Proof. 1. Assume to the contrary that \( |\Omega| = n \) and \( \Omega = \{Q_1, \ldots, Q_n\} \) is not reduced. Then \( n \geq 2 \) and after renumbering (if necessary) we may assume that either \( \sqrt{Q_1} = \sqrt{Q_2} \) or \( Q_2 \cap \ldots \cap Q_n \subset Q_1 \).

We set \( \Omega_1 = \{Q_1 \cap Q_2, Q_3, \ldots, Q_n\} \) if \( \sqrt{Q_1} = \sqrt{Q_2} \), and \( \Omega_1 = \{Q_2, \ldots, Q_n\} \) if \( Q_2 \cap \ldots \cap Q_n \subset Q_1 \).

Then \( \Omega_1 \) is a primary decomposition of \( J \) satisfying \( |\Omega_1| = n - 1 \), a contradiction.

2. Suppose that \( \Omega = \{Q_1, \ldots, Q_n\} \), where \( n \in \mathbb{N}_0 \) and \( Q_1, \ldots, Q_n \) are distinct. If \( n = 0 \), then \( J = D \), and there is nothing to do. If \( n = 1 \), then \( \Omega = \{J\} \), and the assertion follows by Theorem 1.3.3. Thus we may assume that \( n \geq 2 \).

(a) ⇒ (b) Assume that \( P = \sqrt{Q_1} \). If \( z \in (Q_2 \cap \ldots \cap Q_n) \setminus Q_1 \), then \( (Q_1 : z) = D \) for all \( i \in \{2, n\} \), and \( (J : z) = (Q_1 \cap \ldots \cap Q_n : z) = \bigcap_{i=1}^{n} (Q_i : z) = (Q_1 : z) \) is \( P \)-primary by Theorem 1.3.3.3(b).
(b) ⇒ (a) Let \( z \in D \setminus J \) be such that \( P = \sqrt{(J:z)} \). Then
\[
P = \sqrt{(Q_1 \cap \ldots \cap Q_n : z)} = \bigcap_{i=1}^{n} \sqrt{(Q_i : z)} = \bigcap_{i=1}^{n} \sqrt{Q_i},
\]
and therefore \( P = Q_i \) for some \( i \in [1,n] \).

3. By 2. it follows that \( \{ \sqrt{Q} \mid Q \in \mathcal{Q} \} = \{ \sqrt{Q} \mid Q \in \mathcal{Q}' \} \) consists of all prime ideals of the form \( (J:z) \) for some \( z \in D \setminus J \). Therefore there exists a bijective map \( \sigma : \mathcal{Q} \to \mathcal{Q}' \) such that \( \sqrt{\sigma(Q)} = \sqrt{Q} \) for all \( Q \in \mathcal{Q} \).

Assume now that \( \mathcal{Q} = \{ Q_1, \ldots, Q_n \} \), where \( n \in \mathbb{N}_0 \), \( Q_1, \ldots, Q_n \) are distinct, and let \( \sqrt{Q_1} \) be minimal in the set \( \{ \sqrt{Q_1}, \ldots, \sqrt{Q_n} \} \). By symmetry, it suffices to prove that \( \sigma(Q_1) \subset Q_1 \). Assume the contrary, and consider the ideal \( B = \sigma(Q_2) \cap \ldots \cap \sigma(Q_n) \). Since \( Q_1 \supset J = B \cap \sigma(Q_1) \supset B \sigma(Q_1) \), it follows that \( B \subset \sqrt{Q_1} \) and thus \( \sqrt{Q_1} = \sqrt{\sigma(Q_1)} \subset \sqrt{Q_1} \) for some \( i \in [2,n] \), a contradiction, since \( \sqrt{Q_1} \) was minimal and \( \sqrt{Q_i} \neq \sqrt{Q_1} \).

\[\square\]

**Theorem 1.3.6.** Let \( T \subset D^* \) a multiplicatively closed subset and \( j_T : D \to T^{-1}D \) the natural embedding.

1. If \( J \subset D \) is an ideal, then \( T^{-1} \sqrt{J} = \sqrt{T^{-1}J} \).

2. The assignment \( Q \mapsto T^{-1}Q \) defines an inclusion-preserving bijective map
\[
j_T : \{ Q \subset D \mid Q \text{ is a primary ideal, } Q \cap T = \emptyset \} \to \{ \sqrt{Q} \subset T^{-1}D \mid \sqrt{Q} \text{ is a primary ideal} \}.
\]
Its inverse is given by \( \sqrt{Q} \mapsto j_T^{-1}(\sqrt{Q}) \), and if \( Q \subset D \) is a primary ideal, then \( T^{-1} \sqrt{Q} = \sqrt{T^{-1}Q} \).

In particular:

- \( j_T \) induces an inclusion-preserving bijective map from the set of all prime ideals \( P \subset D \) such that \( P \cap T = \emptyset \) onto the set of all prime ideals of \( T^{-1}D \).

- If \( P \subset D \) is a prime ideal and \( P \cap T = \emptyset \), then \( j_T \) induces an inclusion-preserving bijective map from the set of all \( P \)-primary ideals of \( D \) onto the set of all \( T^{-1}P \)-primary ideals of \( T^{-1}D \).

3. Let \( J \subset D \) be an ideal and \( \mathcal{Q} \) is a reduced primary decomposition of \( J \). Then
\[
\mathcal{Q}_T = \{ T^{-1}Q \mid Q \in \mathcal{Q}, Q \cap T = \emptyset \}
\]
is a reduced primary decomposition of \( T^{-1}J \).

**Proof.**

1. Let \( J \subset D \) be an ideal. If \( x \in T^{-1} \sqrt{J} \), then \( x = \frac{a}{t} \), where \( a \in \sqrt{J} \) and \( t \in T \). If \( n \in \mathbb{N} \) is such that \( a^n \in J \), then
\[
x^n = \frac{a^n}{t^n} \in T^{-1}J \quad \text{and} \quad x \in \sqrt{T^{-1}J}.
\]
Conversely, suppose that \( x = \frac{a}{t} \in \sqrt{T^{-1}J} \), where \( a \in D \) and \( t \in T \), and let \( n \in \mathbb{N} \) be such that \( x^n \in T^{-1}J \). Then
\[
x^n = \frac{a^n}{t^n} = \frac{c}{s} \quad \text{for some} \quad c \in J \quad \text{and} \quad s \in T.
\]
Let \( w \in T \) be such that \( wsln = wct^n \in J \). Then \( (ws)q^n = (ws)^{n-1}wsa^n \in J \), hence \( ws \in \sqrt{J} \) and
\[
x = \frac{wsa}{ws} \in T^{-1} \sqrt{J}.
\]

2. It suffices to prove the following assertion:

**A.** If \( Q \subset D \) is a primary ideal and \( Q \cap T = \emptyset \), then \( T^{-1}Q \) is primary, and \( Q = j_T^{-1}(T^{-1}Q) \).
Indeed, suppose that $A$ holds. If $\mathcal{Q} \subset T^{-1}D$ is a primary ideal, then $j_{T^{-1}}^{-1}(\mathcal{Q}) \subset D$ is primary and $\mathcal{Q} = T^{-1}j_{T^{-1}}^{-1}(\mathcal{Q})$ by the Theorems 1.3.3.4 and 1.2.2.4. Moreover, for every ideal $Q \subset D$ we have $T^{-1}\sqrt{Q} = \sqrt{T^{-1}Q}$ by 1., and the assertions follow.

Proof of A. Let $Q \subset D$ be a prime ideal and $Q \cap T = \emptyset$. Let $x, y \in T^{-1}D$ be such that $xy \in T^{-1}Q$ and $x \notin T^{-1}Q$. We set $x = \frac{a}{t}$, $y = \frac{b}{s}$ and $xy = \frac{c}{w}$, where $a, b \in D$, $c \in Q$, $t, s, w \in T$ and $a \notin Q$.

Then there exists some $v \in T$ such that $vwab = vtsc \in Q$, and as $a \notin Q$, we obtain $vwab \in \sqrt{Q}$. If $n \in \mathbb{N}$ is such that $(vwab)^n \in Q$, then

$$y^n = \left(\frac{(vwab)^n}{(vws)^n}\right) \in T^{-1}Q.$$ 

Hence $T^{-1}Q$ is primary.

Obviously, $j_{T^{-1}}^{-1}(T^{-1}Q) \supset Q$. To prove the reverse inclusion, let $c \in j_{T^{-1}}^{-1}(T^{-1}Q)$. Then $\frac{c}{q} = \frac{a}{t}$ for some $a \in Q$ and $t \in T$, and there exists some $s \in T$ such that $cst = sa \in Q$. If $c \notin Q$, then there is some $n \in \mathbb{N}$ such that $(st)^n \in Q \cap T$, a contradiction.

3. By 1. and 2., $\Omega_T$ is a primary decomposition of $T^{-1}J$, since

$$J = \bigcap_{Q \in \Omega} Q \quad \text{implies} \quad T^{-1}J = \bigcap_{Q \in \Omega_T} T^{-1}Q = \bigcap_{Q \in \Omega_T} T^{-1}Q.$$ 

We must prove that $\Omega_T$ is reduced. Assume first that $Q, Q' \in \Omega_T$ are such that $\sqrt{T^{-1}Q} = \sqrt{T^{-1}Q'}$. Then $\sqrt{Q} = j_{T^{-1}}^{-1}(T^{-1}Q) = \sqrt{T^{-1}Q'} = j_{T^{-1}}^{-1}(T^{-1}Q) = \sqrt{Q'}$ and therefore $Q = Q'$. If $Q_1 \in \Omega_T$, then

$$\bigcap_{Q \in \Omega_T} T^{-1}Q \subset T^{-1}Q_1 \quad \text{implies} \quad \bigcap_{Q \in \Omega_T} Q = \bigcap_{Q \in \Omega_T} j_{T^{-1}}^{-1}(T^{-1}Q) = j_{T^{-1}}^{-1}\left(\bigcap_{Q \in \Omega_T} T^{-1}Q\right) \subset j_{T^{-1}}^{-1}(T^{-1}Q_1) = Q_1,$$

which is impossible. Hence $\Omega_T$ is reduced. $\square$

Definition 1.3.7. Let $P \subset D$ be a prime ideal and $K \supset D$ an overmonoid. Then the monoid $K_P = (D \setminus P)^{-1}K$ is called the localization of $K$ at $P$. We denote by $j_P = j_{D \setminus P} : K \rightarrow K_P$ the natural embedding, and for $X \subset K$, we set $X_P = (D \setminus P)^{-1}X \subset K_P$.

Theorem 1.3.8. Let $P \subset D$ be a prime ideal, $T \subset D^*$ a multiplicatively closed subset and $P \cap T = \emptyset$. If $a \in D$ and $s \in T$, then $\frac{a}{s} \in T^{-1}P$ if and only if $a \in P$. In particular, $T^{-1}(D \setminus P) = T^{-1}D \setminus T^{-1}P$, and there is an isomorphism

$$\Phi : (T^{-1}D)_{T^{-1}P} \xrightarrow{\sim} D_P,$$

given by $\Phi\left(\frac{a}{t}\right) = \frac{at}{cs}$ for all $a \in D$, $c \in D \setminus P$ and $s, t \in T$.

In particular, if $D$ is cancellative, then $(T^{-1}D)_{T^{-1}P} = D_P \subset q(D)$.

Proof. Clearly, $a \in P$ and $s \in T$ implies $\frac{a}{s} \in T^{-1}P$. Conversely, if $a \in D$ and $s \in T$ are such that $\frac{a}{s} \in T^{-1}P$, then $\frac{a}{t} = \frac{s}{t} \frac{a}{s} \in T^{-1}P$ and thus $a \in P$ by Theorem 1.3.6. Hence $T^{-1}(D \setminus P) = T^{-1}D \setminus T^{-1}P$, and Theorem 1.2.4.3, applied with $S = D \setminus P$, gives the asserted isomorphism. $\square$

Theorem 1.3.9. Let $D$ be a cancellative monoid, $K = q(D)$, and let $P, Q \subset D$ be prime ideals.

1. If $Q \notin P$, then $(D : Q) \subset D_P$.
2. If $P \notin Q$, then $D_P \subseteq (D : Q)$. 
3. If $I \subset D$ is an ideal such that $I = \sqrt{I} \subset P$, then $(P : P) \subset (I : I)$. 

1.4. Fractional subsets

**Definition 1.4.1.** Let \( D \) be a monoid, \( K = q(D) \) its total quotient monoid and \( X \subset K \).

1. \( X \) is called \( D \)-fractional if there exists some \( a \in D^* \) such that \( aX \subset D \).

   Every finite subset of \( K \) is \( D \)-fractional, and every subset of a \( D \)-fractional set is \( D \)-fractional.

2. \( X \) is called a fractional (semigroup) ideal of \( D \) if \( X \) is \( D \)-fractional, \( 0 \in X \) and \( DX \subset X \) (then \( DX = X \)).

3. \( X \) is called a fractional principal ideal of \( D \) if \( X = Da \) for some \( a \in K \).

   By definition, if \( X \subset D \), then \( X \) is a fractional [principal] ideal of \( D \) if and only if \( X \) is a [principal] ideal of \( D \).

**Theorem 1.4.2.** Let \( D \) be a monoid, \( K = q(D) \) its total quotient monoid and \( X, Y \subset K \).

1. If \( c \in K \) and \( X \) is \( D \)-fractional, then \( cX \) is \( D \)-fractional.

2. \( X \) is \( D \)-fractional if and only if there exists some \( c \in K^\times \) such that \( cX \subset D \).

3. If \( X, Y \subset K \) are \( D \)-fractional, then \( X \cup Y, X \cap Y \) and \( XY \) are also \( D \)-fractional.

4. If \( X \) is \( D \)-fractional and \( Y \cap K^\times \neq \emptyset \), then \( (X : Y) \) is \( D \)-fractional.

5. Let \( T \subset D^* \) be a multiplicatively closed subset \( \{ \text{and } T^{-1}D \subset q(D) \} \). Then \( X \) is \( T^{-1}D \)-fractional if and only if \( cX \subset T^{-1}D \) for some \( c \in D^* \). In particular, if \( Y \subset K \) is \( D \)-fractional, then \( T^{-1}Y \) is \( T^{-1}D \)- fractional.

6. Let \( C \) be a monoid such that \( D \subset C \subset K \). If \( C \) is \( D \)-fractional, then every \( C \)-fractional subset \( X \subset K \) is \( D \)-fractional.

**Proof.** 1. Let \( c = b^{-1}d \in K \) (where \( b \in D^* \) and \( d \in D \)). If \( X \) is \( D \)-fractional and \( a \in D^* \) is such that \( aX \subset D \), then \( ba \in D^* \) and \( ba(cX) = daX \subset dD \subset D \). Hence \( cX \) is \( D \)-fractional.

2. If \( X \) is \( D \)-fractional, then there exists some \( c \in D^* \subset K^\times \) such that \( cX \subset D \). Conversely, let \( c = b^{-1}d \in K^\times \) (where \( b, d \in D^* \)) be such \( cX \subset D \). Then \( dx \subset bcX \subset bD \subset D \), and thus \( X \) is \( D \)-fractional.

3. Let \( a, b \in D^* \) be such that \( aX \subset D \) and \( bY \subset D \). Then \( a(X \cap Y) \subset D \), \( ab(X \cup Y) \subset D \) and \( abXY \subset D \). Hence \( X \cap Y, X \cup Y \) and \( XY \) are \( D \)-fractional.

4. If \( y \in Y \cap K^\times \), then \( y^{-1}X \) is \( D \)-fractional by 1., and since \( (X : Y) \subset y^{-1}X \), it follows that \( (X : Y) \) is \( D \)-fractional.

5. Let \( X \) be \( T^{-1}D \)-fractional and \( z = (T^{-1}D)^* = T^{-1}D^* \) such that \( zX \subset T^{-1}D \). Then \( z = t^{-1}c \), where \( t \in T \) and \( c \in D^* \), and \( cX = tzX \subset T^{-1}D \). The converse is obvious, since \( D^* \subset (T^{-1}D)^* \). If \( Y \subset K \) is \( D \)-fractional and \( c \in D^* \) is such that \( cY \subset D \), then \( ct^{-1}Y = T^{-1}cY \subset T^{-1}D \), and thus \( T^{-1}Y \) is \( T^{-1}D \)-fractional.

6. Let \( a \in K^\times \) be such that \( aC \subset D \). If \( X \subset K \) is \( C \)-fractional and \( c \in K^\times \) is such that \( cX \subset C \), then \( ac \in K^\times \) and \( acX \subset D \). □
1.5. Free monoids, factorial monoids and GCD-monoids

Throughout this section, let \( D \) be a cancellative monoid and \( K = q(D) \).

**Definition 1.5.1.**

1. Let \( X \subset D \). An element \( d \in D \) is called a *greatest common divisor* of \( D \) if \( dd \) is the smallest principal ideal containing \( X \). Equivalently, \( \{x \mid x \in X \text{ and } \exists e \in D \text{ such that } x = ed \} \) is a principal ideal. We denote by \( \text{GCD}(X) = \text{GCD}_D(X) \), the set of all greatest common divisors of \( X \).

By definition, \( \text{GCD}(X) = \{0\} \) if and only if \( X \) is a principal ideal. If \( d \in \text{GCD}(X) \), then \( \text{gcd}(X) = dd \). Consequently, if \( D \) is reduced, then \( |\text{GCD}(X)| \leq 1 \), and we write \( d = \text{gcd}(X) \) instead of \( \text{GCD}(X) = \{0\} \). If \( X = \{a_1, \ldots, a_n\} \) for some \( n \) and \( a_1, \ldots, a_n \in D \), we set \( \text{GCD}(a_1, \ldots, a_n) = \text{GCD}(X) \). In particular, \( \text{GCD}(a) = aD^\times \) for all \( a \in D \). Two elements \( a, b \in D \) are called *coprime* if \( \text{GCD}(a, b) = D^\times \).

If \( X \subset D \), \( d \in \text{GCD}(X) \) and \( \varepsilon : D \to D/D^\times \) denotes the reduction homomorphism, then \( \varepsilon(d) = dD^\times = \text{gcd}(\pi(X)) \).

2. \( D \) is called a GCD-monoid if \( \text{GCD}(E) \neq \emptyset \) for all \( E \in \mathbb{P}(D) \). Hence \( D \) is a GCD-monoid if and only if \( D/D^\times \) is a GCD-monoid. Every divisible monoid is a GCD-monoid.

3. A homomorphism \( \varphi : D \to D' \) of GCD-monoids is called a *GCD-homomorphism* if \( \varphi(\text{GCD}(E)) \subset \text{GCD}(\varphi(E)) \) for every \( E \in \mathbb{P}(D) \).

We denote by \( \text{Hom}_{\text{GCD}}(D, D') \) the set of all GCD-homomorphisms \( \varphi : D \to D' \).

**Theorem 1.5.2.**

1. Let \( (X_\lambda)_{\lambda \in \Lambda} \) be a family of subsets of \( D \), \( b_\lambda \in \text{GCD}(X_\lambda) \) for every \( \lambda \in \Lambda \), and \( B = \{b_\lambda \mid \lambda \in \Lambda\} \). Then

\[
X = \bigcup_{\lambda \in \Lambda} X_\lambda \quad \text{implies} \quad \text{GCD}(X) = \text{GCD}(B).
\]

In particular, \( D \) is a GCD-monoid if and only if \( \text{GCD}(a, b) \neq \emptyset \) for all \( a, b \in D^\bullet \).

2. If \( X \subset D \), \( a \in D \) and \( \text{GCD}(aX) \neq \emptyset \), then \( \text{GCD}(aX) = a \text{GCD}(X) \).

**Proof.** 1. It suffices to prove that \( X \) and \( B \) are contained in the same principal ideals of \( D \). If \( b \in D \), then

\[
X \subset bD \iff X_\lambda \subset bD \quad \text{for all } \lambda \in \Lambda \iff b_\lambda D \subset bD \quad \text{for all } \lambda \in \Lambda \iff B \subset bD.
\]

If \( D \) is a GCD-monoid, then \( \text{GCD}(a, b) \neq \emptyset \) for all \( a, b \in D \). Conversely, suppose that \( \text{GCD}(a, b) \neq \emptyset \) for all \( a, b \in D^\bullet \), and let \( E \in \mathbb{P}(D) \). We must prove that \( \text{GCD}(E) \neq \emptyset \), and since \( \text{GCD}(E) = \text{GCD}(E \setminus \{0\}) \), we may assume that \( E \subset D^\bullet \). We use induction on \( |E| \). If \( |E| \leq 2 \), there is nothing to do. Thus assume that \( |E| \geq 3 \) and \( a \in E \). If \( b \in \text{GCD}(E \setminus \{a\}) \) and \( d \in \text{GCD}(a, b) \), then \( d \in \text{GCD}(E) \).

2. It suffices to prove that \( \text{GCD}(aX) \subset a \text{GCD}(X) \). For \( a = 0 \), this is obvious. Thus suppose that \( a \in D^\bullet \), and let \( c \in \text{GCD}(aX) \). Then \( aX \subset aD \) implies \( cD \subset aD \), hence \( c = ab \) for some \( b \in D \), and \( X \subset bD \). If \( b' \in D \) is such that \( X \subset b'D \), then \( aX \subset ab'D \), hence \( cD = abD \subset ab'D \) and therefore \( bD \subset b'D \). Consequently, \( bD \) is the smallest principal ideal containing \( X \), \( b \in \text{GCD}(X) \), and \( c = ab \in a \text{GCD}(X) \).

**Theorem 1.5.3.** Let \( D \) be a GCD-monoid.

1. If \( E, F \in \mathbb{P}(D) \) and \( b \in D \), then \( \text{GCD}(EF) = \text{GCD}(E) \text{GCD}(F) \) and \( \text{GCD}(bE) = b \text{GCD}(E) \).

2. Let \( a, b, c \in D \) be such that \( a \mid bc \). Then there exist \( b', c' \in D \) such that \( a = b'c' \), \( b' \mid b \) and \( c' \mid c \). In particular, if \( \text{GCD}(a, b) = D^\times \), then \( a \mid c \). \[\square\]
3. Every \( z \in K \) has a representation in the form \( z = a^{-1}b \) with \( a \in D^* \) and \( b \in D \) such that \( \text{GCD}(a, b) = D^x \). In this representation \( aD^x \) and \( bD^x \) are uniquely determined by \( z \).

**Proof.** We use Theorem 1.5.2.
1. Suppose that \( e \in \text{GCD}(E) \) and \( f \in \text{GCD}(F) \), and observe that
   \[
   EF = \bigcup_{b \in E} bF.
   \]
   For every \( b \in E \), we have \( bf \in \text{GCD}(bF) \), and since \( \{bf : b \in E\} = Ef \), we obtain \( ef \in \text{GCD}(EF) \).

2. Let \( b' \in \text{GCD}(a, b) \) and \( c' \in D \) such that \( a = b'c' \). Then it follows that \( b'c \in \text{GCD}(ac, bc) \), and \( b' \text{GCD}(c', c) = \text{GCD}(b'c', b'c) = \text{GCD}(a, ac, bc) = \text{GCD}(a, bc) = aD^x = b'c'D^x \), which implies that \( \text{GCD}(c', c) = c'D^x \) and therefore \( c | c' \). In particular, if \( \text{GCD}(a, b) = D^x \), we may assume that \( b' = 1 \), and then \( a = c' | c \).

3. If \( z \in K \), then \( z = a^{-1}b_1 \), where \( a_1 \in D^* \) and \( b_1 \in D \). If \( d \in \text{GCD}(a_1, b_1) \), then \( a_1 = ad \) and \( b_1 = bd \), where \( a, b \in D \), and \( d = \text{GCD}(ad, bd) = d \text{GCD}(a, b) \). Hence \( \text{GCD}(a, b) = D^x \) and \( z = a^{-1}b \). To prove uniqueness, suppose that \( z = a^{-1}b' \), where \( a' \in D^* \), \( b' \in D \) and \( \text{GCD}(a', b') = D^x \). Then \( a'b = ab' \), and since \( \text{GCD}(a, b) = \text{GCD}(a', b') = D^x \), it follows that \( a | a' \), \( b | b' \), \( a' | a \) and \( a | a' \). Hence \( aD = a'D \) and \( bD = b'D \). \( \square \)

**Definition 1.5.4.**
1. An element \( q \in D^* \) is called
   - an **atom** if \( q \notin D^x \) and, for all \( a, b \in D \), \( q = ab \) implies \( a \in D^x \) or \( b \in D^x \) [equivalently, \( qD \) is maximal in the set \( \{aD : a \in D \setminus D^x\} \)];
   - a **prime element** if \( q \notin D^x \) and, for all \( a, b \in D^* \), \( q | ab \) implies \( q | a \) or \( q | b \) [equivalently, \( qD \) is a prime ideal].

2. \( D \) is called
   - **atomic** if every \( a \in D^* \setminus D^x \) is a product of atoms;
   - **factorial** if every \( a \in D^* \setminus D^x \) is a product of prime elements.

3. \( D \) is said to satisfy the **ACCP** (ascending chain condition for principal ideals) if there is no sequence \( (a_nD)_{n \geq 0} \) of principal ideals of \( D \) such that \( a_nD \subsetneq a_{n+1}D \) for all \( n \in \mathbb{N} \) [equivalently, every non-empty set of principal ideals of \( D \) contains a maximal element].

4. \( D \) is called **free** with basis \( P \subset D \) if the map
   \[
   \chi_P : \mathbb{N}_0^{(P)} \to D^*, \quad \text{defined by} \quad \chi((n_p)_{p \in P}) = \prod_{p \in P} p^{n_p}, \quad \text{is bijective.}
   \]

5. A subset \( P \subset D \) is called a **complete set of primes** if every \( p \in P \) is a prime element and, for every prime element \( p \in D \) there is a unique \( p_0 \in P \) such that \( pD = p_0D \) [equivalently, \( p = p_0u \) for some \( u \in D^x \)].

**Theorem 1.5.5.**
1. If \( D \) satisfies the ACCP, then \( D \) is atomic.

2. Every prime element of \( D \) is an atom, and if \( D \) is a GCD-monoid, then every atom is a prime element.

3. \( D \) is factorial if and only if \( D \) is atomic and every atom is a prime element.
1. Generalities on Monoids

Proof. 1. Let $\Omega$ be the set of all principal ideals $aD$, where $a \in D^* \setminus D^\times$ is not a product of atoms. Assume that, contrary to the assertion, $\Omega \neq \emptyset$. Since $D$ satisfies the ACCP, $\Omega$ contains a maximal element $aD$, and since $a$ is not an atom, it has a factorization $a = bc$, where $b, c \in D \setminus D^\times$. In particular, it follows that $aD \subseteq bD$ and $aD \subseteq cD$, and therefore $bD, cD \notin \Omega$. Hence both $b$ and $c$ are products of atoms, and therefore $a = bc$ is also a product of atoms, a contradiction.

2. Let $p \in D$ be a prime element and $a \in D \setminus D^\times$ such that $pD \subseteq aD$. We must prove that $pD = aD$. Since $p = au$ for some $u \in D$ and therefore $p | au$, it follows that $p | a$ or $p | u$. If $p | a$, then $aD = pD$ and we are done. If $p | u$, then $u = pv$ for some $v \in D$, hence $p = apv$, and from $1 = av$ it follows that $a \in D^\times$, a contradiction.

Assume now that $D$ is a GCD-monoid, and let $q \in D$ be an atom. If $a, b \in D$ and $q | ab$, then Theorem 1.5.3.2 implies that there exist $a', b' \in D$ such that $a' | a$, $b' | b$ and $q = a'b'$. Hence it follows that $a' \in D^\times$ or $b' \in D^\times$, say $a' \in D^\times$. But then $b' | b$ implies $q | b$.

3. If $D$ is atomic and every atom is a prime element, then every $a \in D \setminus D^\times$ is product of prime elements and thus $D$ is factorial.

If $D$ is factorial, then $D$ is atomic, since every prime element is an atom. If $q \in D$ is an atom, then $q = p_1 \cdots p_r$, where $r \in \mathbb{N}$ and $p_1, \ldots, p_r$ are prime elements. But then it follows that $r = 1$ and $q = p_1$ is a prime element. \hfill $\square$

Theorem und Definition 1.5.6.

1. For a subset $P \subseteq D$, the following assertions are equivalent:
   (a) $D$ is factorial and $P$ is a complete set of primes.
   (b) Every $a \in D^*$ has a unique representation
       \[ a = u \prod_{p \in P} p^{v_p(a)}, \quad \text{where} \quad u \in D^\times, \quad v_p(a) \in \mathbb{N}_0 \quad \text{and} \quad v_p(a) = 0 \quad \text{for almost all} \quad p \in P. \]
   (c) $D/D^\times$ is free with basis $\varepsilon(P)$, where $\varepsilon : D \to D/D^\times$ denotes the canonical epimorphism.

   For $a \in D^*$, we call $v_p(a)$ the $p$-adic exponent of $a$, and we set $v_p(0) = \infty$.

2. $D$ is free with basis $P$ if and only if $D$ is factorial and reduced and $P$ is the set of prime elements of $D$.

3. Let $D$ be factorial, $P$ a complete set of primes and $\emptyset \neq X \subset D^*$. Then
   \[ d = \prod_{p \in P} p^{\min\{v_p(x) \mid x \in X\}} \in \text{GCD}(X), \]
   and there exists some $E \in \mathcal{P}(X)$ such that $d \in \text{GCD}(E)$.

4. $D$ is factorial if and only if $D$ is an atomic GCD-monoid.

Proof. 1. (a) $\Rightarrow$ (b) Let $a \in D^*$. Then $a = u'p'_1 \cdots p'_r$, where $r \in \mathbb{N}_0$, $u' \in D^\times$, and $p'_1, \ldots, p'_r \in D$ are prime elements. For $i \in [1, r]$, let $p_i \in P$ and $u_i \in D^\times$ be such that $p'_i = p_iu_i$. Then $u = u'u_1 \cdots u_r \in D^\times$, and $a = up_1 \cdots p_r$. For $p \in P$, let $n_p = |\{i \in [1, r] \mid p_i = p\}| \in \mathbb{N}_0$. Then $n_p = 0$ for almost all $p \in P$, and
   \[ a = u \prod_{p \in P} p^{n_p}. \]

   We must prove uniqueness. Thus assume that
   \[ a = u \prod_{p \in P} p^{n_p} = u' \prod_{p \in P} p^{n'_p}, \]

   ...
where \( u, u' \in D^\times \), \( n_p, n'_p \in \mathbb{N}_0 \) for all \( p \in P \), and \( n_p = n'_p = 0 \) for almost all \( p \in P \). Then we obtain
\[
\left(u^{-1}u'\right) \prod_{\substack{p \in P \\mid \\n_p > n_p \\land \n'_p < n_p}} \prod_{\substack{p \in P \\mid \\n'_p > n_p \\land \n_p < n'_p}} p^{n_p - n'_p} = \prod_{\substack{p \in P \\mid \\n'_p > n_p}} p^{n_p - n'_p}.
\]
Assume now that there is some \( q \in P \) such that \( n'_q > n_q \). Then it follows that
\[
\prod_{\substack{p \in P \\mid \\n'_p < n_p \\land \n_p > n'_p}} p^{n_p - n'_p} \in qP,
\]
and therefore \( p \in qP \) for some \( p \in P \) such that \( n'_p < n_p \), a contradiction. Hence there is no \( p \in P \) such that \( n'_p > n_p \), and for the same reason there is no \( p \in P \) such that \( n'_p > n_p \). Hence it follows that \( n_p = n'_p \) for all \( p \in P \), and consequently \( u = u' \).

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(b) \( \Leftrightarrow \) (c) By definition, \( \varepsilon \mid P : P \rightarrow \varepsilon(P) \) is bijective, and if \( a \in D^\bullet \), \( u \in D^\times \) and \( (n_p)_{p \in P} \in \mathbb{N}_0^{(P)} \), then
\[
a = u \prod_{p \in P} p^{n_p} \quad \text{if and only if} \quad \varepsilon(a) = \prod_{p \in P} \varepsilon(p)^{n_p}.
\]

(b) \( \Rightarrow \) (a) It suffices to prove that \( P \) is a complete set of primes. From the uniqueness in (b) we obtain:

- If \( a, b \in D^\bullet \), then \( v_p(ab) = v_p(a) + v_p(b) \).
- If \( a \in D^\bullet \) and \( p \in P \), then \( a \in pD \) if and only if \( v_p(a) > 0 \).

Hence every \( p \in P \) is a prime element. Indeed, if \( p \in P \) and \( a, b \in D^\bullet \) are such that \( ab \in pD \), then \( v_p(ab) = v_p(a) + v_p(b) > 0 \), hence \( v_p(a) > 0 \) or \( v_p(b) > 0 \) and therefore \( a \in pD \) or \( p \in pD \).

If \( q \in D \) is a prime element, then \( q \in pD \) for every \( p \in P \) such that \( v_p(D) > 0 \). But if \( q \in pD \), then \( qD = pD \), since \( q \) is an atom and \( qD \) is a maximal principal ideal. Hence there is a unique \( p \in P \) such that \( qD = pD \).

2. Obviously by 1.

3. Clearly, \( \min\{v_p(x) \mid x \in X\} \in \mathbb{N}_0 \) for all \( p \in P \), and \( \min\{v_p(x) \mid x \in X\} = 0 \) for almost all \( p \in P \). Hence \( d \in D^\bullet \). If \( b \in D^\bullet \), then \( X \subset bD \) holds if and only if \( v_p(b) \leq v_p(x) \) for all \( x \in X \) and \( p \in P \). Therefore we obtain \( d \in \text{GCD}(X) \).

Let now \( b \in X \) be arbitrary. Then \( v_p(d) \leq v_p(b) \), and the set \( P_0 = \{p \in P \mid v_p(b) \neq 0\} \) is finite. For every \( p \in P_0 \) there is some \( x_p \in X \) such that \( v_p(x_p) = v_p(d_p) \). If \( E = \{b\} \cup \{x_p \mid p \in P_0\} \), then \( d \in \text{GCD}(E) \).

4. If \( D \) is factorial, then \( D \) is atomic by Theorem 1.5.5, and \( D \) is a GCD-monoid by 3. If \( D \) is an atomic GCD-monoid, then every atom is a prime element and therefore \( D \) is factorial, again by Theorem 1.5.5.

\[ \square \]
The formalism of module and ideal systems

2.1. Weak module and ideal systems

Definition 2.1.1. Let $K$ be a monoid.

1. A weak module system on $K$ is a map $r : \mathbb{P}(K) \to \mathbb{P}(K)$, $X \mapsto X_r$ such that, for all $c \in K$ and $X, Y \in \mathbb{P}(K)$ the following conditions are fulfilled:
   1. $M_1$. $X \cup \{0\} \subset X_r$.
   2. $M_2$. If $X \subset Y_r$, then $X_r \subset Y_r$.
   3. $M_3$. $cX_r \subset (cX)_r$.

2. A module system on $K$ is a weak module system $r$ on $K$ such that equality holds in $M_3$ for all $c \in K$ and $X \in \mathbb{P}(K)$.

3. Let $r$ be a weak module system on $K$. A subset $J \subset K$ is called an $r$-module if $J = X_r$ for some subset $X \subset K$ (then $X \cup \{0\} \subset J$ by $M_1$). An $r$-module $J \subset K$ is called $r$-finitely generated if $J = E_r$ for some finite subset $E \subset K$.

We denote by
   - $\mathcal{M}_r(K)$ the set of all $r$-modules in $K$, and by
   - $\mathcal{M}_{r,f}(K)$ the set of all $r$-finitely generated $r$-modules in $K$.

A submonoid $D \subset K$ is called an $r$-monoid if it is an $r$-module.

4. For two $r$-modules $J_1, J_2 \subset K$, we define their $r$-product by $J_1 \cdot_r J_2 = (J_1 J_2)_r$, and we call $\cdot_r$ the $r$-multiplication.

Theorem 2.1.2. Let $K$ be a monoid, $r$ be a weak module system on $K$ and $X, Y \subset K$.

1. $(X_r)_r = X_r$. In particular, $X$ is an $r$-module if and only if $X = X_r$.
2. If $X \subset Y$, then $X_r \subset Y_r$. In particular,
   \[ X_r = \bigcap_{J \supset X, J \in \mathcal{M}_r(K)} J \]
   is the smallest $r$-module containing $X$.
3. $X_r = (X \cup \{0\})_r$, $\emptyset_r = \{0\}_r$, and if $r$ is a module system, then $\emptyset_r = \{0\}_r = \{0\}$.
4. The intersection of any family of $r$-modules is again an $r$-module.
5. For every family $(X_\lambda)_{\lambda \in \Lambda}$ in $\mathbb{P}(K)$ we have
   \[ \bigcup_{\lambda \in \Lambda} (X_\lambda)_r \subset \left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r = \left( \bigcup_{\lambda \in \Lambda} (X_\lambda)_r \right)_r. \]
6. \((XY)_r = (X,Y)_r = (XY,r)_r = (X,Y)_r\). If \(T \subset K\) and \(1 \in T\), then the following assertions are equivalent:

(a) \(X_r = TX_r\)  
(b) \(X_r = (TX)_r\)  
(c) \(X_r = T_rX_r\).

In particular, if \(TX = X\), then \(T_rX_r = X_r\).

7. Equipped with the \(r\)-multiplication, \(\mathcal{M}_r(K)\) is a monoid with unit element \(\{1\}_r\), zero element \(\emptyset_r\), and \(\mathcal{M}_r(f)(K) \subset \mathcal{M}_r(K)\) is a submonoid.

8. For every family \((X_\lambda)_{\lambda \in \Lambda}\) in \(\mathcal{P}(K)\) we have the distributive law

\[
\left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r = \left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r \cap Y_r = \left( \bigcup_{\lambda \in \Lambda} (X_\lambda)_r \cap Y_r \right)_r.
\]

9. \((X : Y)_r \subset (X_r : Y) = (X_r : Y)_r\). In particular, if \(X\) is an \(r\)-module, then \((X : Y)_r\) is also an \(r\)-module.

**Proof.**

1. **M1** implies \(X_r \subset (X)_r\), and since \(X_r \subset X_r\), we obtain \((X)_r \subset X\) by **M2**. Hence \((X)_r = X_r\).

If \(X = X_r\), then \(X\) is an \(r\)-module by definition. Conversely, if \(X\) is an \(r\)-module, then \(X = Z_r\) for some subset \(Z \subset K\), and hence \((X)_r = (Z)_r = Z_r = X\).

2. If \(X \subset Y\), then \(X \subset Y\) and therefore \(X_r \subset Y_r\), again by **M1** and **M2**.

If \(J \in \mathcal{M}_r(K)\) and \(X \subset J\), then \(X_r \subset J_r = J\), and therefore

\[
X_r \subset \bigcap_{J \in \mathcal{M}_r(K)} J.
\]

Since \(X_r = (X)_r \in \mathcal{M}_r(K)\), the reverse inclusion is obvious.

3. By **M1** we have \(X \cup \{0\} \subset X_r\), hence \((X \cup \{0\})_r \subset X_r\) by **M2**, and since \(X_r \subset (X \cup \{0\})_r\) by 2., equality follows. If \(r\) is a module system, then \(\{0\} = 0\{1\}_r = \{0\}_r\).

4. Let \((J_\lambda)_{\lambda \in \Lambda}\) be a family of \(r\)-modules, and

\[
X = \bigcap_{\lambda \in \Lambda} J_\lambda.
\]

Then \(\{J_\lambda : \lambda \in \Lambda\} \subset \{J \in \mathcal{M}_r(K) : J \supset X\}\) and therefore

\[
X_r = \bigcap_{J \in \mathcal{M}_r(K)} J \subset \bigcap_{\lambda \in \Lambda} J_\lambda = X \subset X_r,
\]

which implies equality.

5. For each \(\alpha \in \Lambda\) we have

\[
X_\alpha \subset \bigcup_{\lambda \in \Lambda} X_\lambda \subset \left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r, \quad \text{hence} \quad (X_\alpha)_r \subset \left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r \quad \text{and} \quad \bigcup_{\lambda \in \Lambda} (X_\lambda)_r \subset \left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r.
\]

Now it follows by **M2** that

\[
\left( \bigcup_{\lambda \in \Lambda} (X_\lambda)_r \right)_r \subset \left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r, \quad \text{and} \quad \bigcup_{\lambda \in \Lambda} X_\lambda \subset \bigcup_{\lambda \in \Lambda} (X_\lambda)_r \quad \text{implies} \quad \left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r \subset \left( \bigcup_{\lambda \in \Lambda} (X_\lambda)_r \right)_r.
\]

6. Using **M3**, we obtain

\[
XY_r = \bigcup_{x \in X} xY_r \subset \bigcup_{x \in X} (xY)_r \subset (XY)_r \quad \text{and} \quad X_rY_r = \bigcup_{y \in Y_r} X_r y \subset \bigcup_{y \in Y_r} (Xy)_r \subset (XY)_r.
\]

Hence it follows, using **M2**, that \((X_rY)_r \subset (XY)_r \subset (XY)_r \subset X_rY_r \subset (X_rY_r)_r\), and thus equality holds throughout.

(a) \(\Rightarrow\) (b) From \(TX \subset TX_r = X_r\) we obtain \((TX)_r \subset X_r \subset (TX)_r\), and thus \(X_r = (TX)_r\).
(b) \(\Rightarrow\) (c) From \(X_r \subset T_r X_r \subset (T_r X_r)_r = (TX)_r = X_r\) we obtain \(X_r = T_r X_r\).

(c) \(\Rightarrow\) (a) From \(X_r \subset TX_r \subset T_r X_r = X_r\) we obtain \(X_r = TX_r\).

If \(TX = X\), then \((TX)_r = X_r\) and therefore \(T_r X_r = X_r\).

7. Obviously, \(\cdot_r\) is commutative, and for every subset \(X \subset K\) we have \((1X)_r = X_r\) and \((\emptyset X)_r = \emptyset_r\).

If \(J_1, J_2, J_3 \in M_r(K)\), then \((J_1 \cdot_r J_2)_r \cdot_r J_3 = ((J_1 \cdot_r J_2)_r)_r = (J_1 \cdot_r (J_2 \cdot_r J_3))\).

Hence \(\cdot_r\) is associative, and \(M_r(K)\) is a monoid with unit element \(\{1\}_r\) and zero element \(\emptyset_r\).

If \(J_1, J_2 \in M_{r,f}(K)\), then there exist finite subsets \(E_1, E_2 \subset K\) such that \(J_1 = (E_1)_r\) and \(J_2 = (E_2)_r\).

Hence it follows that \(J_1 \cdot_r J_2 = ((E_1)_r (E_2)_r)_r = (E_1 E_2)_r \in M_{r,f}(K)\).

8. If \((X_\lambda)_{\lambda \in \Lambda}\) is a family in \(P(K)\), then \(5.\) implies that
\[
\left( \bigcup_{\lambda \in \Lambda} X_\lambda Y \right)_r = \left( \left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r \right)_r = \left( \bigcup_{\lambda \in \Lambda} X_\lambda \cdot_r Y \right)_r = \left( \bigcup_{\lambda \in \Lambda} (X_\lambda)_r \cdot_r Y_r \right)_r.
\]

9. Since \((X_r : Y)_r Y \subset (X_r : Y), Y \subset (X_r : Y), Y \subset ((X_r : Y)_r)_r = (X_r)_r = X_r\), it follows that
\[(X_r : Y)_r \subset (X_r : Y) \subset (X_r : Y) \subset (X_r : Y) \subset (X_r : Y) \subset (X_r : Y)_r = (X_r : Y)_r \subset (X_r : Y)\).

Since \((X : Y) \subset (X_r : Y) = (X_r : Y)_r\), it follows that \((X : Y)_r \subset (X_r : Y)\).

If \(X\) is an \(r\)-module, then \((X : Y)_r = (X_r : Y)_r = (X_r : Y) = (X : Y)\), and therefore \((X : Y)\) is also an \(r\)-module. \(\square\)

**Remarks and Definition 2.1.3.** Let \(K\) be a monoid and \(D \subset K\) a submonoid.

1. A (weak) module system \(r: P(K) \to P(K)\) is called a
   - (weak) \(D\)-module system if \(DJ \subset J\) (and thus \(DJ = J\)) for every \(J \in M_r(K)\).
   - (weak) ideal system of \(D\) if it is a (weak) \(D\)-module system and \(D_r = D\).

   In this case, we say more precisely that \(r\) is a (weak) ideal system of \(D\) defined on \(K\).

   Whenever it does not matter on which overmonoid of \(D\) the ideal system \(r\) is defined, we say that \(r\) is an ideal system of \(D\).

   If \(r\) is a (weak) ideal system of \(D\) defined on \(K\), then \(r \upharpoonright P(D): P(D) \to P(D)\) is also a (weak) ideal system of \(D\).

2. Let \(r: P(K) \to P(K)\) be a weak ideal system of \(D\). An \(r\)-modules \(J \in M_r(K)\) is called an \(r\)-ideal of \(D\) if \(J \subset D\). If \(J\) is an \(r\)-ideal of \(D\), then \(0 \in J\) and \(DJ = J\), and thus \(J\) is a (semigroup) ideal of \(D\). We denote by
   - \(I_r(D) = \{ J \in M_r(K) \mid J \subset D \}\) the set of all \(r\)-ideals of \(D\) and by
   - \(I_{r,f}(D) = I_r(D) \cap M_{r,f}(K)\) the set of all \(r\)-finitely generated \(r\)-ideals of \(D\).

   By definition, \(I_{r,f}(D) \subset I_r(D) \subset M_r(K)\) are submonoids.

3. Let again \(r: P(K) \to P(K)\) be a weak ideal system of \(D\), and assume that \(K = q(D)\). Then an \(r\)-module \(J \in M_r(K)\) is called a fractional \(r\)-ideal of \(D\) if \(J\) is \(D\)-fractional.

   If \(J\) is a fractional \(r\)-ideal of \(D\), then \(0 \in J\) and \(DJ = J\), and thus \(J\) is a fractional (semigroup) ideal of \(D\). We denote by
   - \(F_r(D) = \{ J \in M_r(K) \mid J \text{ is } D\text{-fractional} \}\) the set of all fractional \(r\)-ideals of \(D\),

   and we assert that \(M_{r,f}(K) \subset F_r(D)\) [Proof: If \(J \in M_{r,f}(K)\), then \(J = E_r\) for some \(E \in P(K)\). Hence there exists some \(a \in D^*\) such that \(aE \subset D\), and therefore \(aJ = aE_r \subset (aE)_r \subset D_r = D\)].

   Consequently, we denote by
   - \(F_{r,f}(D) = M_{r,f}(K)\) the set of all \(r\)-finitely generated fractional \(r\)-ideals of \(D\).

   By definition, \(F_{r,f}(D) = M_{r,f}(K) \subset F_r(D) \subset M_r(K)\), and \(M_{r,f}(K) \subset M_r(K)\) is a submonoid. We assert that also \(F_r(D) \subset M_r(K)\) is a submonoid. [Proof: If \(J_1, J_2 \in F_r(D)\), then \(J_1 J_2\) is \(D\)-fractional by Theorem 1.4.2.3. Hence there exists some \(c \in D^*\) such that \(cJ_1 J_2 \subset D\), and then \(c(J_1 J_2)_r \subset (cJ_1 J_2)_r \subset D_r = D\) implies that \(J_1 \cdot_r J_2 = (J_1 J_2)_r \in F_r(D)\).]
Consequently, $I_r(D) \subset \mathcal{F}_r(D)$ and $I_{r,f}(D) \subset \mathcal{F}_{r,f}(D)$ are also submonoids.

**Theorem 2.1.4.** Let $K$ be a monoid and $D \subset K$ a submonoid. Assume that $K = q(D)$, and let $r: \mathbb{P}(K) \to \mathbb{P}(K)$ be an ideal system of $D$. Then

$$\mathcal{F}_r(D) = \{a^{-1}I \mid I \in I_r(D), \ a \in D^* \} \supset \{J \in \mathbb{P}(K) \mid aJ \in I_r(D) \ \text{for some} \ a \in D^* \}$$

and

$$\mathcal{F}_{r,f}(D) = \{a^{-1}I \mid I \in I_{r,f}(D), \ a \in D^* \} \supset \{J \in \mathbb{P}(K) \mid aJ \in I_{r,f}(D) \ \text{for some} \ a \in D^* \}$$

**Proof.** We show that

$$\mathcal{F}_r(D) \subset \{a^{-1}I \mid I \in I_r(D), \ a \in D^* \} \subset \{J \in \mathbb{P}(K) \mid aJ \in I_r(D) \ \text{for some} \ a \in D^* \} \subset \mathcal{F}_r(D).$$

If $J \in \mathcal{F}_r(D)$, then there exists some $a \in D^*$ such that $I = aJ \subset D$, and $I_r = aJ_r = aI = I$. Hence $I \in I_r(D)$ and $J = a^{-1}I$. If $I \in I_r(D)$ and $a \in D^*$, then $J = a^{-1}I \subset K$ and $I = aJ$. If $J \subset K$, $a \in D^*$ and $I = aJ \in I_r(D)$, then $J$ is $D$-fractional, and $J_r = (a^{-1})_r = a^{-1}I_r = a^{-1}I = J$, hence $J \in \mathcal{F}_r(D)$.

In all arguments above, $J$ is $r$-finitely generated if and only if $I$ is $r$-finitely generated, and thus also the second set of equalities holds.

**Examples 2.1.5 (Some (weak) ideal systems).**

1. **Trivial systems.** Let $K$ be a monoid. There are two trivial weak ideal systems $y, y_1$ on $K$, defined as follows.

   $y_1: \mathbb{P}(K) \to \mathbb{P}(K)$, defined by $X_{y_1} = K$ for all subsets $X \subset K$.

   $y: \mathbb{P}(K) \to \mathbb{P}(K)$, defined by $X_y = \{0\}$ if $X \subset \{0\}$, and $X_y = K$ if $X \not\subset \{0\}$.

   It is easily checked that $y$ and $y_1$ are weak ideal systems of $K$.

   Let $K$ be divisible. Then $K$ and $\{0\}$ are the only semigroup ideals of $K$. Hence $y$ and $y_1$ are the only weak ideal systems of $K$.

2. **The semigroup system.** Let $K$ be a monoid and $D \subset K$ a submonoid. The *semigroup system* of $D$ defined on $K$ is the system $s(D): \mathbb{P}(K) \to \mathbb{P}(K)$, defined by

   $$\emptyset_{s(D)} = \{0\}, \quad X_{s(D)} = DX = \bigcup_{a \in X} Da \text{ if } X \not= \emptyset.$$

   It is plain that $s(D)$ is an ideal system of $D$, and $\mathcal{M}_{s(D)}(K) = \{J \subset K \mid 0 \in J \text{ and } DJ = J\}$. In particular, $\mathcal{M}_{s(D)}(D)$ is the set of all semigroup ideals of $D$. If $c \in K$, then $\{c\}_{s(D)} = cD$, the union of any family of $s(D)$-modules is again an $s(D)$-module, and if $J_1, J_2 \in \mathcal{M}_{s(D)}(K)$, then $J_1, J_2 \in \mathcal{M}_{s(D)}(K)$.

   If $K = q(D)$, then $\mathcal{F}_{s(D)}(D)$ is the set $\{J \subset D \mid J \neq \emptyset\}$ of all fractional (semigroup) ideals of $D$, and

   $$\mathcal{F}_{s(D),f}(D) = \{c_1D \cup \cdots \cup c_mD \mid m \in \mathbb{N}, \ c_1, \ldots, c_m \in K\}.$$

   If $K$ is divisible, then $s(K)$ is the only ideal system of $K$. In fact, it coincides with the trivial system $y$ considered in Example 1.

3. **The Dedekind system.** Let $K$ be a ring and $D \subset K$ a subring. The *Dedekind system* of $D$ defined on $K$ is the system $d(D): \mathbb{P}(K) \to \mathbb{P}(K)$, defined by

   $$X_{d(D)} = \{a_1x_1 + \cdots + a_nx_n \mid n \in \mathbb{N}, \ x_1, \ldots, x_n \in X, \ a_1, \ldots, a_n \in K\} = k(X) \text{ for all } X \in \mathbb{P}(K),$$

   where $X_{d(D)}$ is the $D$-submodule of $K$ generated by $X$.

   It is plain that $d(D)$ is an ideal system of $D$, and $\mathcal{M}_{d(D)}(K)$ is the set of all $D$-submodules of $K$. A $D$-module $J \in \mathcal{M}_{d(D)}(K)$ is $d(D)$-finitely generated if and only if it is a finitely generated $D$-module.
$\mathcal{I}_{d(D)}(D)$ is the set of all ideals of $D$, and if $c \in K$, then $\{c\}_{d(D)} = \{c\}_{s(D)} = cD$. For every family $(J_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{M}_{d(D)}(K)$, we have

$$\left( \bigcup_{\lambda \in \Lambda} J_\lambda \right)_{d(K)} = \sum_{\lambda \in \Lambda} J_\lambda.$$ 

If $J_1, J_2 \in \mathcal{I}_{d(K)}(K)$, then $J_1 \cdot d(K) J_2$ is the additive abelian group generated by $J_1 J_2$.

If $K$ is a field, then $d(K) = s(K)$ is the only ideal system of $K$.

4. The system of homogenous ideals. Let $K$ be a graded ring with homogeneous components $(K_i)_{i \geq 0}$, that means,

$$K = \bigoplus_{i \geq 0} K_i$$

as an additive abelian group, and $K_i K_j \subseteq K_{i+j}$ for all $i, j \geq 0$.

An element $x \in K$ is called homogenous of degree $i \geq 0$ if $x \in K_i$. Every $x \in K$ has a unique representation

$$x = \sum_{i \geq 0} x_i, \text{ where } x_i \in K_i \text{ and } x_i = 0 \text{ for almost all } i \geq 0.$$ 

In this representation we call $x_i$ the $i$-th homogenous component of $x$. For every subset $X \subseteq K$ let $X^h$ be the set of all homogeneous components of elements of $X$. An ideal $J \subseteq K$ is called homogenous if $J^h \subseteq J$, equivalently

$$J = \sum_{i \geq 0} J \cap K_i.$$ 

Then $X^h = (X^h)_{d(K)}$ is the smallest homogenous ideal containing $X$, and

$$h : \mathcal{P}(K) \rightarrow \mathcal{P}(K), \quad X \mapsto X^h,$$

is a weak ideal system of $K$.

6. The system of filters. Let $(K, \leq, 0, 1)$ be a lattice. That means, $(K, \leq)$ is a partially ordered set, $0 = \max(K)$, $1 = \min(K)$, and any two elements $a, b \in K$ possess a supremum $ab = a \vee b$ and an infimum $a \wedge b$. Then $K$ is a monoid with unit 1 and zero 0.

If $M$ is a set, then $(K, \leq, 0, 1) = (\mathcal{P}(M), \subseteq, M, \emptyset)$ is a lattice (the subset lattice of $M$).

Let $(K, \leq, 0, 1)$ be a lattice. A non-empty subset $F \subseteq K$ is called a filter if for all $a, b \in K$ the following assertions hold:

- If $a \leq b$ and $a \in F$, then $b \in F$.
- If $a, b \in F$, then $ab \in F$.

For a subset $X \subseteq K$, let $X_f$ be the smallest filter containing $X$. Then $\emptyset_f = \{\emptyset\}$, and if $X \neq \emptyset$, then

$$X_f = \bigcap_{F \subseteq K \text{ is a filter}} F = \{x \in K \mid \text{ there exist } x_1, \ldots, x_r \in X \text{ such that } x \geq x_1 \cdot \ldots \cdot x_r\}.$$ 

The map $f : \mathcal{P}(X) \rightarrow \mathcal{P}(X), \quad X \mapsto X_f$, is a weak ideal system on $K$, and for every $c \in K$ it follows that $\{c\}_f = \{x \in K \mid x \geq c\} = cK$. [All this is easily checked, observing that, for all $x, y \in K$, $x \leq y$ holds if and only if $xy = y$].

Theorem 2.1.6. Let $K$ be a monoid, $D \subseteq K$ a submonoid and $r$ a weak module system on $K$.

1. $D_r$ is an $r$-monoid. In particular, $\{1\}_r$ is the smallest $r$-monoid in $K$, and if $D \subseteq \{1\}_r$, then $\{1\}_r = D_r$.

2. Let $r$ be a weak $D$-module system. Then $r$ is a weak $D_r$-module system, $\{1\}_r = D_r$, and if $X \subseteq K$, then $X_r = DX_r = D_r X_r = (DX)_r$ and $D_r \subseteq (X_r : X)$.

3. $r$ is a weak $D$-module system if and only if $D \subseteq \{1\}_r$. In particular:
   
   (a) $r$ is a weak $\{1\}_r$-module system.
In particular, $D$  

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thus an $\{ c \}_r = cD$, for all $c \in K$.

(c) If $r$ is an ideal system of $D$, then $\{ c \}_r = cD$ for all $c \in K$.

4. If $r$ is a weak ideal system of $D$ and $I$, $J \in \mathcal{I}_r(D)$, then $I \cdot r, J \subset I \cap J$.

Proof. 1. By Theorem 2.1.2.6, $D D = D$ implies $D_r D_r = D_r$. Hence $D_r \subset K$ is a submonoid and thus an $r$-monoid. In particular, $\{ 1 \}_r = \{ 0, 1 \}_r$ is an $r$-monoid, and it is the smallest $r$-monoid in $K$.

If $D \subset \{ 1 \}_r$, then $D_r \subset \{ 1 \}_r \subset D_r$, and therefore $\{ 1 \}_r = D_r$.

2. If $X \subset K$, then $X_r = DX_r$ by definition, and therefore $X_r = D_r X_r = (DX)_r$ by Theorem 2.1.2.6. In particular, $D_r \subset D_r \{ 1 \}_r = \{ 1 \}_r$ and therefore $D_r = \{ 1 \}_r$. If $J \in \mathcal{M}_r(K)$, then $J = J_r$ and $J = DJ$ implies $J = D_r J$, and therefore $r$ is a weak $D_r$-module system.

If $X \subset K$, then $(X_r : X)$ is an $r$-module and $1 \in (X_r : X)$. Hence it follows that $D_r = \{ 1 \}_r \subset (X_r : X)$.

3. If $r$ is a weak $D$-module system, then $\{ 1 \}_r = D \{ 1 \}_r \subset D$. Conversely, if $D \subset \{ 1 \}_r$ and $J \in \mathcal{M}_r(K)$, then $J \subset DJ = \{ 1 \}_r J \subset J_r = J$, and thus $r$ is a weak $D$-module system.

(a) Since $r$ is obviously a $\{ 0, 1 \}$-module system and $\{ 0, 1 \}_r = \{ 1 \}_r$, it is also an $\{ 1 \}_r$-module system.

(b), (c) If $r$ is a $D$-module system, then $\{ c \}_r = c \{ 1 \}_r = cD_r$, and if $r$ is an ideal system of $D$, then $D_r = D$.

4. Let $r$ be a weak ideal system of $D$ and $I, J \in \mathcal{I}_r(D)$. Then $I \cap J \in \mathcal{I}_r(D)$, and since $I$ and $J$ are semigroup ideals, it follows that $IJ \subset I \cap J$, and consequently $I \cdot r, J = (IJ)_r \subset I \cap J$. $\square$

2.2. Finitary and noetherian (weak) module systems

Theorem and Definition 2.2.1. Let $K$ be a monoid and $r$ a weak module system on $K$.

1. The following assertions are equivalent:

(a) For every subset $X \subset K$, we have

$$X_r = \bigcup_{E \in \mathcal{P}_r(X)} E_r.$$

(b) For all $X \subset K$ and $a \in X$, there exists a finite subset $E \subset X$ such that $a \in E_r$.

(c) For every directed family $(X_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{P}(K)$ we have

$$\left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r = \bigcup_{\lambda \in \Lambda} (X_\lambda)_r.$$

(d) The union of every directed family of $r$-modules is again an $r$-module.

(e) If $X \subset K$, $J \in \mathcal{M}_r(K)$ and $J \subset X_r$, then there exists some $E \in \mathcal{P}_r(X)$ such that $J \subset E_r$.

If $r$ satisfies these equivalent conditions, then $r$ is called finitary.

2. If $r$ is finitary, $X \subset K$ and $X_r \in \mathcal{M}_r(K)$, then there exists some $E \in \mathcal{P}_r(X)$ such that $E_r = X_r$.

3. If $r$ and $q$ are finitary weak module systems on $K$, then $r = q$ if and only if $E_r = E_q$ for all $E \in \mathcal{P}_r(X)$.

Proof. 1. (a) $\Rightarrow$ (b) Obvious.

(b) $\Rightarrow$ (c) If

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda,$$

then $X_r \supset \bigcup_{\lambda \in \Lambda} (X_\lambda)_r$. 

To prove the converse, let \( x \in X_r \) and \( E \subset X \) finite such that \( x \in E_r \). Since \( (X_\alpha)_{\alpha \in \Lambda} \) is directed, there exists some \( \alpha \in \Lambda \) such that \( E \subset X_\alpha \), hence \( E_r \subset (X_\alpha)_r \), and consequently
\[ x \in E_r \subset \bigcup_{\alpha \in \Lambda} (X_\alpha)_r. \]

(c) \( \Rightarrow \) (d) Let \( (X_\alpha)_{\alpha \in \Lambda} \) be a directed family of \( r \)-modules. Then
\[ \left( \bigcup_{\alpha \in \Lambda} X_\alpha \right)_r = \bigcup_{\alpha \in \Lambda} (X_\alpha)_r = \bigcup_{\alpha \in \Lambda} X_\alpha. \]

(d) \( \Rightarrow \) (a) Obviously,
\[ \bigcup_{E \in \mathcal{P}_f(X)} E_r \subset X_r. \]

For \( E, F \in \mathcal{P}_f(X) \), we have \( E \cup F \subset (E \cup F)_r \). Hence \( (E_r)_{E \in \mathcal{P}_f(X)} \) is directed, and we obtain
\[ X_r = \left( \bigcup_{E \in \mathcal{P}_f(X)} E \right)_r \subset \left( \bigcup_{E \in \mathcal{P}_f(X)} E_r \right) = \bigcup_{E \in \mathcal{P}_f(X)} E_r. \]

(b) \( \Rightarrow \) (e) Suppose that \( X \subset K \) and \( J = F_r \subset X_r \), where \( F \in \mathcal{P}_f(K) \). For every \( c \in F \), there is some \( E(c) \in \mathcal{P}_f(X) \) such that \( c \in E(c)_r \). Then
\[ E = \bigcup_{c \in E} E(c) \in \mathcal{P}_f(X), \quad F = \bigcup_{c \in E} E(c)_r \subset E_r \quad \text{and thus} \quad J = F_r \subset E_r. \]

(e) \( \Rightarrow \) (b) If \( X \subset K \) and \( a \in X_r \), then \( \{a\}_r \in \mathcal{M}_{r, f}(K) \) and \( \{a\}_r \subset X_r \). Hence there exists a finite subset \( E \subset X \) such that \( a \in \{a\}_r \subset E_r \).

1. If \( r \) is finitary, \( X \subset K \) and \( X_r \in \mathcal{M}_{r, f}(K) \), then we apply 1.(e) with \( J = X_r \in \mathcal{M}_{r, f} \) to obtain \( X_r \subset E_r \) for some \( E \in \mathcal{P}_f(X) \), and thus \( X_r = E_r. \)

2. By 1.(a), two finitary weak module systems coincide if and only if they coincide on finite sets. \( \Box \)

**Theorem and Definition 2.2.2.** Let \( K \) be a monoid and \( D \subset K \) a submonoid.

1. Let \( r: \mathcal{P}_f(K) \to \mathcal{P}(K) \) be a map such that, for all \( c \in K \) and \( E, F \in \mathcal{P}_f(K) \) the following conditions are fulfilled:

\begin{align*}
\text{M1}_r, & \quad E \cup \{0\} \subset E_r, \\
\text{M2}_r, & \quad \text{If } E \subset F_r, \text{ then } E_r \subset F_r, \\
\text{M3}_r, & \quad cE_r \subset (cE)_r.
\end{align*}

Then there exists a unique finitary weak module system \( \mathfrak{r} \) on \( K \) satisfying \( \mathfrak{r} | \mathcal{P}_f(K) = r \). It is given by
\[ X_{\mathfrak{r}} = \bigcup_{E \in \mathcal{P}_f(X)} E_r \quad \text{for all } \ X \subset K. \]

\( \mathfrak{r} \) is a weak \( D \)-module system if and only if \( cD \subset \{c\}_r \) for all \( c \in K \), and it is a module system if and only if \( (cE)_r = cE_r \) for all \( c \in K \) and \( E \in \mathcal{P}_f(K) \).

\( \mathfrak{r} \) is called the total system defined by \( r \) and is usually again denoted by \( r \).

2. Let \( r \) be a weak module system on \( K \). Then there exists a unique finitary weak module system \( r_{\mathfrak{t}} \) on \( K \) such that \( E_r = E_{\mathfrak{t}_r} \) for all finite subsets of \( K \). It is given by
\[ X_{\mathfrak{t}_r} = \bigcup_{E \in \mathcal{P}_f(X)} E_r \quad \text{for all } \ X \subset K, \]
and it has the following properties:

(a) \( X_{\mathfrak{t}_r} \subset X_r \) for all \( X \in \mathcal{P}(K) \), \( \mathcal{M}_r(K) \subset \mathcal{M}_{\mathfrak{t}_r}(K) \), and \( \mathcal{M}_{r, f}(K) = \mathcal{M}_{\mathfrak{t}, f}(K). \)
(b) \((r_I)_I = r_I\), and \(r\) is finitary if and only if \(r = r_I\).

(c) If \(r\) is a module system, then \(r_I\) is a module system, too.

(d) \(r_I\) is a weak \(D\)-module system \(\{\) a weak ideal system of \(D\) \(\}\) if and only if \(r\) is a weak \(D\)-module system \(\{\) a weak ideal system of \(D\) \(\}\).

\(r_I\) is called the \(\) finitary system associated with \(r\).

**Proof.** 1. Let \(\tilde{\tau}: \mathcal{P}(K) \to \mathcal{P}(K)\) be defined by

\[
X_{\tilde{\tau}} = \bigcup_{E \in \mathcal{P}_I(X)} E_r \quad \text{for all} \quad X \subset K.
\]

We prove that \(\tilde{\tau}\) satisfies the properties \(\mathbf{M1}, \mathbf{M2}, \mathbf{M3}\) for all \(c \in K\) and \(X, Y \subset K\). Once this is done, it is obvious that \(E_{\tilde{\tau}} = E_r\) for all \(E \in \mathcal{P}_I(X)\). Hence \(\tilde{\tau} \mid \mathcal{P}_I(K) = r\), and \(\tilde{\tau}\) is finitary.

\(\mathbf{M1}\). Since \(E \cup \{0\} \subset E_r\) for all \(E \in \mathcal{P}_I(X)\), we obtain \(X \cup \{0\} \subset X_{\tilde{\tau}}\).

\(\mathbf{M2}\). Suppose that \(X \subset Y_{\tilde{\tau}}\), and let \(x \in X_r\). There exists some \(E \in \mathcal{P}_I(X)\) such that \(x \in E_r\), and

\[
E \subset Y_{\tilde{\tau}} = \bigcup_{F \in \mathcal{P}_I(Y)} F_r.
\]

For every \(e \in E\), there exists some \(F(e) \in \mathcal{P}_I(Y)\) such that \(e \in F(e)_r\), and we obtain

\[
F = \bigcup_{e \in E} F(e) \in \mathcal{P}_I(Y), \quad \text{and} \quad E \subset \bigcup_{e \in E} F(e)_r \subset F_r,
\]

hence \(E_r \subset F_r \subset Y_{\tilde{\tau}}\) and \(x \in Y_{\tilde{\tau}}\).

\(\mathbf{M3}\). Note that \(\mathcal{P}_I(cX) = \{cE \mid E \in \mathcal{P}_I(X)\}\). Hence it follows that

\[
cX_{\tilde{\tau}} = \bigcup_{E \in \mathcal{P}_I(X)} cE_r \subset \bigcup_{E \in \mathcal{P}_I(X)} (cE)_r = \bigcup_{F \in \mathcal{P}_I(cX)} F_r = (cX)_{\tilde{\tau}},
\]

and \(cX_{\tilde{\tau}} = (cX)_{\tilde{\tau}}\) holds if and only if \(cE_r = (cE)_r\) for all \(E \in \mathcal{P}_I(X)\). Consequently, \(\tilde{\tau}\) is a module system if and only if \(cE_r = (cE)_r\) for all \(E \in \mathcal{P}_I(X)\). By Theorem 2.1.6.3 it follows that \(\tilde{\tau}\) is a weak \(D\)-module system if and only if \(cD \subset \{c\}_r\) for all \(c \in K\).

It remains to prove the uniqueness of \(\tilde{\tau}\). If \(\tilde{\tau}\) is any finitary weak module system on \(K\) satisfying \(\tilde{\tau} \mid \mathcal{P}_I(K) = r\), then

\[
X_{\tilde{\tau}} = \bigcup_{E \in \mathcal{P}_I(X)} E_{\tilde{\tau}} = \bigcup_{E \in \mathcal{P}_I(X)} E_r = X_{\tilde{\tau}} \quad \text{for all} \quad X \subset K, \quad \text{and therefore} \quad \tilde{\tau} = \tilde{\tau}.
\]

2. By 1., applied with \(r \mid \mathcal{P}_I(X)\), there exists a unique weak module system \(r_I\) on \(K\) such that \(E_I = E_r\) for all \(E \in \mathcal{P}_I(X)\), and if \(X \subset K\), then \(X_{r_I}\) is given as asserted.

(a) If \(X \in \mathcal{P}(K)\), then \(E_r \subset X_r\) for all \(E \in \mathcal{P}_I(X)\), and therefore \(X_{r_I} \subset X_r\). If \(X \in \mathcal{M}_r(K)\), then \(X_{r_I} \subset X_r = X\) and therefore \(X = X_{r_I} \in \mathcal{M}_r(K)\). Since \(E_r = E_{r_I}\) for all \(E \in \mathcal{P}_I(K)\), it follows that \(\mathcal{M}_{r,I}(K) = \mathcal{M}_{r,I}(K)\).

(b) By the uniqueness of \(r_I\) it follows that \(r_I = r\) if and only if \(r\) is finitary, and since \(r_I\) is finitary, we obtain \((r_I)_I = r_I\).

(c) If \(r\) is a module system, then \((cE)_r = cE_r\) for all \(c \in K\) and \(E \in \mathcal{P}_I(K)\), and then \(r_I\) is a module system by 1.

(d) Since \(\{1\}_r = \{1\}_{r_I}\), Theorem 2.1.6.3 implies that \(r_I\) is a weak \(D\)-module system if and only if \(r\) is a weak \(D\)-module system. In this case, \(D_r = \{1\}_r = \{1\}_{r_I} = D_{r_I}\), and therefore \(r_I\) is a weak ideal system of \(D\) if and only if \(r\) is a weak ideal system of \(D\).
Remark 2.2.3.
1. Let $K$ be a monoid, $D \subset K$ a submonoid and $s(D)$ the semigroup system of $D$ defined on $K$ (see Example 2.1.5.2). If $\emptyset \neq X \subset K$, then

$$X_{s(D)} = DX = \bigcup_{a \in X} Da \subset \bigcup_{E \in \mathcal{F}_\tau(X)} DE = \bigcup_{E \in \mathcal{F}_\tau(X)} E_{s(D)} \subset DX,$$

and therefore $s(D)$ is finitary.

2. Let $K$ be a ring, $D \subset K$ a subring and $d(D)$ the Dedekind system of $D$ defined on $K$ (see Example 2.1.5.3). Since every $D$-module is the union of its finitely generated submodules, the system $d(D)$ is finitary.

Example 2.2.4. Let $K$ be a topological monoid (that is, a monoid equipped with a topology such that the multiplication $K \times K \to K$, $(x, y) \mapsto xy$, is continuous). Let $c: \mathbb{P}(K) \to \mathbb{P}(K)$ be defined by

$$X_c = \overline{X_{s(K)}} = \begin{cases} \{0\} & \text{if } X = \emptyset, \\ X^K & \text{if } X \neq \emptyset. \end{cases}$$

Then $X_c$ is the smallest closed semigroup ideal of $K$ containing $X$. If $\emptyset \neq X \subset K$ and $z \in K$, then $zX^K \subset zXK \subset X^K$, and therefore $c$ is a weak ideal system on $K$. If $z \in K$ is such that the map $\tau_z: K \to K$, defined by $\tau_z(x) = zx$, is closed, then $(zX)_c = zX_c$ for all $X \in \mathbb{P}(K)$. In particular, if $\tau_z$ is a closed map for all $z \in K$, then $c$ is an ideal system of $K$. In particular, this holds if $K$ is compact. In general however, $c$ is not finitary.

We consider the additive monoid $\mathbb{R}_{\geq 0}$. For every $z \in \mathbb{R}_{\geq 0}$, the map $x \mapsto z + x$ is closed, and thus $c$ is an ideal system on $\mathbb{R}_{\geq 0}$. If $\gamma \in \mathbb{R}_{\geq 0}$ and $X = (\gamma, \infty)$, then $X_c = [\gamma, \infty)$, but for every finite subset $E \subset (\gamma, \infty)$, it follows that $E_c = \min(E, \infty) \subset (\gamma, \infty)$. Hence $X_{c\gamma} = X$, $c \neq \alpha$, and $c$ is not finitary.

Theorem und Definition 2.2.5. Let $K$ be a monoid, $D \subset K$ a submonoid and $r: \mathbb{P}(K) \to \mathbb{P}(K)$ a weak ideal system of $D$ defined on $K$.

1. The following conditions are equivalent:
   (a) $\mathcal{I}_r(D)$ satisfies the ACC:
      - For every sequence $(J_n)_{n \geq 0}$ in $\mathcal{I}_r(D)$ satisfying $J_n \subset J_{n+1}$ for all $n \geq 0$, there exists some $m \geq 0$ such that $J_n = J_m$ for all $n \geq m$.
      - Every non-empty set of $r$-ideals has a maximal element.
   (b) For every subset $X \subset D$, there exists some $E \in \mathcal{F}_\tau(X)$ such that $X \subset E_r$ (and then $X_r = E_r$).
   (c) $r | \mathbb{P}(D)$ is finitary, and $\mathcal{I}_r(D) = \mathcal{I}_{r\tau}(D)$.

   If these conditions are fulfilled, then $r$ is called a noetherian weak ideal system, and $D$ is called $r$-noetherian.

2. $D$ is $r$-noetherian if and only if $D$ is $r_\tau$-noetherian.

3. If $K = q(D)$ and $D$ is $r$-noetherian, then $\mathcal{F}_r(D) = \mathcal{F}_{r\tau}(D)$ (that is, every fractional $r$-ideal is $r$-finitely generated).

Proof. 1. (a) $\Rightarrow$ (b) Let $X \subset D$ and $\Omega = \{E_r \mid E \in \mathcal{P}_\tau(X)\}$. By assumption, there exists some $E \in \mathcal{P}_\tau(X)$ such that $E_r$ is maximal in $\Omega$, and we assert that $E_r = X_r$. Indeed, if $E_r \subset X_r$, then $X \not\subset E_r$, and if $c \in X \setminus E_r$, then $E_r \subset (E \cup \{c\})_r$, which contradicts the maximality of $E_r$.

Clearly, if $E \in \mathcal{P}_\tau(X)$, then $X \subset E_r$ if and only if $X_r = E_r$.

(b) $\Rightarrow$ (c) By (b), every $r$-ideal is $r$-finitely generated. Hence $\mathcal{I}_r(D) = \mathcal{I}_{r\tau}(D)$, and $r$ is finitary.
2. THE FORMALISM OF MODULE AND IDEAL SYSTEMS

If \( r \) is an \( r \)-ideal (since \( r \mid P(D) \) is finitary), and there exists some \( E \in P(J) \) such that \( J = E_r \). There is some \( m \in \mathbb{N} \) such that \( E \subseteq J_m \), and then it follows that \( J_n = J_m \) for all \( n \geq m \).

2. If \( D \) is \( r \)-noetherian, then \( D \) is \( r \)-noetherian, since \( I_r(D) \subseteq I_n(D) \) by Theorem 2.2.2.2(a). If \( D \) is \( r \)-noetherian, then \( r \mid P(D) = r_t \mid P(D) \) by 1.(c), and thus \( D \) is \( r \)-noetherian.

3. Since \( I_r(D) = I_{r,t}(D) \), Theorem 2.1.4 implies

\[
\mathcal{F}_r(D) = \{ a^{-1} I \mid I \in I_r(D) , \ a \in D^* \} = \{ a^{-1} I \mid I \in I_{r,t}(D) , \ a \in D^* \} = \mathcal{F}_{r,t}(D). \quad \square
\]

2.3. Comparison and mappings of module systems

**Definition 2.3.1.** Let \( K \) be a monoid, and let \( r \) and \( q \) be weak module systems on \( K \). We call \( q \) **finer** than \( r \) and \( r \) **coarser** than \( q \) and write \( r \leq q \) if \( X_r \subseteq X_q \) for all subsets \( X \subseteq K \).

\( \leq \) is a partial order on the set of all weak module systems on \( K \).

**Theorem 2.3.2.** Let \( K \) be a monoid, and let \( r \) and \( q \) be weak module systems on \( K \). Then \( r_t \leq r \), and the following assertions are equivalent:

(a) \( r \leq q \).
(b) \( X_r = (X_q)_q \) for all subsets \( X \subseteq K \).
(c) \( M_q(K) \subseteq M_r(K) \).

If \( r \) is finitary, then there are also equivalent:

(d) \( E_r \subseteq E_q \) for all finite subsets \( E \subseteq K \).
(e) \( M_q(K) \subseteq M_r(K) \).
(f) \( M_r(K) \subseteq M_r(K) \).
(g) \( r \leq q_t \).

**Proof.** It follows by Theorem 2.2.2 that \( r_t \leq r \).

(a) \( \Rightarrow \) (b) If \( X \subseteq K \), then \( X_r \subseteq X_q \) by assumption, hence \( (X_r)_q \subseteq X_q \), and since \( X \subseteq X_r \), it follows that \( X_q \subseteq (X_r)_q \).

(b) \( \Rightarrow \) (c) If \( J \in M_q(K) \), then \( J_r \subseteq (J_q)_q = J_q = J_r \), and therefore \( J = J_r \in M_r(K) \).

(c) \( \Rightarrow \) (a) If \( X \subseteq K \), then \( X_q \in M_q(K) \subseteq M_r(K) \), and therefore \( X_q = (X_q)_r \supset X_r \).

Assume now that \( r \) is finitary.

(a) \( \Rightarrow \) (d) Obvious.

(d) \( \Rightarrow \) (e) If \( J \in M_q(K) \), then

\[
J = J_q = \bigcup_{E \in \mathcal{P}(J)} E_q \supset \bigcup_{E \in \mathcal{P}(J)} E_r = J_r \supset J \quad \text{implies that} \quad J = J_r \in M_r(K).
\]

(e) \( \Rightarrow \) (f) \( M_{q,t}(K) = M_{q_t}(K) \subseteq M_{q_t}(K) \subseteq M_r(K) \).

(f) \( \Rightarrow \) (g) If \( E \in \mathcal{P}(K) \), then \( E_q \in M_q(K) \subseteq M_r(K) \), and therefore \( E_q = (E_q)_r \supset E_r \). Consequently, if \( X \subseteq K \), then

\[
X_r = \bigcup_{E \in \mathcal{P}(X)} E_r \subseteq \bigcup_{E \in \mathcal{P}(X)} E_q = X_q, \quad \text{and therefore} \quad r \leq q_t.
\]

(g) \( \Rightarrow \) (a) \( r \leq q_t \leq q \). \( \square \)
Theorem 2.3.3. Let $K$ be a monoid and $D \subset K$ a submonoid.

1. Let $r: \mathbb{P}(K) \to \mathbb{P}(K)$ be a weak module system on $K$. Then $r$ is a $D$-module system if and only if $s(D) \leq r$.
2. Let $r$ and $q$ be ideal systems of $D$ such that $r \leq q$. If $D$ is $r$-noetherian, then $D$ is $q$-noetherian.

**Proof.** 1. By definition, $r$ is a $D$-module system if and only if $\mathcal{M}_r(K) \subset \mathcal{M}_{s(D)}(K)$, and by Theorem 2.3.2 this is equivalent to $s(D) \leq r$.
2. If $r \leq q$, then $I_q(D) \subset I_r(D)$. □

**Definition 2.3.4.** Let $\varphi: K \to L$ be a monoid homomorphism, $r$ a weak module system on $K$ and $q$ a weak module system on $L$.

1. Let $\varphi^*q: \mathbb{P}(K) \to \mathbb{P}(K)$ be defined by $X_{\varphi^*q} = \varphi^{-1}(\varphi(X)_q)$. $\varphi^*q$ is called the pullback of $q$ under $\varphi$.
2. $\varphi$ is called an $(r,q)$-homomorphism if $\varphi(X_i) \subset \varphi(X)_q$ for all subsets $X \subset K$. We denote by $\text{Hom}_{(r,q)}(K,L)$ the set of all $(r,q)$-homomorphisms $\varphi: K \to L$.

**Remarks 2.3.5.** Let $\varphi: K \to L$ and $\psi: L \to M$ be monoid homomorphisms, $r$ a weak module system on $K$, $q$ a weak module system on $L$ and $y$ a weak module system on $M$.

1. Let $r$ be finitary. Then $\varphi$ is an $(r,q)$-homomorphism if and only if $\varphi(E_r) \subset \varphi(E)_r$ for all $E \in \mathbb{P}(K)$.
2. $(\psi \circ \varphi)^*y = \varphi^*(\psi^*y)$.
3. If $\varphi$ is an $(r,q)$-homomorphism and $\psi$ is a $(q,y)$-homomorphism, then $\psi \circ \varphi$ is an $(r,y)$-homomorphism.

In particular, monoids together with weak module systems form a category.

**Theorem 2.3.6.** Let $\varphi: K \to L$ a monoid homomorphism, $r$ a weak module system on $K$ and $q$ a weak module system on $L$.

1. $\varphi^*q$ is a weak module system on $K$, $\mathcal{M}_{\varphi^*q}(K) = \{\varphi^{-1}(J) \mid J \in \mathcal{M}_q(L)\}$, and if $q$ is finitary, then $\varphi^*q$ is also finitary.

   If $B \subset L$ is a submonoid and $q$ is a weak $B$-module system, then $\varphi^*q$ is a weak $\varphi^{-1}(B)$-module system.

2. $\varphi$ is an $(r,q)$-homomorphism if and only if $r \leq \varphi^*q$ [that is, if and only if $\varphi^{-1}(J) \in \mathcal{M}_r(K)$ for all $J \in \mathcal{M}_q(L)$].

**Proof.** 1. We check the properties M1, M2 and M3 for $\varphi^*q$. Let $X, Y \subset K$ and $c \in K$.

   M1. $X_{\varphi^*q} = \varphi^{-1}(\varphi(X)_q) \supset \varphi^{-1}(\varphi(X) \cup \{0\}) \supset X \cup \{0\}$.

   M2. If $X \subset Y_{\varphi^*q} = \varphi^{-1}(\varphi(Y)_q)$, then $\varphi(X) \subset (\varphi(Y)_q)$, hence $\varphi(X)_q \subset (\varphi(Y)_q)_q$, and therefore $X_{\varphi^*q} = \varphi^{-1}(\varphi(X)_q) \subset \varphi^{-1}(\varphi(Y)_q) = Y_{\varphi^*q}$.

   M3. $cX_{\varphi^*q} = \varphi(c)\varphi(X_{\varphi^*q}) \subset \varphi(c)\varphi(X)_q \subset [\varphi(c)\varphi(X)]_q = \varphi(cX)_q$. Hence it follows that $cX_{\varphi^*q} \subset \varphi^{-1}(\varphi(cX)_q) = \varphi(c)^*q$.

   Let $q$ be finitary and $X \subset K$. Then $\mathbb{P}_r(X) = \{\varphi(E) \mid E \in \mathbb{P}_r(X)\}$ and therefore $X_{\varphi^*q} = \varphi^{-1}(\varphi(X)_q) = \varphi^{-1}\left(\bigcup_{E \in \mathbb{P}_r(X)} \varphi(E)_q\right) = \bigcup_{E \in \mathbb{P}_r(X)} \varphi^{-1}(\varphi(E)_q) = \bigcup_{E \in \mathbb{P}_r(X)} E_{\varphi^*q}$.

   Hence $\varphi^*q$ is finitary.
Let $B \subseteq L$ be a submonoid such that $q$ is a weak $B$-module homomorphism. If $X \subseteq K$, then $\varphi^{-1}(B)X_{\varphi q} = \varphi^{-1}(B)\varphi^{-1}(\varphi(X)_q) \subseteq \varphi^{-1}(B\varphi(X)_q) = \varphi^{-1}(\varphi(X)_q) = X_{\varphi q}$, and therefore $\varphi^*q$ is a weak $\varphi^{-1}(B)$-module system.

It remains to prove that $\mathcal{M}_{\varphi q}(K) = \{\varphi^{-1}(J) \mid J \in \mathcal{M}_q(L)\}$. If $I \in \mathcal{M}_{\varphi q}(K)$, then $\varphi(I) \in \mathcal{M}_q(L)$, and $I = I_{\varphi q} = \varphi^{-1}(\varphi(I)_q)$. Conversely, if $J \in \mathcal{M}_q(L)$, then $\varphi^{-1}(J)_{\varphi q} = \varphi^{-1}(\varphi^{-1}(J)_q) \subseteq \varphi^{-1}(J_q) = \varphi^{-1}(J) \subseteq \varphi^{-1}(J)_{\varphi q}$.

Hence equality holds, and $\varphi^{-1}(J) \in \mathcal{M}_{\varphi q}(K)$.

2. If $X \subseteq K$, then $\varphi(X) \subseteq \varphi(X)_q$ holds if and only if $X \subseteq \varphi^{-1}(\varphi(X)_q) = X_{\varphi q}$. Consequently, $\varphi$ is an $(r, q)$-homomorphism if and only if $r \leq \varphi^* q$.

**Theorem und Definition 2.3.7.** Let $\varepsilon : K \rightarrow K'$ be a surjective monoid homomorphism, $D \subseteq K$ a submonoid, $D' = \varepsilon(D)$, and $G \subseteq D'^*$ a subgroup such that $\varepsilon^{-1}(\varepsilon(x)) = xG$ for all $x \in K$. If $\pi : K \rightarrow K/G$ denotes the natural epimorphism, defined by $\pi(a) = aG$, then $\varepsilon$ factorizes in the form $\varepsilon : K \xrightarrow{\pi} K/G \xrightarrow{\sim} K'$ and induces an isomorphism $D/G \xrightarrow{\sim} D'$.

For a weak $D$-module system $r$ on $K$ we define $\varepsilon(r) : \mathbb{P}(K') \rightarrow \mathbb{P}(K')$ by $X_{\varepsilon r} = \varepsilon^{-1}(X)_r$ for all $X \subseteq K'$.

1. $\varepsilon(r)$ is a weak $D'$-module system on $K'$. If $X \subseteq K$, then $\varepsilon(X)_{\varepsilon r} = \varepsilon(X)_r$, and $\varepsilon^*\varepsilon(r) = r$.

2. The assignment $r \mapsto \varepsilon(r)$ defines a bijective map from the set of all weak $D$-module systems on $K$ onto the set of all weak $D'$-module systems on $K'$. If $r'$ is a weak $D'$-module system on $K'$, then $r' = \varepsilon(r')$ is a weak $D$-module system on $K$, and $\varepsilon^* r' = r$.

3. If $r$ is a weak $D$-module system on $K$, then the maps $\mathcal{M}_r(K) \rightarrow \mathcal{M}_{\varepsilon r}(K')$, $J \mapsto \varepsilon(J)$ and $\mathcal{M}_{\varepsilon r}(K') \rightarrow \mathcal{M}_r(K)$, $J' \mapsto \varepsilon^{-1}(J')$ are inclusion-preserving, bijective and inverse to each other. In particular, if $r$ is a weak ideal system of $D$, then $D$ is $r$-noetherian if and only if $D'$ is $\varepsilon(r)$-noetherian.

**Proof.** By definition, $\varepsilon$ factors as asserted and induces isomorphisms $K/G \rightarrow K'$ and $D/G \rightarrow D'$. For every subset $X \subseteq K$, we have $\varepsilon^{-1}(\varepsilon(X)) = XG$, and $X_r = GX_r \subseteq (GX)_r$. [indeed, since $r$ is a $D$-module system, we have $X_r \subseteq GX_r \subseteq (GX)_r \subseteq (DX)_r = X_r$].

1. If $X \subseteq K$, then $\varepsilon(X)_{\varepsilon r} = \varepsilon([\varepsilon^{-1}(\varepsilon(X))_r]) = \varepsilon[(XG)_r] = \varepsilon(X)_r$. We prove that $\varepsilon(r)$ satisfies the properties $M_1$, $M_2$, $M_3$ for all $\varepsilon' \in K'$ and $X', Y' \subseteq K'$. We may assume that $\varepsilon' = \varepsilon(c)$, $X' = \varepsilon(X)$ and $Y' = \varepsilon(Y)$, where $c \in K$ and $X, Y \subseteq K$.

$M_1$. $X'_{\varepsilon r} = \varepsilon(X)_{\varepsilon r} = \varepsilon(X_r) \supseteq \varepsilon(X \cup \{0\}) = X' \cup \{0\}$.

$M_2$. If $X' \subseteq Y_{\varepsilon r}$, then $\varepsilon(X) \subseteq \varepsilon(Y_r)$, hence $X \subseteq Y_rG = Y_r$, $X_r \subseteq Y_r$, and therefore we obtain $X'_{\varepsilon r} = \varepsilon(X)_r \subseteq \varepsilon(Y)_r = Y'_{\varepsilon r}$.

$M_3$. Since $\varepsilon'X' = \varepsilon(\varepsilon(X))$, we obtain $(\varepsilon')^* = \varepsilon([\varepsilon c X_r] \supseteq \varepsilon(c)\varepsilon(X_r) = c\varepsilon(X)_r = c\varepsilon'_{\varepsilon r})$, and equality holds if and only if $(\varepsilon c)_rG = cX_rG$, that is, if and only if $(\varepsilon c)_r = cX_r$.

Hence $\varepsilon(r)$ is a weak module system, it is a module system if and only if $r$ is a module system, and it is a $D'$-module system since $D'X'_{\varepsilon r} = \varepsilon(D)\varepsilon(X_r) = \varepsilon(DX_r) = \varepsilon(X_r) = X'_{\varepsilon r}$.

If $X' = \varepsilon(X) \subseteq K'$, then $\mathbb{P}(X') = \{\varepsilon(E) \mid E \in \mathbb{P}(X)\}$, and therefore $X'_{\varepsilon r} = \bigcup_{E' \in \mathbb{P}(X')} E'_{\varepsilon r} = \bigcup_{E \in \mathbb{P}(X)} \varepsilon(E)_{\varepsilon r} = \varepsilon\left(\bigcup_{E \in \mathbb{P}(X)} E_{\varepsilon r}\right) = \varepsilon(X_{\varepsilon r}) = X'_{\varepsilon r}$. 


2.4. Quotient monoids and module systems

Theorem 2.4.1. Let $K$ be a monoid, $D \subseteq K$ a submonoid and $T \subseteq D$ a multiplicatively closed subset. Let $j_T: K \to T^{-1}K$ be the natural embedding and $r$ a finitary weak $D$-module system on $K$.

1. There exists a unique finitary weak $T^{-1}D$-module system $T^{-1}r$ on $T^{-1}K$ such that
   
   \[ j_T(E)_{T^{-1}r} = T^{-1}E_r \quad \text{for all finite subsets } E \subseteq K. \]

2. On finite subsets of $T^{-1}K$ is given by
   \[
   \left\{ \frac{a_1}{t_1}, \ldots, \frac{a_m}{t_m} \right\}_{T^{-1}r} = T^{-1}\left\{a_1, \ldots, a_m\right\}_r \quad \text{for all } m \in \mathbb{N}, \ a_1, \ldots, a_m \in K \text{ and } t_1, \ldots, t_m \in T. \]

3. If $r$ is a weak ideal system of $D$, then $T^{-1}r$ is a weak ideal system of $T^{-1}D$, and if $r$ is a module system, then $T^{-1}r$ is a module system, too.

4. If $X \subseteq K$, then $T^{-1}X = (T^{-1}X)_{T^{-1}r} = j_T(X)_{T^{-1}r}$.

5. If $V \in \mathcal{M}_{T^{-1}r}(T^{-1}K)$, then $J = j_T^{-1}(V) \in \mathcal{M}_r(K)$, and $V = T^{-1}J$. The map
   \[ j_T^*: \mathcal{M}_r(K) \to \mathcal{M}_{T^{-1}r}(T^{-1}K), \]
   \[ \text{defined by } j_T^*(J) = T^{-1}J, \]
   is an inclusion-preserving monoid epimorphism satisfying $j_T^*(\mathcal{M}_r(I)) = \mathcal{M}_{T^{-1}r}(T^{-1}I)$ and $T^{-1}(J_1 \cup J_2) = T^{-1}J_1 \cap T^{-1}J_2$ for all $J_1, J_2 \in \mathcal{M}_r(K)$.

Let $r$ be a weak ideal system of $D$. If $V \in \mathcal{I}_{T^{-1}r}(T^{-1}D)$, then $J = j_T^{-1}(V) \cap D \in \mathcal{I}_r(D)$, and $V = T^{-1}J$. In particular, $j_T^*(\mathcal{I}_r(D)) = \mathcal{I}_{T^{-1}r}(T^{-1}D)$, $j_T^*(\mathcal{I}_r(D)) = \mathcal{I}_{T^{-1}r}(T^{-1}D)$, and if $D$ is $r$-noetherian, then $T^{-1}D$ is $T^{-1}r$-noetherian.

Proof. 1. We prove first:

A. $T^{-1}\left\{a_1, \ldots, a_m\right\}_r = T^{-1}\left\{t_1a_1, \ldots, t_ma_m\right\}_r$ (for $m \in \mathbb{N}$, $a_1, \ldots, a_m \in K$ and $t_1, \ldots, t_m \in T$).

Proof of A. By Theorem 2.1.6.2, \( \left\{t_1a_1, \ldots, t_ma_m\right\}_r \subset (D\{a_1, \ldots, a_m\})_r = \{a_1, \ldots, a_m\}_r \), which implies $T^{-1}\{t_1a_1, \ldots, t_ma_m\}_r \subset T^{-1}\{a_1, \ldots, a_m\}_r$. To prove the reverse inclusion, let $c \in \{a_1, \ldots, a_m\}_r$ and $t \in T$. Since $t_1 \cdot \cdots \cdot t_mc = t_1 \cdot \cdots \cdot t_m \cdot \frac{c}{t} \in T^{-1}\{a_1, \ldots, a_m\}_r$, we obtain
   \[
   \frac{c}{t} = \frac{t_1 \cdot \cdots \cdot t_m c}{t_1 \cdot \cdots \cdot t_m} \in T^{-1}\{t_1a_1, \ldots, t_ma_m\}_r. \]

Now we define a map $T^{-1}r: \mathbb{P}(T^{-1}K) \to \mathbb{P}(T^{-1}K)$ by
\[
\left\{ \frac{a_1}{t_1}, \ldots, \frac{a_m}{t_m} \right\}_{T^{-1}r} = T^{-1}\left\{a_1, \ldots, a_m\right\}_r \quad \text{for all } m \in \mathbb{N}_0, \ a_1, \ldots, a_m \in K \text{ and } t_1, \ldots, t_m \in T, \]
and we must prove that this assignment does not depend on the choice of representatives. We show that, for all \(m \in \mathbb{N}\), \(a_1, \ldots, a_m, a'_1, \ldots, a'_m \in K\) and \(t_1, \ldots, t_m, t'_1, \ldots, t'_m \in T\),

\[
\frac{a_j}{t_j} = \frac{a'_j}{t'_j} \quad \text{for all } j \in [1, m] \implies T^{-1}\{a_1, \ldots, a_m\}_r = T^{-1}\{a'_1, \ldots, a'_m\}_r.
\]

For \(j \in [1, m]\), let \(s_j \in T\) be such that \(s_j t'_ja_j = s_j t_j a'_j\). Then \(A\) implies

\[
T^{-1}\{a_1, \ldots, a_m\}_r = T^{-1}\{st'_1 a_1, \ldots, st'_m a_m\}_r = T^{-1}\{st'_1 a_1, \ldots, st'_m a_m\}_r = T^{-1}\{a'_1, \ldots, a'_m\}_r.
\]

We shall prove that \(T^{-1}r\) satisfies \(M_1, M_2, M_3\) and \(\{c\} T^{-1}r \supset c T^{-1}D\) for all \(c \in T^{-1}K\) and \(E, F \in \mathbb{P}_t(T^{-1}K)\), and that equality holds in \(M_3\) if \(r\) is a module system.

Once this is done, Theorem 2.2.2 implies the existence of a finitary weak \(T^{-1}D\)-module system on \(T^{-1}K\), again denoted by \(T^{-1}r\), such that \(j_T(E) T^{-1} r = T^{-1} E_r\) for all \(E \in \mathbb{P}_t(K)\), and that \(T^{-1}r\) is a module system if \(r\) is a module system. If \(r\) is a weak ideal system of \(D\), then \(\{1\}_r = D\), hence \(\{\frac{1}{T}\} T^{-1} r = T^{-1}\{1\}_r = T^{-1} D\), and therefore \(T^{-1} r\) is a weak ideal system of \(T^{-1} D\).

Assume that

\[
E = \left\{ \left\{ \frac{a_1}{t_1}, \ldots, \frac{a_m}{t_m} \right\} \right\}, \quad F = \left\{ \left\{ \frac{b_1}{s_1}, \ldots, \frac{b_n}{s_n} \right\} \right\} \quad \text{and} \quad c = \frac{a}{t},
\]

where \(m, n \in \mathbb{N}_0\), \(a_1, \ldots, a_m, b_1, \ldots, b_n \in K\) and \(t_1, \ldots, t_m, s_1, \ldots, s_n, t \in T\).

**\(M_1\)\:** For \(j \in [1, m]\),

\[
\frac{a_j}{t_j} \in T^{-1}\{a_1, \ldots, a_m\} \subset T^{-1}\{a_1, \ldots, a_m\}_r \implies E \subset E_{T^{-1}r},
\]

and \(0 \in \{a_1, \ldots, a_m\}_r\) implies \(0 \in T^{-1}E_{T^{-1}r}\).

**\(M_2\)\:** Suppose that \(E \subset F_{T^{-1}r} = T^{-1}\{b_1, \ldots, b_n\}_r\), say

\[
\frac{a_j}{t_j} = \frac{c_j}{v_j} \quad \text{for all } j \in [1, m], \quad \text{where } c_j \in \{b_1, \ldots, b_n\}_r \quad \text{and } v_j \in T.
\]

For \(j \in [1, m]\), let \(w_j \in T\) be such that \(w_j v_j a_j = w_j t_j c_j\). Then \(w_j v_j a_j \in \{b_1, \ldots, b_n\}_r\), and therefore

\[
E_{T^{-1}r} = T^{-1}\{a_1, \ldots, a_m\}_r = T^{-1}\{w_1 v_1 a_1, \ldots, w_m v_m a_m\}_r \subset T^{-1}\{b_1, \ldots, b_n\}_r = F_{T^{-1}r}.
\]

**\(M_3\)\:** We have

\[
\{cE\} T^{-1} r = \left\{ \left\{ \frac{aa_1}{tt_1}, \ldots, \frac{aa_m}{tt_m} \right\} \right\}_{T^{-1}r} = T^{-1}\{aa_1, \ldots, a_m\}_r
\]

\[
\supset T^{-1} a\{a_1, \ldots, a_m\}_r = c T^{-1}\{a_1, \ldots, a_m\}_r = cE_{T^{-1}r},
\]

and equality holds if \(r\) is a module system.

Since \(r\) is a weak \(D\)-module system, it follows that \(\{c\} T^{-1}r = T^{-1}\{a\}_r \supset T^{-1} a D \supset c T^{-1} D\).

It remains to prove the uniqueness of \(T^{-1} r\). Thus let \(\tilde{r}\) be a finitary weak \(T^{-1} D\)-module system on \(T^{-1} K\) satisfying \(j_T(E) \tilde{r} = T^{-1} E_r\) for all finite subsets \(E \subset K\). By Theorem 2.2.1.3 it suffices to prove that \(F_{\tilde{r}} = F_{T^{-1} r}\) for every finite subset \(F \subset T^{-1} K\).

Thus assume that

\[
F = \left\{ \frac{a_1}{t_1}, \ldots, \frac{a_m}{t_m} \right\}, \quad \text{where } m \in \mathbb{N}, \quad a_1, \ldots, a_m \in K \quad \text{and } t_1, \ldots, t_m \in T.
\]

If \(E = \{a_1, \ldots, a_m\}\), then \(T^{-1}D F = \langle T^{-1}D j_T(E) \rangle\), and

\[
F_{\tilde{r}} = \langle ((T^{-1}D) F)_{\tilde{r}} \rangle = \langle ((T^{-1}D) j_T(E))_{\tilde{r}} \rangle = j_T(E)_{\tilde{r}} = T^{-1} E_r = F_{T^{-1} r}.
\]

2. Observing that \(\mathbb{P}_t(j_T(X)) = \{j_T(E) \mid E \in \mathbb{P}_t(K)\}\), we obtain

\[
T^{-1} X_r = T^{-1} \bigcup_{E \in \mathbb{P}_t(X)} E_r = \bigcup_{E \in \mathbb{P}_t(X)} T^{-1} E_r = \bigcup_{E \in \mathbb{P}_t(X)} j_T(E)_{T^{-1}r} = \bigcup_{F \in \mathbb{P}_t(j_T(X))} F_{T^{-1} r} = j_T(X)_{T^{-1}r}.
\]

Since \((T^{-1}D)(T^{-1} X) = (T^{-1}D) j_T(X)\) and \(T^{-1} r\) is a weak \(T^{-1} D\)-module system, it follows that \((T^{-1} X)_{T^{-1}r} = (j_T(X))_{T^{-1}r}\).
3. Let $V \in M_{T^{-1}}(T^{-1}K)$ and $J = j^{-1}_T(V)$. We prove first that $V = T^{-1}J$.
   If $\frac{x}{t} \in V$, where $x \in K$ and $t \in T$, then $\frac{x}{t} = \frac{x}{t} \in (T^{-1}D)V = V$, hence $x \in J$ and $\frac{x}{t} \in T^{-1}J$.
   Conversely, if $x \in J$ and $t \in T$, then $\frac{x}{t} \in V$, $\frac{1}{t} \in T^{-1}D$ and $\frac{1}{t} \frac{x}{t} \in T^{-1}DV = V$.
   It remains to prove that $J \in M_r(K)$, and for this it suffices to show that $J_r \subset J$. If $a \in J_r$, then $\frac{a}{1} \in T^{-1}J_r = (T^{-1}J)_T^{-1}r = V_T^{-1}r = V$ and therefore $a \in J$.

4. If $J \in M_r(K)$, then $(T^{-1}J)_{T^{-1}r} = T^{-1}J_r = T^{-1}J \in M_{T^{-1}}(T^{-1}K)$. If $J \in M_{r,T}(K)$, then $J = E_r$ for some $E \in P_{r}(K)$, and $T^{-1}J = j_T(E)_{T^{-1}r} \in M_{r,T}(T^{-1}K)$. Hence $j_T^* \in T$ is an inclusion-preserving map as asserted, and by 3. it is surjective.

If $V \in M_{T^{-1},T}(T^{-1}K)$, then $V = \{\frac{a_1}{t_1}, \ldots, \frac{a_m}{t_m}\}_{T^{-1}r}$ for some $m \in \mathbb{N}$, $a_1, \ldots, a_m \in K$ and $t_1, \ldots, t_m \in T$. If $E = \{a_1, \ldots, a_m\} \subset K$, then $V = j_T(E)_{T^{-1}r} \in j_T^*(M_{r,T}(K))$.

If $J_1, J_2 \in M_r(K)$, then $T^{-1}(J_1 \cap J_2) = T^{-1}J_1 \cap T^{-1}J_2$, since $TJ_1 = J_1$ and $TJ_2 = J_2$. Moreover, $T^{-1}(J_1 \cap J_2) = T^{-1}(J_1J_2)_r = ((T^{-1}J_1)(T^{-1}J_2)_{T^{-1}r} = (T^{-1}J_1)_{T^{-1}r} (T^{-1}J_2)_r$, and therefore $j_T^*$ is a homomorphism.

5. By 1., $T^{-1}r$ is a weak ideal system of $T^{-1}D$. If $V \in \mathcal{I}_{T^{-1}r}(T^{-1}D)$, then $j_T^{-1}(V) \in M_r(K)$ by 3., and consequently $J = j_T^{-1}(V) \cap D \in \mathcal{I}_r(D)$. If $a \in J$ and $t \in T$, then $\frac{a}{t} = \frac{a}{1} \in T^{-1}DV = V$, and therefore $T^{-1}J \subset V$. To prove the reverse inclusion, assume that $\frac{a}{t} \in V$, where $a \in D$ and $t \in T$. Then it follows that $\frac{a}{t} = \frac{a}{1} \frac{1}{t} \in V$, hence $a \in j_T(V) \cap D$, and $\frac{1}{t} \in T^{-1}J$. If $V \in \mathcal{I}_{T^{-1},r}(T^{-1}D)$, then $V = \{\frac{a_1}{t_1}, \ldots, \frac{a_m}{t_m}\}_{T^{-1}r}$ for some $m \in \mathbb{N}$, $a_1, \ldots, a_m \in D$ and $t_1, \ldots, t_m \in T$. If $E = \{a_1, \ldots, a_m\} \subset D$, then $V = j_T(E)_{T^{-1}r} \in j_T^*(\mathcal{I}_r(D))$.

Clearly, $j_T^*(\mathcal{I}_r(D)) \subset \mathcal{I}_{T^{-1},r}(T^{-1}D)$ and $j_T^*(\mathcal{I}_{r,T}(D)) \subset \mathcal{I}_{T^{-1},r,T}(T^{-1}D)$, and as we have just proved, equality holds. In particular, if $D$ is $r$-noetherian, then $\mathcal{I}_r(D) = \mathcal{I}_{r,T}(D)$, hence $\mathcal{I}_{T^{-1},r}(T^{-1}D) = \mathcal{I}_{T^{-1},r,T}(T^{-1}D)$, and therefore $T^{-1}D$ is $T^{-1}r$-noetherian. \hfill \square

Theorem 2.4.2.

1. Let $K$ be a monoid, $D \subset K$ a submonoid, $s(D) : P(K) \rightarrow P(K)$ the semigroup system of $D$ defined on $K$ and $T \subset D$ a multiplicatively closed subset. Then $T^{-1}s(D) = s(T^{-1}D) : T^{-1}K \rightarrow T^{-1}K$ is the semigroup system of $T^{-1}D$ defined on $T^{-1}K$.

2. Let $K$ be a ring, $D \subset K$ a subring, $d(D) : P(K) \rightarrow P(K)$ the Dedekind system of $D$ defined on $K$ and $T \subset D$ a multiplicatively closed subset. Then $T^{-1}d(D) = d(T^{-1}D) : T^{-1}K \rightarrow T^{-1}K$ is the Dedekind system of $T^{-1}D$ defined on $T^{-1}K$.

Proof. 1. We prove that $j_T(E)_{s(T^{-1}D)} = T^{-1}E_{s(D)}$ for all $E \in P_{T}(K)$. The assertion follows from the uniqueness of $T^{-1}s(D)$ in Theorem 2.4.1.

If $E = \{a_1, \ldots, a_m\}$, where $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in K$, then $j_T(E)_{s(T^{-1}D)} = \bigcup_{j=1}^{m} T^{-1}D a_j = \bigcup_{j=1}^{m} T^{-1}(Da_j) = T^{-1}\bigcup_{j=1}^{m} Da_j = T^{-1}E_{s(D)}$.

2. As in 1. it suffices to prove that $j_T(E)_{d(T^{-1}D)} = T^{-1}E_{d(D)}$ for all $E \in P_{T}(K)$.

If $E = \{a_1, \ldots, a_m\}$, where $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in K$, then $j_T(E)_{d(T^{-1}D)} = \sum_{j=1}^{m} T^{-1}D a_j = \sum_{j=1}^{m} T^{-1}(Da_j) = T^{-1}\sum_{j=1}^{m} Da_j = T^{-1}E_{s(D)}$. \hfill \square
2.5. Extension and restriction of module systems

Definition 2.5.1. Let $K$ be a monoid, $D \subset K$ a submonoid and $r$ a weak module system on $K$. Then we define

$$r[D]: \mathbb{P}(K) \to \mathbb{P}(K) \quad \text{by} \quad X_{r[D]} = (XD)_r$$

for all $X \subset K$.

$$r_{(D)}: \mathbb{P}(D) \to \mathbb{P}(D) \quad \text{by} \quad X_{r_{(D)}} = X_r \cap D \quad \text{for all} \quad X \subset D, \quad \text{and we set} \quad r_D = r[D]_{(D)}: \mathbb{P}(D) \to \mathbb{P}(D).$$

By definition, we have $X_{r_D} = X_{r[D]} \cap D = (XD)_r \cap D \quad \text{for all} \quad X \subset D$, and if $T \subset K$ is another submonoid, then $r[D][T] = r[DT]$.

We call $r[D]$ the extension of $r$ by $D$ and $r_D$ the weak ideal system induced by $r$ on $D$ (see Theorem 2.5.2).

Theorem 2.5.2. Let $K$ be a monoid, $D \subset K$ a submonoid and $r$ a weak module system on $K$.

1. $r_{(D)}$ is a weak module system on $D$. If $r$ is finitary, then $r_{(D)}$ is also finitary, and if $r$ is a weak $D$-module system, then $r_{(D)}$ is a weak ideal system of $D$.

2. If $D_r = D$, then $r_{(D)} = r_{D} \mathbb{P}(D)$, and if $r$ is a module system [an ideal system of $D$], then $r_{(D)}$ is also a module system [an ideal system of $D$].

3. $r[D]$ is a weak $D$-module system on $K$. If $r$ is a module system, then $r[D]$ is also a module system, and if $r$ is finitary, then $r[D]$ is also finitary. Moreover, we have $r \leq r[D]$, $\mathcal{M}_{r[D]}(K) = \{J \in \mathcal{M}_r(K) \mid DJ = J\}$, and $r = r[D]$ if and only if $r$ is a weak $D$-module system.

4. $r_D = r[D]_{(D)}$ is a weak ideal system on $D$ and if $J \in \mathcal{M}_{r[D]}(K)$, then $J \cap D \in \mathcal{I}_D(D)$.

If $r$ is finitary, then $r_D$ is also finitary. If $D_r = D$, then $r[D]$ is an ideal system of $D$, and $r_D = r[D] | \mathbb{P}(D)$. In particular, if $r$ is a weak ideal system of $D$, then $r_D = r[D]$.

Proof. 1. We check the properties $\textbf{M1, M2, M3}$ for $r(D)$. Let $X, Y \in \mathbb{P}(D)$ and $c \in D$.

**M1.** $X_{r(D)} = X_r \cap D \supset X \cup \{0\}$.

**M2.** If $X \subset Y_{r(D)} = Y_r \cap D$, then $X_{r(D)} = X_r \cap D \subset Y_r \cap D = Y_{r(D)}$.

**M3.** $(cX)_{r(D)} = (cX)_r \cap D \supset cX_r \cap D = cX_{r(D)}$.

Let $r$ be finitary, $X \subset D$ and $a \in X_{r(D)} = X_r \cap D$. Then there exists some $E \in \mathbb{P}(X)$ such that $a \in E_r \cap D = E_{r(D)}$. Hence $r_{(D)}(D)$ is finitary.

If $r$ is a weak $D$-module system and $X \subset D$, then $DX_{r(D)} = D(X_r \cap D) \subset DX_r \cap D = X_r \cap D \subset X_{r(D)}$. Hence $r_{(D)}$ is a weak $D$-module system, and since $D_{r(D)} = D_r \cap D = D$, it is a weak ideal system of $D$.

2. If $X \subset D$, then $X_r \subset D$ and $X_{r(D)} = X_r \cap D = X_r$. Hence $r_{(D)} = r | \mathbb{P}(D)$, and if $r$ is a module system [an ideal system of $D$], then $r_{(D)}$ is also a module system [an ideal system of $D$].

3. We check the properties $\textbf{M1, M2, M3}$ for $r[D]$. Let $X, Y \in \mathbb{P}(K)$ and $c \in K$.

**M1.** $X_{r[D]} = (DX)_r \supset DX \cup \{0\} \supset X \cup \{0\}$.

**M2.** If $X \subset Y_{r(D)} = (DY)_r$, then $DX \subset D(DY)_r \subset (DY)_r$, and $X_{r[D]} = (DX)_r \subset (DY)_r = Y_{r[D]}$.

**M3.** $(cX)_{r[D]} = (cDX)_r \supset c(X)_r \supset cX_{r[D]}$, and equality holds if $r$ is a module system.

Hence $r[D]$ is a weak module system on $K$, and it is a module system if $r$ is a module system. If $r$ is finitary, $X \subset K$ and $a \in X_{r[D]} = (DX)_r$, then there exists some $E \in \mathbb{P}(X)$ such that $a \in (DE)_r = E_{r[D]}$, and therefore $r[D]$ is also finitary.

Next we prove that $\mathcal{M}_{r[D]}(K) = \{J \in \mathcal{M}_r(K) \mid DJ = J\}$. Once this is done, it follows that $r[D]$ is a $D$-module system, $r \leq r[D]$, and $r = r[D]$ if and only if $r$ is a weak $D$-module system.

If $J \in \mathcal{M}_r(K)$ and $DJ = J$, then $J_{r[D]} = (DJ)_r = J \in \mathcal{M}_{r[D]}(K)$. Conversely, if $J \in \mathcal{M}_{r[D]}(K)$, then $J = J_{r[D]} = (DJ)_r \in \mathcal{M}_r(K)$, and $DJ = (DJ)_r = J_{r[D]} = J$. 

4. It suffices to prove that \( J \in \mathcal{M}_{r[D]}(K) \) implies \( J \cap D \in \mathcal{I}_{r,D}(D) \). The remaining assertions follow by 1., 2. and 3.

If \( J \in \mathcal{M}_{r[D]}(K) \), then \( DJ = J \) and therefore \( (J \cap D)_{r,D} = ((J \cap D)D)_r \cap D \subset (JD)_r \cap D = J \cap D \). Hence \( (J \cap D)_{r,D} = J \cap D \) is an \( r_D \)-ideal.

\[ \square \]

**Examples 2.5.3.**

1. Let \( K \) be a monoid, and let \( D \subset T \subset K \) be submonoids. If \( s(D) : \mathbb{P}(K) \to \mathbb{P}(K) \) is the semigroup system of \( D \) defined on \( K \), then \( s(D)[T] : \mathbb{P}(K) \to \mathbb{P}(K) \) is the semigroup system of \( T \) defined on \( K \), and \( s(D)_T = s(D) \mid \mathbb{P}(T) : \mathbb{P}(T) \to \mathbb{P}(T) \) is the semigroup system of \( D \) defined on \( T \).

2. Let \( K \) be a ring, and let \( D \subset T \subset K \) be subrings. If \( d(D) : \mathbb{P}(K) \to \mathbb{P}(K) \) is the Dedekind system of \( D \) defined on \( K \), then \( d(D)[T] : \mathbb{P}(K) \to \mathbb{P}(K) \) is the Dedekind system of \( T \) defined on \( K \), and \( d(D)_T = d(D) \mid \mathbb{P}(T) : \mathbb{P}(T) \to \mathbb{P}(T) \) is the Dedekind system of \( D \) defined on \( T \).

**Theorem 2.5.4.** Let \( K \) be a monoid, \( D \subset K \) a submonoid, \( r \) a finitary \( D \)-module system on \( K \) and \( T \subset K^\times \cap D \) a multiplicatively closed subset (then \( T \subset K^\times \) and \( T^{-1}D \subset T^{-1}K = K \)). Then

\[
T^{-1}r = r[T^{-1}D] \quad \text{and} \quad r_{T^{-1}D} = T^{-1}r_D.
\]

In particular:

1. If \( X \subset K \), then \( X_{T^{-1}r} = T^{-1}X_r = (T^{-1}X)_r = X_{r[T^{-1}D]} \).

2. If \( X \subset D \), then \( X_{r_{T^{-1}D}} = T^{-1}X_{r_D} \).

**Proof.** It suffices to prove 1. and 2. Indeed, 1. implies that \( T^{-1}r = r[T^{-1}D] \), and from 2. and the uniqueness of \( T^{-1}r_D \) in Theorem 2.4.1 it follows that \( r_{T^{-1}D} = T^{-1}r_D \).

1. We start with a preliminary remark. If \( Y \subset K \) and \( TY = Y \), then

\[
T^{-1}Y = \bigcup_{t \in T} t^{-1}Y \quad \text{and} \quad (T^{-1}Y)_r = \bigcup_{t \in T} t^{-1}Y_r,
\]

since the family \( (t^{-1}Y)_{t \in T} \) is directed [indeed, if \( t_1, t_2 \in Y \), then \( t_1^{-1}Y = (t_1t_2)^{-1}(t_1t_2)^{-1}(t_1t_2)^{-1}Y \).

If \( X \subset K \), then \( TDX = DX \) and \( TX_r = X_r \). By the preliminary remark we obtain

\[
(T^{-1}X)_r = (T^{-1}DX)_r = \bigcup_{t \in T} (t^{-1}DX)_r = \bigcup_{t \in T} t^{-1}X_r = T^{-1}X_r = X_{T^{-1}r}.
\]

2. If \( X \subset D \), then

\[
X_{r_{T^{-1}D}} = X_{r[T^{-1}D]} \cap T^{-1}D = (T^{-1}DX)_r \cap T^{-1}D = T^{-1}X_r \cap T^{-1}D = T^{-1}(X_r \cap D)T^{-1}X_{r_D} \quad \square
\]

**Theorem 2.5.5.** Let \( K \) be a monoid, \( D \subset K \) a submonoid, \( K = q(D) \), \( r : \mathbb{P}(K) \to \mathbb{P}(K) \) a finitary ideal system of \( D \) and \( T \subset D^* \) a multiplicatively closed subset (then \( T \subset K^\times \) and \( T^{-1}D \subset T^{-1}K = K \)).

1. \( (T^{-1}D)_r = T^{-1}D, \quad T^{-1}r : \mathbb{P}(K) \to \mathbb{P}(K) \) is a finitary ideal system of \( T^{-1}D \),

\[
\mathcal{F}_{T^{-1}r}(T^{-1}D) = \{ a^{-1}T^{-1}I \mid I \in \mathcal{I}_r(D), \quad a \in D^* \} = \{ T^{-1}J \mid J \in \mathcal{F}_r(D) \},
\]

\[
\mathcal{F}_{T^{-1}r}(T^{-1}D) = \{ a^{-1}T^{-1}I \mid I \in \mathcal{I}_r(D), \quad a \in D^* \} = \{ T^{-1}J \mid J \in \mathcal{F}_r(D) \},
\]

and the map \( j^*_T : \mathcal{F}_r(D) \to \mathcal{F}_{T^{-1}r}(T^{-1}D) \), defined by \( j^*_T(J) = T^{-1}J \), is a surjective monoid homomorphism satisfying \( j^*_T(\mathcal{F}_r(D)) = \mathcal{F}_{T^{-1}r}(T^{-1}D) \) and \( T^{-1}(J_1 \cap J_2) = T^{-1}J_1 \cap T^{-1}J_2 \) for all \( J_1, J_2 \in \mathcal{F}_r(D) \).

2. Let \( D \) be \( r \)-noetherian. If \( J \in \mathcal{F}_r(D) \) and \( X \subset K \) is \( D \)-fractional, then

\[
T^{-1}(J : X) = (T^{-1}J : T^{-1}X) = (T^{-1}J : X).
\]
2. The Formalism of Module and Ideal Systems

Proof. 1. By Theorem 2.5.4.1, \((T^{-1}D)_{r} = T^{-1}D_{r} = T^{-1}D\), and by Theorem 2.4.1 \(T^{-1}r \) is a finitary ideal system of \(T^{-1}D\). Next we prove that

\[ \mathcal{F}_{T^{-1}r}(T^{-1}D) \subset \{a^{-1}T^{-1}I \mid I \in \mathcal{I}_{r}(D), \ a \in D^{*}\} \subset \{T^{-1}J \mid J \in \mathcal{F}_{r}(D)\} \subset \mathcal{F}_{T^{-1}r}(T^{-1}D) \]

and

\[ \mathcal{F}_{T^{-1}r, f}(T^{-1}D) \subset \{a^{-1}T^{-1}I \mid I \in \mathcal{I}_{r, f}(D), \ a \in D^{*}\} \subset \{T^{-1}J \mid J \in \mathcal{F}_{r, f}(D)\} \subset \mathcal{F}_{T^{-1}r, f}(T^{-1}D) \]

If \(V \in \mathcal{F}_{T^{-1}r}(T^{-1}D)\), then Theorem 2.1.4 implies that \(V = a^{-1}I_{1}\), where \(a_{1} \in (T^{-1}D)^{*}\) and \(I_{1} \in \mathcal{I}_{T^{-1}r}(T^{-1}D)\). By Theorem 1.2.6, \(a_{1} = t^{-1}a\) for some \(t \in T\) and \(a \in D^{*}\), and by Theorem 2.4.1 \(I_{1} = T^{-1}I\) for some \(I \in \mathcal{I}_{r}(D)\). Hence we obtain \(V = ta^{-1}T^{-1}I = a^{-1}T^{-1}I\). If \(V\) is \(T^{-1}r\)-finitely generated, then \(I_{1}\) is also \(T^{-1}r\)-finitely generated and \(I\) is \(r\)-finitely generated.

If \(I \in \mathcal{I}_{r}(D)\) and \(a \in D^{*}\), then \(J = a^{-1}I \in \mathcal{F}_{r}(D)\) and \(a^{-1}T^{-1}I = T^{-1}J\). If \(J\) is \(r\)-finitely generated, then \(J\) is \(r\)-finitely generated, too.

If \(J \in \mathcal{F}_{r}(D)\), then \(T^{-1}J \in \mathcal{M}_{T^{-1}r}(K)\), and there is some \(a \in D^{*}\) such that \(aJ \in \mathcal{I}_{r}(D)\). Then \(T^{-1}aJ = aT^{-1}J \subset \mathcal{I}_{r}(T^{-1}D)\), and since \(a \in (T^{-1}D)^{*}\), it follows that \(T^{-1}J \in \mathcal{F}_{T^{-1}r}(T^{-1}J)\). If \(J\) is \(r\)-finitely generated, then \(T^{-1}J = T^{-1}r\)-finitely generated.

By the above, \(j_{r}^{*}: \mathcal{F}_{r}(D) \to \mathcal{F}_{T^{-1}r}(T^{-1}D)\) is surjective map and \(j_{r}^{*}(\mathcal{F}_{r, f}(D)) = \mathcal{F}_{T^{-1}r, f}(T^{-1}D)\). The proof of the remaining assertions is literally the same as in Theorem 2.4.1.4.

2. Since \(X\) is \(D\)-fractional, it follows that \(X_{r} \in \mathcal{F}_{r}(D) = \mathcal{F}_{r, f}(D)\), and therefore \(X_{r} = E_{r}\) for some \(E \in \mathcal{P}(X)\). Hence \(X_{T^{-1}r} = T^{-1}X_{r} = T^{-1}E_{r} = E_{T^{-1}r} = (T^{-1}E)_{T^{-1}r}\) and, using Theorems 1.2.4.4 and 2.1.2.9,

\[ T^{-1}(J: X_{r}) = T^{-1}(J: X_{r}) = T^{-1}(J: E) = (T^{-1}J: (T^{-1}E)_{T^{-1}r}) = (T^{-1}J: X_{T^{-1}r}) \]

Finally, \(X_{T^{-1}r} = (T^{-1}X)_{T^{-1}r}\) implies \(T^{-1}(J: X) = (T^{-1}J: T^{-1}X)\). \(\square\)

Theorem and Definition 2.5.6. Let \(D\) be a cancellative monoid, \(K = q(D)\) and \(r: \mathcal{P}(D) \to \mathcal{P}(D)\) a module system on \(D\).

1. There exists a unique module system \(r_{\infty}\) on \(K\) such that, for all \(X \subset K\),

\[ X_{r_{\infty}} = \begin{cases} K & \text{if } X \text{ is not } D\text{-fractional}, \\ a^{-1}(aX)_{r} & \text{if } a \in D^{*} \text{ and } aX \subset D. \end{cases} \]

\(r_{\infty}\) | \(\mathcal{P}(D) = r\), and if \(r\) is an ideal system of \(D\), then \(r_{\infty}\) is also an ideal system of \(D\). If \(q\) is any module system on \(K\) such that \(q|\mathcal{P}(D) = r\), then \(q \leq r_{\infty}\).

\(r_{\infty}\) is called the trivial extension of \(r\) to \(K\).

2. \((r_{\infty})_{r}\) is the unique finitary module system on \(K\) satisfying \((r_{\infty})_{r}|\mathcal{P}(D) = r_{r}\). If \(r_{r}\) is an ideal system of \(D\), then \((r_{\infty})_{r}\) is also an ideal system of \(D\).

\((r_{\infty})_{r}\) is called the natural extension of \(r_{r}\) to \(K\).

In particular, for every finitary module system \(r\) on \(D\) there exists a unique finitary module system \(\tau\) on \(K\) such that \(\tau|\mathcal{P}(D) = r\).

3. If \(q: \mathcal{P}(K) \to \mathcal{P}(K)\) is any finitary ideal system of \(D\), then \(q = ((q_{D})_{\infty})_{r}\).

Proof. 1. Uniqueness is obvious. We define \(r_{\infty}\) as in the assertion. Note that this definition does not depend on the choice of \(a \in D^{*}\) with \(aX \subset D\). Indeed, if \(X \subset K\) and \(a_{1}, a_{2} \in D^{*}\) are such that \(a_{1}X \subset D\) and \(a_{2}X \subset D\), then \(a_{1}a_{2}X \subset D\) and \((a_{1}a_{2}X)_{r} = a_{1}(a_{2}X)_{r} = a_{2}(a_{1}X)_{r}\) and therefore \(a_{1}^{-1}(a_{2}X)_{r} = a_{1}^{-1}(a_{1}X)_{r}\). By definition, \(r_{\infty}|\mathcal{P}(D) = r\).

We check the conditions \(M_{1}, M_{2}, M_{3}\) for \(r_{\infty}\). Let \(X, Y \subset K\) and \(c = b^{-1}d \in K\), where \(b \in D^{*}\) and \(d \in D\).

\(M_{1}\). If \(X\) is not \(D\)-fractional, then \(X_{r_{\infty}} = K \supset X \cup \{0\}\). If \(a \in D^{*}\) is such that \(aX \subset D\), then \(X_{r_{\infty}} = a^{-1}(aX)_{r} \supset a^{-1}(aX) \cup \{0\} = X \cup \{0\}\).
M2. Suppose that $X \subseteq Y_{r_{\infty}}$. If $Y$ is not $D$-fractional, then $Y_{r_{\infty}} = K \supset X_{r_{\infty}}$. Thus let $a \in D^*$ be such that $aY \subseteq D$. Then $X \subseteq Y_{r_{\infty}} = a^{-1}(aY)_r$, hence $aX \subseteq (aY)_r \subseteq D$, and therefore it follows that $X_{r_{\infty}} = a^{-1}(aX)_r \subseteq a^{-1}(aY)_r = Y_{r_{\infty}}$.

M3. We may assume that $c \neq 0$, hence $c \in K^\times$ and $d \in D^*$. If $X$ is not $D$-fractional, then (by Lemma 1.4.2) also $cX$ is not $D$-fractional, and $(cX)_{r_{\infty}} = K = cK = cX_{r_{\infty}}$.

Thus assume that $aX \subseteq D$ for some $a \in D^*$. Then $ab(cX) = aX \subseteq dD \subseteq D$ and therefore $(cX)_{r_{\infty}} = (ab)^{-1}(abcX)_r = (ab)^{-1}bc(aX)_r = c(aX)_r X_{r_{\infty}}$.

Let $q$ be any module system on $K$ such that $q \mid \mathbb{P}(D) = r$ and $X \subseteq K$. If $a \in D^*$ is such that $aX \subseteq D$, then $X_q = a^{-1}(aX)_q = a^{-1}(aX)_r = X_{r_{\infty}}$, and if $X$ is not $D$-fractional, then $X_q \subseteq K = X_{r_{\infty}}$. Hence it follows that $q \subseteq r_{\infty}$.

Let $r$ be an ideal system of $D$. Since $D_{r_{\infty}} = D_r = D$ and $DX_{r_{\infty}} = X_{r_{\infty}}$ for all $X \subseteq \mathbb{P}(K)$, it follows that $r_{\infty}$ is also an ideal system of $D$.

2. By definition, $(r_{\infty})_t$ is a finitary module system on $K$. If $X \subseteq D$, then

$$X_{(r_{\infty})_t} = \bigcup_{E \subseteq \mathbb{P}(X)} E_{r_{\infty}} = \bigcup_{E \subseteq \mathbb{P}(X)} E_r = X_{r_t},$$

and therefore $(r_{\infty})_t \mid \mathbb{P}(D) = r_t$.

Let now $r_t$ be an ideal system of $D$ and $X \subseteq K$. For $E \subseteq \mathbb{P}(X)$, let $a \in D^*$ be such that $aE \subseteq D$. Then $DE_{r_{\infty}} = Da^{-1}(aE)_r = Da^{-1}(aE)_t = a^{-1}(aE)_r = E_{r_{\infty}}$, and

$$DX_{(r_{\infty})_t} = \bigcup_{E \subseteq \mathbb{P}(X)} DE_{r_{\infty}} = \bigcup_{E \subseteq \mathbb{P}(X)} E_{r_{\infty}} = X_{(r_{\infty})_t}.$$

Hence $(r_{\infty})_t$ is an ideal system of $D$.

To prove uniqueness of $r_t$, let $\tilde{r}$ be a finitary module system on $K$ such that $\tilde{r} \mid \mathbb{P}(D) = r_t$. We must prove that $E_{\tilde{r}} = E_{(r_{\infty})_t}$ for all $E \subseteq \mathbb{P}(K)$. If $E \subseteq \mathbb{P}(K)$, let $c \in D^*$ be such that $cE \subseteq D$. Then $E_{\tilde{r}} = c^{-1}(cE)_t = c^{-1}(cE)_r = E_{r_{\infty}} = E_{(r_{\infty})_t}$.

3. Let $q: \mathbb{P}(K) \to \mathbb{P}(K)$ be a finitary ideal system of $D$. Then $qD = q \mid \mathbb{P}(D)$ by Theorem 2.5.2.4, and it suffices to prove that $E_{(qD_{\infty})_t} = E_q$ for all $E \subseteq \mathbb{P}(K)$. If $E \subseteq \mathbb{P}(K)$ and $a \in D^*$ is such that $aE \subseteq K$, then $E_{(qD_{\infty})_t} = E_{(qD_{\infty})_t} = a^{-1}(aE)_q = a^{-1}(aE)_q = E_q$.

Example 2.5.7. Let $D$ be a domain, $K = q(D)$ and $\mathcal{F}(D) = \mathcal{M}_{d(D)}(K)^*$ the set of all non-zero $D$-submodules of $K$.

A semistar operation of $D$ is a map $*: \mathcal{F}(D) \to \mathcal{F}(D)$, $M \mapsto M^*$, such that, for all $c \in K$ and $M, N \in \mathcal{F}(D)$, the following conditions are satisfied:

1. $M \subseteq M^*$;
2. $M \subseteq N^*$ implies $M^* \subseteq N^*$;
3. $(cM)^* = (cM^*)$.

If moreover $D^* = D$, then $*$ is called a (semijstar operation, and the restriction $*: \mathcal{F}(D) \to \mathcal{F}(D)$ is called a star operation.

Let $*$ be a semistar operation of $D$, and define $r_*: \mathbb{P}(K) \to \mathbb{P}(K)$, $X \mapsto X_{r_*}$, by

$$X_{r_*} = \begin{cases} \{0\} & \text{if } X \subseteq \{0\}, \\ D(X)^* & \text{if } X \not\subseteq \{0\}. \end{cases}$$

Then $r_*$ is a $D$-module system on $K$, $d(D) \leq r_*$ and $D_{r_*} = D^*$. Hence $r_*$ is an ideal system of $D$ if and only if $*$ is a (semi)star operation.

Conversely, let $r: \mathbb{P}(K) \to \mathbb{P}(K)$ be a $D$-module system on $K$ and $d(D) \leq r$. Then $\mathcal{M}_r(K)^* \subseteq \mathcal{F}(D)$, and we define $*: \mathcal{F}(D) \to \mathcal{F}(D)$ by $M^* = M_r$. Then $*$ is a semistar operation, $r_{*r} = r$, and for every semistar operation $*$ of $D$ we have $r_{*r} = *$. 


2.6. The ideal systems \( v \) and \( t \)

Throughout this section, let \( D \) be a cancellative monoid, \( K = \mathbb{q}(D) \), and for \( X \subset K \), let \( X^{-1} = (D:X) \).

**Definition 2.6.1.** If \( D \neq K \), we define \( v = v(D): \mathbb{P}(K) \rightarrow \mathbb{P}(K) \) by \( X_v = (X^{-1})^{-1} \) for all \( X \subset D \), and if \( D = K \), we set \( v(K) = s(K): \mathbb{P}(K) \rightarrow \mathbb{P}(K) \). We shall see in Theorem 2.6.2 that \( v(D) \) is an ideal system of \( D \), and we define \( t = t(D) = v(D)_t: \mathbb{P}(K) \rightarrow \mathbb{P}(K) \).

\( v(D) \) is called the divisorial system and \( t(D) \) is called the total system of \( D \) defined on \( K \).

**Theorem 2.6.2.** Assume that \( D \neq K \), and set \( v = v(D): \mathbb{P}(K) \rightarrow \mathbb{P}(K) \).

1. If \( X \subset K \), then
   - \( X^{-1} = K \) if and only if \( X^* = \emptyset \),
   - \( (X^{-1})^* \neq \emptyset \) if and only if \( X \) is \( D \)-fractional,
   - \( X_v = K \) if and only if \( X \) is not \( D \)-fractional,
   - \( X_v = \{0\} \) if and only if \( X \subset \{0\} \).

   In any case, we have
   \[
   X_v = \bigcap_{X \in zD} zD.
   \]

2. If \( X \subset K \), then \( X \cup \{0\} \subset X_v \), \( X_v^{-1} = X^{-1} = (X^{-1})_v \), and \( (XX^{-1})^{-1} = (X^{-1}:X^{-1}) \).

3. \( v \) is a ideal system of \( D \), \( \mathcal{M}_v(K) = \{X^{-1} | X \subset K\} \), and \( (v)_\infty = v \). If \( q \) is any ideal system of \( D \) defined on \( K \), then \( q \leq v \).

4. The system \( t = t(D) = v(D)_t: \mathbb{P}(K) \rightarrow \mathbb{P}(K) \) is a finitary ideal system of \( D \). If \( q \) is any finitary ideal system of \( D \) defined on \( K \), then \( q \leq t \).

5. Let \( D' \) be another cancellative monoid, \( K' = \mathbb{q}(D') \), \( v' = v(D') \) and \( t' = t(D') \). Let \( \varepsilon: K \rightarrow K' \) be a surjective monoid homomorphism, \( D' = \varepsilon(D) \), and let \( G \subset D^\times \) be a subgroup such that \( \varepsilon^{-1}(\varepsilon(x)) = xG \) for all \( x \in K \). Then we have \( \varepsilon(X^{-1}) = \varepsilon(X^{-1}) \) for all subsets \( X \subset K \), \( v' = \varepsilon(v) \) and \( t' = \varepsilon(t) \).

**Proof.** 1. Let \( X \subset K \).

   If \( X^* = \emptyset \), then \( XK = X \) and \( X^{-1} = K \). If \( z \in X^* \), then \( zK = K \neq D \), and therefore \( X^{-1} \neq K \).

   By definition, \( (X^{-1})^* \neq \emptyset \) if and only if \( X \) is \( D \)-fractional. Therefore we obtain \( X_v = (X^{-1})^{-1} \) if and only if \( (X^{-1})^* = \emptyset \), that is, if and only if \( X \) is not \( D \)-fractional. Similarly, \( X_v = (X^{-1})^{-1} \subset \{0\} \) if and only if \( X^{-1} \) is not \( D \)-fractional which holds if and only if \( X^* = \emptyset \).

   It remains to prove \((*)\). If \( X \subset \{0\} \), \((*)\) holds by Theorem 1.2.8. Thus assume that \( X \neq \{0\} \). Since \( (X^{-1})^* = \{y \in K^\times | yX \subset D\} = \{z^{-1} | z \in K^\times, X \subset zD\} \), we obtain
   \[
   X_v = (D:X^{-1}) = (D:(X^{-1})^*) = \bigcap_{y \in (X^{-1})^*} y^{-1}D = \bigcap_{z \in K^\times} zD = \bigcup_{X \subset zD} zD.
   \]

2. If \( X \subset K \), then \( (X \cup \{0\})^{-1} \subset D \) implies that \( X \cup \{0\} \subset (X^{-1})^{-1} \). Hence we obtain \( X_v^{-1} \subset X^{-1} \subset (X^{-1})_v = [(X^{-1})^{-1}]^{-1} = X_v^{-1} \), and thus \( X_v^{-1} = X^{-1} = (X^{-1})_v \). Finally,
   \[
   (X^{-1}:X^{-1}) = ((D:X):(X^{-1})_v) = (D:XX^{-1}) = (XX^{-1})^{-1}.
   \]

3. We verify the conditions M1, M2 and M3. Let \( X, Y \subset K \) and \( c \in K \).

**M1.** By 1.

**M2.** If \( X \subset Y_v \), then \( Y^{-1} = Y_v^{-1} \subset X^{-1} \), and therefore \( X_v = (X^{-1})^{-1} \subset (Y^{-1})^{-1} = Y_v \).

**M3.** We may assume that \( c \neq 0 \). Then \( cX_v = c(X^{-1})^{-1} = (c^{-1}X^{-1})^{-1} = ((cX)^{-1})^{-1} = (cX)_v \).
2.6. THE IDEAL SYSTEMS v AND t

If \( c \in D \) and \( X \subset K \), then \( cX_v \subset X_v \) by (\(*\)). Hence \( v \) is a \( D \)-module system, and since \( D_v = D \) it is even an ideal system of \( D \). In particular, \( vD = v|\mathbb{P}(D) \), and if \( X \subset K \) is not \( D \)-fractional, then \( X_v = K = X_{(v)} \). Hence it follows that \( v = (v)_\infty \).

If \( X \in \mathcal{M}_v(K) \), then \( X_v = (X^{-1})^{-1} \), and if \( X \subset K \), then \( (X^{-1})_v = X^{-1} \). Hence we obtain \( \mathcal{M}_v(K) = \{X^{-1} \mid X \subset K \} \).

Let \( q \) be any ideal system of \( D \) defined on \( K \) and \( X \subset K \). If \( z \in K \) is such that \( X \subset zD \), then \( X_q \subset zD \), and therefore \( X_q \subset X_v \) by (\(*\)). Hence \( q \leq v \).

4. By Theorem 2.2.2, \( t \) is a finitary ideal system of \( D \). If \( q \) is any finitary ideal system of \( D \) defined on \( K \), then \( q \leq v \) by 3., and therefore \( q = q \leq v_t = t \).

5. If \( X \subset K \) and \( x' = \varepsilon(x) \in K' \), then \( x' \varepsilon(X) = \varepsilon(xX) \subset D' = \varepsilon(D) \) if and only if \( xX \subset D \). Hence we obtain \( \varepsilon(X)^{-1} = \varepsilon(X^{-1}) \), and \( \varepsilon(X)_{x'} = (\varepsilon(X)^{-1})^{-1} = \varepsilon((X^{-1})^{-1}) = \varepsilon(X_v) = \varepsilon(X)_{\varepsilon(x')} \).

Consequently, \( \varepsilon' = \varepsilon(v) \), and by Theorem 2.3.7 it follows that \( \varepsilon(t) = \varepsilon(v_t) = \varepsilon(v)_t = v'_t = t' \). \( \square \)

Theorem 2.6.3. Let \( v = v(D) : \mathbb{P}(K) \to \mathbb{P}(K) \), \( X \subset D \) and \( a, d \in D \).

1. If \( X_v = dD \), then \( \text{GCD}(X) = dD^\times \).
2. If \( \text{GCD}(X) = dD^\times \) and \( \text{GCD}(bX) \neq \emptyset \) for all \( b \in D \), then \( X_v = dD \).
3. The following assertions are equivalent:
   (a) \( \text{GCD}(X) \neq \emptyset \) for all \( X \in \mathbb{P}(D) \).
   (b) Every (fractional) \( v \)-ideal of \( D \) is principal.
4. If \( D \) is a GCD-monoid, \( X \subset D \) and \( d \in D \), then
   \[
   X_v = \bigcap_{a \in D \atop X \subset aD} aD ,
   \]
   and \( X_v = dD \) if and only if \( d \in \text{GCD}(X) \).

Proof. We may assume that \( D \neq K \).

1. If \( X_v = dD \), then \( X \subset dD \), and if \( b \in D \) is such that \( X \subset bD \), then \( dD = X_v \subset bD \). Hence \( dD \) is the smallest principal ideal containing \( X \), and \( dD^\times = \text{GCD}(X) \).

2. If \( \text{GCD}(X) = dD^\times \), then \( X \subset dD \), and therefore
   \[
   X_v = \bigcap_{z \in K \atop \varepsilon(z) \in D} zD \subset dD .
   \]

Hence it suffices to prove that, for all \( z \in K \), \( X \subset zD \) implies \( dD \subset zD \). Thus suppose that \( z = b^{-1}c \in K \), where \( b \in D^\times \) and \( c \in D \), and \( X \subset zD \). Then \( bx \subset cD \), and since \( \text{GCD}(bX) \neq \emptyset \), it follows that \( \text{GCD}(bX) = bdD^\times \). Therefore we obtain \( bdD \subset cD \), and \( dD \subset b^{-1}cD = zD \).

3. Obvious by 1. and 2.

4. Clearly,
   \[
   X = \bigcap_{a \in D \atop X \subset aD} aD \supset \bigcap_{z \in K \atop \varepsilon(z) \in zD} zD = X_v .
   \]

To prove the converse, suppose that \( x \in X \subset \bar{X} \subset D \), and let \( z \in K \) be such that \( X \subset zD \). Then \( z = a^{-1}b \), where \( a \in D^* \), \( b \in D \), \( \text{GCD}(a, b) = D^\times \), and it suffices to prove that \( X \subset bD \). If \( x \in X \), then \( x = zc \) for some \( c \in D \), hence \( ax = bc \), and since \( a \) is coprime to \( b \), it follows that \( a \mid c \), say \( c = ad \) for some \( d \in D \). But then \( x = bd \in bD \).

By 1. and 2. it follows that \( X_v = dD \) if and only if \( d \in \text{GCD}(X) \). \( \square \)
2. THE FORMALISM OF MODULE AND IDEAL SYSTEMS

Theorem 2.6.4. Let \( v = v(D) : \mathbb{P}(K) \to \mathbb{P}(K) \) and \( t = t(D) : \mathbb{P}(K) \to \mathbb{P}(K) \).

1. \( D \) is a GCD-monoid if and only if every \( v \)-finitely generated \( v \)-ideal is principal [equivalently, every \( t \)-finitely generated \( t \)-ideal is principal].

In this case, \( \mathcal{F}_t(D)^* = \mathcal{F}_v(D)^* = \{ aD \mid a \in K^\times \} \cong K^\times / D^\times \) is a group.

2. \( D \) is factorial if and only if every \( t \)-ideal of \( D \) is principal.

In this case, \( \mathcal{F}_t(D)^* = \{ aD \mid a \in K^\times \} \cong K^\times / D^\times \) is a group.

3. Let \( D' \) be another cancellative monoid, \( \varepsilon : D \to D' \) a surjective monoid homomorphism and \( G \subset D^\times \) a subgroup such that \( \varepsilon^{-1}(\varepsilon(x)) = xG \) for all \( x \in D \). Then \( D' \) is factorial [a GCD-monoid] if and only if \( D \) is factorial [a GCD-monoid].

Proof. 1. Let \( D \) be a GCD-monoid and \( J \in \mathcal{I}_v(D) \). Then \( J = E_v \) for some \( E \in \mathcal{P}_v(D) \), and if \( d \in \text{GCD}(E) \), then \( J = E_v = dD \) by Theorem 2.6.3.2.

Conversely, if every \( v \)-finitely generated \( v \)-ideal is principal and \( E \in \mathcal{P}_v(D) \), then \( E_v = dD \) for some \( d \in D \), and then \( d \in \text{GCD}(E) \) by Theorem 1.5.1.2.

2. Let \( D \) be factorial. Then Theorem 1.5.6.3 implies that \( \text{GCD}(X) \neq \emptyset \) for every subset \( X \subset D \). If \( J \in \mathcal{I}_v(D)^* \) and \( d \in \text{GCD}(J) \), then \( J = dD \) by Theorem 2.6.3.2.

Conversely, assume that every \( t \)-ideal is principal. Then \( D \) is \( t \)-noetherian, and as every principal ideal is a \( t \)-ideal, it satisfies the ACCP. By 1., \( D \) is a GCD-monoid, and by Theorem 1.5.5, it is an atomic GCD-monoid and thus it is factorial by Theorem 1.5.6.4.

3. Let \( \varepsilon : K \to K' \) be the extension of \( \varepsilon \) to the quotient monoids and \( t' = t(D') \). By the Theorems 2.6.2 and 2.3.7 we have \( \varepsilon(t) = t' \), \( \varepsilon(X)t' = \varepsilon(X)t \) for all subsets \( X \subset D \), and \( J \to \varepsilon(J) \) defines a bijection \( \mathcal{I}_v(D) \to \mathcal{I}_v(D') \). Hence every \( [t \text{-finitely generated}] \) \( t \)-ideal of \( D \) is principal if and only if every \( [t' \text{-finitely generated}] \) \( t' \)-ideal of \( D' \) is principal, and the assertion follows by 1. and 2. \( \square \)

Theorem 2.6.5. For \( i \in \{1, 2\} \), let \( D_i \) be a GCD-monoid, \( K_i = q(D_i) \), \( t_i = t(D_i) : \mathbb{P}(K_i) \to \mathbb{P}(K_i) \), and let \( \varphi : K_1 \to K_2 \) be a monoid homomorphism. Then \( \varphi \) is a \((t_1, t_2)\)-homomorphism if and only if \( \varphi(D_1) \subset D_2 \) and \( \varphi|_{D_1} : D_1 \to D_2 \) is a GCD-homomorphism. In particular, there is a bijective map

\[
\text{Hom}_{(t_1, t_2)}(K_1, K_2) \to \text{Hom}_{\text{GCD}}(D_1, D_2), \quad \text{given by } \varphi \mapsto \varphi|_{D_1}.
\]

Proof. Let first \( \varphi \) be a \((t_1, t_2)\)-homomorphism. Then

\[
\varphi(D_1) = \varphi(\{1_{D_1}\}t_1) \subset \{ \varphi(1_{D_1}) \}t_2 = \{1_{D_2}\}t_2 = D_2.
\]

Let \( E \subset D_1 \) be finite and \( d \in \text{GCD}(E) \). Then \( E_{t_1} = dD_1 \) and \( \varphi(d) \in \varphi(E_{t_1}) \subset \varphi(E)t_2 = d'D_2 \), where \( d' \in \text{GCD}(\varphi(E)) \). Since \( E \subset dD_1 \), it follows that \( \varphi(E) \subset \varphi(d)D_2 \), hence \( d'D_2 \subset \varphi(d)D_2 \), and since \( \varphi(d) \in d'D_2 \), we obtain \( \varphi(d) \in d'D_2^\times = \text{GCD}(\varphi(E)) \).

Assume now that \( \varphi(D_1) \subset D_2 \), and let \( \varphi|_{D_1} : D_1 \to D_2 \) be a GCD-homomorphism. We must prove that \( \varphi(E_{t_1}) \subset \varphi(E)t_2 \) for all \( E \in \mathcal{P}_v(K_1) \). If \( E \in \mathcal{P}_v(K_1) \) and \( c \in D_1^\times \) such that \( cE \subset D_1 \). If \( d \in \text{GCD}(cE) \), then \( \varphi(d) \in \text{GCD}(\varphi(c)\varphi(E)) \), and therefore

\[
\varphi(E_{t_1}) = \varphi(c^{-1}(cE)_{t_1}) = \varphi(c)^{-1} \varphi(d_{t_1}) \subset \varphi(c)^{-1}\varphi(d_{t_1}) \subset \varphi(c)^{-1}(\varphi(c)\varphi(E))t_2 = \varphi(E)t_2.
\] \( \square \)

Theorem und Definition 2.6.6. Let \( v = v(D) : \mathbb{P}(K) \to \mathbb{P}(K) \) and \( t = t(D) : \mathbb{P}(K) \to \mathbb{P}(K) \).

1. The following assertions are equivalent:
   (a) \( D \) is \( v \)-noetherian.
   (b) \( D \) is \( t \)-noetherian.
(c) For every sequence \((J_n)_{n \geq 0}\) in \(\mathcal{F}_v(D)\) such that \(J_n \supset J_{n+1}\) for all \(n \geq 0\), and \((\bigcap_{n \geq 0} J_n)^* \neq \emptyset\), there exists some \(m \geq 0\) such that \(J_n = J_m\) for all \(n \geq m\).

(d) Every non-empty subset \(\Omega \subset \mathcal{F}_v(D)\) satisfying
\[
(\bigcap_{J \in \Omega} J)^* \neq \emptyset
\]
possesses a minimal element (with respect to inclusion).

(e) For every subset \(X \subset D\) there exists some \(E \in \mathbb{P}_t(X)\) such that \(X^{-1} = E^{-1} \subset K\).

If these conditions are satisfied, then \(D\) is called a Mori monoid.

In particular, if \(D\) is a Mori monoid, then \(X_v = X_t\) for every \(D\)-fractional subset \(X \subset K\), \(\mathcal{F}_v(D) = \mathcal{F}_t(D)\) and \(\mathcal{I}_v(D) = \mathcal{I}_t(D)\).

2. Let \(D\) be a Mori monoid and \(T \subset D\) be a multiplicatively closed subset.
   (a) If \(T^{-1}D\) is a Mori monoid, and \(t(T^{-1}D) = T^{-1}_t: \mathbb{P}(K) \to \mathbb{P}(K)\), and therefore \(X_v = X_t\) for every \(D\)-fractional subset \(X \subset K\), \(\mathcal{F}_v(D) = \mathcal{F}_t(D)\) and \(\mathcal{I}_v(D) = \mathcal{I}_t(D)\).

   (b) Let (c) If \((J_n)_{n \geq 0}\) be a sequence in \(\mathcal{F}_v(D)\) such that \(J_n \supset J_{n+1}\) for all \(n \geq 0\), and let \(c \in K^*\) be such that \(c \in J_n\) for all \(n \geq 0\). Then \((cJ_n^{-1})_{n \geq 0}\) is an ascending sequence in \(\mathcal{I}_v(D)\). Hence it becomes stationary, and therefore the sequence \((J_n)_{n \geq 0}\) becomes stationary, too.

   (c) Assume to the contrary that there exists a subset \(\emptyset \neq \Omega \subset \mathcal{F}_v(D)\) without a smallest element, and that there is some \(c \in K^*\) such that \(c \in J\) for all \(J \in \Omega\). Consequently, for every \(J \in \Omega\) there exists some \(J' \in \Omega\) such that \(J' \subsetneq J\). If \(J_0 \in \Omega\) is arbitrary and \((J_n)_{n \geq 0}\) is recursively defined by \(J_{n+1} = J_n^*\) for all \(n \geq 0\), then the sequence \((J_n)_{n \geq 0}\) contradicts (c).

   (d) Assume to the contrary that there exists a subset \(\emptyset \neq \Omega \subset \mathcal{F}_v(D)\) without a smallest element, and that there is some \(c \in K^*\) such that \(c \in J\) for all \(J \in \Omega\). Consequently, for every \(J \in \Omega\) there exists some \(J' \in \Omega\) such that \(J' \subsetneq J\). If \(J_0 \in \Omega\) is arbitrary and \((J_n)_{n \geq 0}\) is recursively defined by \(J_{n+1} = J_n^*\) for all \(n \geq 0\), then the sequence \((J_n)_{n \geq 0}\) contradicts (c).

   (e) Assume to the contrary that there exists a subset \(\emptyset \neq \Omega \subset \mathcal{F}_v(D)\) without a smallest element, and that there is some \(c \in K^*\) such that \(c \in J\) for all \(J \in \Omega\). Consequently, for every \(J \in \Omega\) there exists some \(J' \in \Omega\) such that \(J' \subsetneq J\). If \(J_0 \in \Omega\) is arbitrary and \((J_n)_{n \geq 0}\) is recursively defined by \(J_{n+1} = J_n^*\) for all \(n \geq 0\), then the sequence \((J_n)_{n \geq 0}\) contradicts (c).

   (e) If \(X \subset D\) and \(X^* = \emptyset\), we set \(E = X\). Thus assume that \(X \subset D\), \(X^* \neq \emptyset\), and set \(\Omega = \{F^{-1} \mid F \in \mathbb{P}_t(X), F^* \neq \emptyset\}\). Then \(\Omega \neq \emptyset\), and if \(F \in \mathbb{P}_t(X)\) and \(F^* \neq \emptyset\), then \(F^{-1} \in \mathcal{F}_v(D)\) and \(1 \in F^{-1}\). Thus by (d) there exists some \(E \in \mathbb{P}_t(X)\) such that \(E^* \neq \emptyset\) and \(E^{-1}\) is minimal in \(\Omega\). Clearly, \(X^{-1} \subset E^{-1}\), and we assert that equality holds. Indeed, suppose to the contrary that there is some \(c \in E^{-1} \setminus X^{-1}\), and let \(a \in X\) be such that \(ca \notin D\). Then \((E \cup \{a\})^{-1} \in \Omega\), \(c \notin (E \cup \{a\})^{-1}\), and therefore \((E \cup \{a\})^{-1} \subseteq E^{-1}\), a contradiction.

   (e) If \(X \subset D\), there exists some \(E \in \mathbb{P}_t(X)\) such that \(E^{-1} = X^{-1}\) and thus \(E_v = X_v\). Hence \(D\) is \(v\)-noetherian.

2. (a),(b) By Theorem 2.4.1.5 \(T^{-1}D\) is \(T^{-1}t\)-noetherian, and thus it is a Mori monoid. If \(X \subset K\) is \(D\)-fractional, then \(T^{-1}(D:X) = (T^{-1}D:T^{-1}X) = (T^{-1}D:X)\) by Theorem 2.5.5.2, and therefore
\[
T^{-1}X_v = T^{-1}(D:(D:X)) = (T^{-1}D:(T^{-1}D:T^{-1}X)) = (T^{-1}X)(T^{-1}D)
\]
In particular, if \(E \in \mathbb{P}_t(K)\), then \(E_{T^{-1}t} = T^{-1}E_t = T^{-1}E_v = E_v(T^{-1}D) = E_t(T^{-1}D)\), and therefore \(T^{-1}t = t(T^{-1}D)\).
(c) If $P \in v\text{-}\text{spec}(D)$, then $(T^{-1}P)_{v(T^{-1}D)} = T^{-1}P_v = T^{-1}P \in v(T^{-1}D)\text{-}\text{spec}(T^{-1}D)$. Conversely, if $T^{-1}P \in v(T^{-1}D)\text{-}\text{spec}(T^{-1}D) = t(T^{-1}D)\text{-}\text{spec}(T^{-1}D)$, then $t \leq t(T^{-1}D)$ implies $(T^{-1}P)_t = T^{-1}P$, hence $P_t = (T^{-1}P \cap D)_t = T^{-1}P \cap D = P$, and consequently $P \in t\text{-}\text{spec}(D) = v\text{-}\text{spec}(D)$.

3. Since $t[C]$ is an ideal system of $C$, it follows that $t \leq t[C] \leq t(C)$, and therefore we obtain $F_{t[C]}(C) \subset M_{t[C]}(K) \subset M_{t}(K)$. By Theorem 1.4.2.6 every $C$-fractional subset of $K$ is $D$-fractional, and therefore it follows that $F_{t[C]}(C) \subset F_{t}(D)$. □
CHAPTER 3

Prime Ideals and Valuation Monoids

Throughout this chapter, let \( D \) be a monoid, \( K = q(D) \), \( s = s(D) : \mathbb{P}(K) \to \mathbb{P}(K) \), and if \( D \) is cancellative, then \( v = v(D) : \mathbb{P}(K) \to \mathbb{P}(K) \) and \( t = t(D) : \mathbb{P}(K) \to \mathbb{P}(K) \).

3.1. Prime ideals and Krull’s Theorem

**Definition 3.1.1.** Let \( r \) be a weak ideal system of \( D \).

1. We denote by
   - \( r\text{-spec}(D) \subset \mathcal{I}_r(D) \) the set of all prime \( r \)-ideals (in particular, \( s\text{-spec}(D) \) is the set of all prime ideals of \( D \));
   - \( \mathcal{X}(D) \) the set of all minimal non-zero prime ideals of \( D \);
   - \( r\text{-max}(D) \) the set of all maximal elements of \( \mathcal{I}_r(D) \setminus \{D\} \) (they are called \( r \)-maximal ideals).

2. An \( r \)-ideal \( Q \in \mathcal{I}_r(D) \) is called \( r \)-irreducible if \( Q \neq D \) and, for all \( I, J \in \mathcal{I}_r(D) \), \( Q = I \cap J \) implies \( Q = I \) or \( Q = J \).

3. \( D \) is called \( r \)-local if \( |r\text{-max}(D)| = 1 \).

   If \( D \setminus D^\times \in \mathcal{I}_r(D) \), then \( r\text{-max}(D) = \{D \setminus D^\times\} \), and \( D \) is \( r \)-local. In particular, \( D \setminus D^\times \in \mathcal{I}_s(D) \), and \( D \) is \( s \)-local.

**Theorem 3.1.2** (Krull). Let \( r \) be a weak ideal system of \( D \). Let \( \emptyset \neq \mathcal{L} \subset \mathbb{P}(D) \) be such that, for all \( M, N \in \mathcal{L} \) it follows that \( MN \in \mathcal{L} \), and set \( \Omega = \{C \in \mathcal{I}_r(D) \mid M \not\subset C \text{ for all } M \in \mathcal{L}\} \).

1. Every (with respect to the inclusion) maximal element of \( \Omega \) is a prime ideal.

2. Suppose that \( r \) is finitary and \( M_r \in \mathcal{I}_{r.(D)} \) for all \( M \in \mathcal{L} \). For every \( C_0 \in \Omega \), there exists a maximal element \( P \in \Omega \) such that \( C_0 \subset P \).

   In particular, there exists some \( P \in \Omega \cap r\text{-spec}(D) \) such that \( C_0 \subset P \).

**Proof.** 1. Assume to the contrary that there is a maximal element \( P \in \Omega \) which is not a prime ideal. As \( \mathcal{L} \neq \emptyset \), it follows that \( P \neq D \). Let \( a, b \in D \setminus P \) be such that \( ab \in P \). Then it follows by the maximality of \( P \) that \( (P \cup \{a\})_r, (P \cup \{b\})_r \notin \Omega \), and there exist \( M, N \in \mathcal{L} \) such that \( M \subset (P \cup \{a\})_r \) and \( N \subset (P \cup \{b\})_r \). Hence we obtain \( MN \subset (P \cup \{a\})_r (P \cup \{b\})_r \subset (P^2 \cup Pa \cup Pb \cup \{ab\})_r \subset P \), a contradiction, since \( MN \in \mathcal{L} \).

2. By assumption, \( \Omega_1 = \{C \in \Omega \mid C_0 \subset C \} \neq \emptyset \), and we prove that every chain in \( (\Omega_1, \subset) \) has an upper bound in \( \Omega_1 \). Then the assertion follows by 1. and Zorn’s Lemma. Let \( \Sigma \subset \Omega_1 \) be a chain, and

\[
P = \bigcup_{C \in \Sigma} C.
\]

Then \( P \in \mathcal{I}_r(D) \), and we assert that \( P \in \Omega_1 \). Clearly, \( C_0 \subset P \), and we assume to the contrary that \( M \subset P \) for some \( M \in \mathcal{L} \). Then there is some \( E \in \mathbb{P}_r(D) \) such that \( M_r = E_r \), hence \( E \subset P \), and
exists some $P$.
By Theorem 3.1.2 there exists some $P$ as $\Sigma$ is a chain, we obtain $E \subset C$ for some $C \in \Sigma$. But then it follows that $M \subset M_r = E_r \subset C$, a contradiction. \hfill $\square$

**Corollary 3.1.3.** Let $r$ be a weak ideal system of $D$, $T \subset D^\times$ a multiplicatively closed subset and $\Omega = \{C \in \mathcal{I}_r(D) \mid C \cap T = \emptyset\}$.

1. Every (with respect to the inclusion) maximal element of $\Omega$ is a prime ideal.
2. Suppose that $r$ is finitary and $C_0 \in \Omega$. Then there exists a maximal element $P \in \Omega$ such that $C_0 \subset P$. In particular, there exists some $P \in \Omega \cap r\text{-spec}(D)$ such that $C_0 \subset P$.

**Proof.** By Theorem 3.1.2, applied with $L = \{\{a\} \mid a \in T\}$.

**Corollary 3.1.4.** Let $r$ be a weak ideal system of $D$.

1. $r\text{-max}(D) \subset r\text{-spec}(D)$.\hfill $\square$
2. If $r$ is finitary and $J \in \mathcal{I}_r(D) \setminus \{D\}$, then there exists some $M \in r\text{-max}(D)$ such that $J \subset M$. In particular, if $\emptyset_r \neq D$, then $r\text{-max}(D) \neq \emptyset$.

**Proof.** We apply Corollary 3.1.3 with $T = D^\times$.
1. If $M \in r\text{-max}(D)$, then $M$ is maximal in $\{C \in \mathcal{I}_r(D) \mid C \cap D^\times = \emptyset\}$.
2. If $J \in \mathcal{I}_r(D)$, and $M$ is maximal in $\{C \in \mathcal{I}_r(D) \mid J \subset C, J \cap D^\times = \emptyset\}$, then $M \in r\text{-max}(D)$. \hfill $\square$

**Corollary 3.1.5.** Let $r$ be a finitary ideal system of $D$. If $D$ is $r$-local, then $r\text{-max}(D) = \{D\} \setminus D^\times$.

**Proof.** Let $D$ be $r$-local and $r\text{-max}(D) = \{M\}$. If $a \in D \setminus D^\times$, then $aD \in \mathcal{I}_r(D)$ and $aD \neq D$. By Corollary 3.1.4 there exists some $P \in r\text{-max}(D)$ such that $aD \subset P$, and by assumption we have $P = M$. \hfill $\square$

**Theorem 3.1.6.** Let $r$ be a finitary weak ideal system of $D$ and $J \in \mathcal{I}_r(D) \setminus \{D\}$.

1. $\mathcal{P}(J) \subset r\text{-spec}(D)$.\hfill $\square$
2. If $\mathcal{P}(J) \cap r\text{-spec}(D) \subset \mathcal{I}_r(D)$, then $\mathcal{P}(J)$ is finite.
3. Suppose that every principal ideal of $D$ is an $r$-ideal. Then $\mathcal{X}(D) \subset r\text{-spec}(D)$. In particular, if $D$ is cancellative, then $\mathcal{X}(D) \subset t\text{-spec}(D)$.
4. If $r$ is finitary, then $\sqrt{J} \in \mathcal{I}_r(D)$. If $I \in \mathcal{I}_r(D)$ and $I \subset \sqrt{J}$, then there is some $n \in \mathbb{N}$ such that $I^n \subset J$.
5. If $r$ is finitary, then $\sqrt{r}: \mathcal{P}(D) \to \mathcal{P}(D)$, defined by $X_{\sqrt{r}} = \sqrt{X_r}$, is a finitary weak ideal system of $D$, and $\sqrt{r} \leq r$.

**Proof.** 1. If $P \in \mathcal{P}(J)$, then $D \setminus P$ is multiplicatively closed, and by Corollary 3.1.3 there exists some $P_0 \in r\text{-spec}(D)$ such that $J \subset P_0 \subset P$. Hence $P_0 \in \Sigma(J)$ and therefore $P_0 = P \in r\text{-spec}(D)$.
2. Let $\mathcal{L} = \{P_1, \ldots, P_m \mid m \in \mathbb{N}, P_1, \ldots, P_m \in \Sigma(J)\}$, $\Omega = \{C \in \mathcal{I}_r(D) \mid L \not\subset C$ for all $L \in \mathcal{L}\}$, and assume that $J \in \Omega$. For every $L \in \mathcal{L}$, we have $L_r \in \mathcal{I}_r(D)$, and if $L_1, L_2 \in \mathcal{L}$, then $L_1L_2 \in \mathcal{L}$. By Theorem 3.1.2 there exists some $P \in r\text{-spec}(D) \cap \Omega$ such that $J \subset P$, and by Theorem 1.3.2 there exists some $P_0 \in \mathcal{P}(J)$ such that $P_0 \subset P$, which implies $P_0 \in \Omega \cap \mathcal{L}$, a contradiction. Hence there exists some $L \in \mathcal{L}$ such that $L \subset J$, say $L = P_1 \cdot \ldots \cdot P_m$, where $m \in \mathbb{N}$ and $P_1, \ldots, P_m \in \mathcal{P}(J)$. We assert that $\mathcal{P}(J) \subset \{P_1, \ldots, P_m\}$. Indeed, if $P \in \mathcal{P}(J)$, then $P_1 \cdot \ldots \cdot P_m \subset J \subset P$ implies $P_j \subset P$ for some $j \in [1, m]$ and hence $P = P_j$ by the minimality of $P$.
3. If $P \in \mathcal{X}(D)$ and $a \in P^\times$, then $aD \in \mathcal{I}_r(D)$ and $P \in \mathcal{P}(aD) \subset r\text{-spec}(D)$.
4. By Theorem 1.3.2,
\[ \sqrt{J} = \bigcap_{P \in \mathcal{P}(J)} P, \]
and as \( \mathcal{P}(J) \subset r\text{-spec}(D) \), we obtain \( \sqrt{J} \in \mathcal{I}_r(D) \).

Assume now that \( I \in \mathcal{I}_r(D) \) and \( I \subset \sqrt{J} \), say \( I = E_r \), where \( E = \{a_1, \ldots, a_m\} \in \mathcal{P}(J) \). For \( j \in [1, m] \), let \( n_j \in \mathbb{N} \) be such that \( a_j^{n_j} \in J \), and set \( n = n_1 + \ldots + n_m \). We assert that \( E^n \subset J \). Indeed, if \( a \in E^n \), then \( a = a_1^{\nu_1} \cdots a_m^{\nu_m} \), where \( \nu_1, \ldots, \nu_m \in \mathbb{N}_0 \), \( \nu_1 + \ldots + \nu_m = n \), and there is some \( j \in [1, m] \) such that \( \nu_j \geq n_j \), which implies \( a \in J \). Now it follows that \( I^n = E^n \in \mathcal{I}_r(D) \subset J \).

5. We verify the properties M1, M2, and M3. Let \( X, Y \subset D \) and \( c \in D \).

**M1.** \( X_\sqrt{r} = \sqrt{X_r} \supset X_r \supset X \cup \{0\} \).

**M2.** If \( X \subset Y_\sqrt{r} = \sqrt{Y_r} \) (since \( Y_r \in \mathcal{I}_r(D) \)), and consequently \( \sqrt{X_r} \subset \sqrt{Y_r} \).

**M3.** If \( x \in X_\sqrt{r} = \sqrt{X_r} \) and \( n \in \mathbb{N} \) is such that \( x^n \in X_r \), then \( (cx)^n \in c^n X_r \subset cX_r \subset (cX)_r \) and therefore \( cx \in \sqrt{(cX)_r} = (cX)_{\sqrt{r}} \). Hence \( cX_\sqrt{r} \subset (cX)_{\sqrt{r}} \).

Clearly, \( X_r \subset X_\sqrt{r} \) implies \( \sqrt{r} \leq r \). If \( X \subset D \) and \( x \in X_\sqrt{r} \), let \( n \in \mathbb{N} \) be such that \( x^n \in X_r \). As \( r \) is finitary, there exists some \( E \in \mathcal{P}(X) \) such that \( x^n \in E_r \) and consequently \( x \in E_{\sqrt{r}} \). Hence \( \sqrt{r} \) is finitary. If \( X \subset D \), then \( \sqrt{X_r} \subset D \) is an ideal, and therefore \( \sqrt{r} \) is a weak ideal system of \( D \). \( \square \)

**Theorem 3.1.7.** Let \( r \) be a finitary weak ideal system of \( D \). Then \( D \) is \( \sqrt{r} \)-noetherian if and only if \( r\text{-spec}(D) \) satisfies the ACC and for every \( J \in \mathcal{I}_r(D) \) the set \( \mathcal{P}(J) \) is finite.

**Proof.** Assume first that \( D \) is \( \sqrt{r} \)-noetherian. As \( r\text{-spec}(D) \subset \{ J \in \mathcal{I}_r(D) \mid \sqrt{J} = J \} = \mathcal{I}_r(\sqrt{D}) \), it satisfies the ACC. If \( J \in \mathcal{I}_r(\sqrt{D}) \), then \( \sqrt{J} \in \mathcal{I}_r(D) \), and \( \mathcal{P}(J) = \mathcal{P}(\sqrt{J}) \subset \mathcal{I}_r(D) = \mathcal{I}_r(\sqrt{D}) \). Hence \( \mathcal{P}(J) \) is finite by Theorem 3.1.6.2.

Assume now that \( r\text{-spec}(D) \) satisfies the ACC, \( \mathcal{P}(J) \) is finite for all \( J \in \mathcal{I}_r(D) \), and yet there exists a properly ascending sequence \( (J_n)_{n \geq 0} \) in \( \mathcal{I}_r(D) \). As \( \sqrt{r} \) is finitary, we obtain
\[ J = \bigcup_{n \geq 0} J_n \in \mathcal{I}_r(\sqrt{D}). \]

Let \( \mathcal{P}(D) = \{ J^{(1)}, \ldots, J^{(N)} \} \). For \( n \geq 0 \), let \( \{ P \in \mathcal{P}(J_n) \mid J \not\subset P \} = \{ P^{(1)}_n, \ldots, P^{(N_n)}_n \} \). By Theorem 1.3.2.3 it follows that \( J = P^{(1)} \cap \ldots \cap P^{(N)} \) and \( J_n = J \cap P^{(1)}_n \cap \ldots \cap P^{(N_n)}_n \). We denote by \( L_n \) the (finite) set of all sequences \( (\nu_0, \ldots, \nu_n) \in [1, N_0] \times \ldots \times [1, N_n] \) such that \( P^{(\nu_0)}_0 \subset P^{(\nu_1)}_1 \subset \ldots \subset P^{(\nu_n)}_n \), and we assert that \( L_n \neq \emptyset \).

We proceed by induction on \( n \). For \( n = 0 \), there is nothing to do. Thus suppose that \( n \geq 1 \) and \( \nu_0 \in [1, N_0] \). Since \( J_{n-1} = J \cap P^{(1)}_{n-1} \cap \ldots \cap P^{(N_{n-1})}_{n-1} \subset J_n \subset P^{(\nu_n)}_n \) and \( J \not\subset P^{(\nu_n)}_n \), it follows that \( P^{(\nu_{n-1})}_{n-1} \subset P^{(\nu_n)}_n \) for some \( \nu_{n-1} \in [1, N_{n-1}] \), and the induction hypothesis yields the complementary sequence \( (\nu_0, \ldots, \nu_{n-1}) \).

Now the assignment \( (\nu_0, \ldots, \nu_n) \mapsto (\nu_0, \ldots, \nu_{n-1}) \) defines a map \( L_n \to L_{n-1} \), and as the projective limit of a system of non-empty finite sets is not empty, there exists a sequence
\[ (\nu_n)_{n \geq 0} \subset \lim_{n \geq 0} L_n. \]

By construction, \( (P^{(\nu_n)}_n)_{n \geq 0} \) is an ascending sequence in \( r\text{-spec}(D) \). Hence there exists some \( m \geq 0 \) such that \( P^{(\nu_n)}_n = P^{(\nu_m)}_m \) for all \( m \geq n \), and consequently
\[ J = \bigcup_{n \geq 0} J_n \subset \bigcup_{n \geq 0} P^{(\nu_n)}_n = P^{(\nu_m)}_m \not\subset J, \]
a contradiction. \( \square \)
3.2. Associated primes, localizations and primary decompositions

Throughout this section, we set \((X : Y) = (X :_D Y)\) for all subsets \(X, Y \subset D\).

**Definition 3.2.1.** Let \(B \supset D\) be an overmonoid and \(P \subset D\) be a prime ideal. Recall from Definition 1.3.7 that the localization \(B_P\) of \(B\) at \(P\) is defined by \(B_P = (D \setminus P)^{-1}B\), that \(j_P : B \to B_P\) denotes the natural embedding, and for every subset \(X \subset B\), \(X_P = (D \setminus P)^{-1}X\).

For a finitary weak module system \(r\) on \(B\), we define \(r_P = (D \setminus P)^{-1}r : \mathcal{P}(B_P) \to \mathcal{P}(B_P)\).

If \(r\) is a finitary weak module system of \(B\), then \(r_D\) is a finitary weak module system on \(D\) by Theorem 2.5.2, \(r_P\) is a finitary weak module system on \(B_P\) and if \(X \subset B\), then \((X_r)_P = j_P(X)_r = (X_P)_r\) by Theorem 2.4.1.

**Theorem 3.2.2.** Let \(B \supset D\) be an overmonoid, \(r\) a finitary weak module system on \(B\), and for \(P \in r_D\)-spec\((D)\), let \(j_P : B \to B_P\) be the natural embedding. If \(A \in \mathcal{M}_r(B)\) is a \(D\)-module, then

\[
A = \bigcap_{P \in r_D\text{-max}(D)} j_P^{-1}(A_P).
\]

In particular:

- If \(A, A' \in \mathcal{M}_r(B)\) are \(D\)-modules and \(A_P = A'_P\) for all \(P \in r_D\)-max\((D)\), then \(A = A'\).
- Assume that \(D^\bullet \subset B^\times\). Then \(B = B_P \supset A_P \supset A\), \(j_P = \text{id}_B\) for all \(P \in r_D\)-spec\((D)\), and

\[
A = \bigcap_{P \in r_D\text{-max}(D)} A_P.
\]

**Proof.** By Theorem 2.5.2.4, \(r_D\) is a finitary weak ideal system on \(D\). Obviously, \(A \subset j_P^{-1}(A_P)\) for all \(P \in r_D\)-max\((D)\). Thus assume that \(z \in B\), \(j_P(z) \in A_P\) for all \(P \in r_D\)-max\((D)\), and set \(J = (A : z) \cap D\). Then \(J \subset D\) is an \(r_D\)-ideal, and therefore it suffices to prove that \(J \not\subset P\) for all \(P \in r_D\)-max\((D)\), for then \(J = D\) by Corollary 3.1.4.2, hence \(1 \in J\) and therefore \(z \in A\).

If \(P \in r_D\)-max\((D)\), then

\[
\frac{z}{1} = \frac{a}{t} \quad \text{for some} \quad a \in A \quad \text{and} \quad t \in D \setminus P,
\]

and there exists some \(s \in D \setminus P\) such that \(stz = sa \in A\) and therefore \(st \in J \setminus P\).

**Theorem 3.2.3.** Let \(r\) be a finitary weak ideal system of \(D\) and \(P \in r\)-spec\((D)\).

1. \(D_P\) is \(r_P\)-local with \(r_P\)-maximal ideal \(P_P = D_P \setminus D_P^r\).
2. If \(J \in \mathcal{I}_r(D)\) and \(\sqrt{J} \in r\)-max\((D)\), then \(J\) is primary.
3. Let \(J \in \mathcal{I}_r(D)\) and \(P \in \mathcal{P}(J)\).
   
   (a) \(P_P\) is the only \(r_P\)-ideal of \(D_P\) containing \(J_P\), \(J_P\) is \(P_P\)-primary, and \(j_P^{-1}(J_P)\) is the smallest \(P\)-primary \(r\)-ideal of \(D\) which contains \(J\).

   (b) Assume that \(P = \sqrt{J}\) and \(J_M\) is \(P_M\)-primary for all \(M \in r\)-max\((D)\) such that \(M \supset J\). Then \(J = j_P^{-1}(J_P)\) is \(P\)-primary.

**Proof.**

1. By Theorem 1.2.4, \(D_P^r = (D \setminus P)^{-1}(D \setminus P) = D_P \setminus P_P\), and therefore \(P_P\) is the greatest ideal of \(D_P\).

2. Let \(a, b \in D\) be such that \(ab \in J\) and \(a \notin J\). Then \((J:a) \in \mathcal{I}_r(D)\), and \(J \cup \{b\} \subset (J:a) \subset D\). By Corollary 3.1.4.2 there exists some \(M \in r\)-max\((D)\) such that \((J:a) \subset M\). Now \(J \subset M\) implies \(\sqrt{J} \subset M\), hence \(\sqrt{J} = M\) and \(b \in \sqrt{J}\).

3. (a) Let \(Q \in r_P\)-spec\((D_P)\) be such that \(J_P \subset Q_P\). By Theorem 1.3.6.2 we have \(Q = Q_P\) for some \(Q \in r\)-spec\((D)\) such that \(Q \subset P\). Now \(J_P \subset Q_P \subset P_P\) implies \(J \subset j_P^{-1}(J_P) \subset Q \subset P\), hence \(Q = P\) and...
therefore $\overline{Q} = P_P$. Hence $P_P$ is the only prime $r_P$-ideal containing $J_P$, $P_P = \sqrt{J_P}$, and $J_P$ is $P_P$-primary by 1. By Theorem 1.3.6 $j^{-1}_P(J_P)$ is primary, and $\sqrt{j^{-1}_P(J_P)} = j^{-1}(\sqrt{J_P}) = P$. If $Q$ is any $P$-primary $r$-ideal containing $J$, then $J_P \subset Q_P \subset P_P$, and $j^{-1}_P(J_P) \subset j^{-1}_P(Q_P) = Q$.

(b) Let $J_M$ be $P_M$-primary for all $M \in r$-$\text{max}(D)$ satisfying $M \supset J$. It suffices to prove that $j^{-1}_M(J_M) \subset J$. If $a \in j^{-1}_M(J_M)$, then $\frac{a}{1} = \frac{t}{1}$ for some $c \in J$ and $t \in D \setminus P$, and therefore there exists some $s \in D \setminus P$ such that $sta = sc \in J$. By Theorem 3.2.5 it follows that $J = \bigcap_{M \in r$-$\text{max}(D)} j^{-1}_M(J_M)$, and therefore it suffices to prove that $a \in j^{-1}_M(J_M)$ for all $M \in r$-$\text{max}(D)$. If $M \in r$-$\text{max}(D)$ and $J \notin M$, then $j^{-1}_M(J_M) = D$ and there is nothing to do. If $M \in r$-$\text{max}(D)$ and $J \subset M$, then $\frac{a}{1} \in J_M$, and we assert that $\frac{a}{1} \in J_M$ (which implies $a \in j^{-1}_M(J_M)$). Indeed, if $\frac{a}{1} \notin J_M$, then $\frac{a}{1} \in P_M$ and $st \in j^{-1}_M(P_M) = P$, a contradiction. \hfill \Box

**Definition 3.2.4.** Let $r$ be a weak ideal system of $D$ and $J \in \mathcal{I}_r(D)$.

1. A prime ideal $P \subset D$ is called an \textit{associated prime of $J$} if $P = (J : z)$ for some $z \in D \setminus J$. Let $\text{Ass}_D(J) = \text{Ass}(J) \subset r$-$\text{spec}(D)$ the set of all associated primes of $J$.

If $D$ is cancellative, $K = q(D)$ and $z \in K^\times$, then $(J : z) = z^{-1} J \cap D$.

2. A primary decomposition $\mathcal{Q}$ of $J$ is called an \textit{$r$-primary decomposition} if $\mathcal{Q} \subset \mathcal{I}_r(D)$.

By definition, a primary decomposition is just an $s$-primary decomposition. If $J$ possesses an $r$-primary decomposition, then it also possesses a reduced one (this is proved as in Theorem 1.3.5). If $\mathcal{Q}$ is a reduced $r$-primary decomposition of $J$, then $\{ \sqrt{Q} \mid Q \in \mathcal{Q} \} \subset \text{Ass}(J)$ by Theorem 1.3.5.2.

3. $D$ is called \textit{$r$-laskerian} if every $r$-ideal of $D$ possesses an $r$-primary decomposition.

**Theorem 3.2.5.** Let $r$ be a weak ideal system of $D$ and $J \in \mathcal{I}_r(D)$.

1. Every maximal element in the set $\{(J : z) \mid z \in D \setminus J\}$ belongs to $\text{Ass}(J)$.

2. Let $r$ be finitary, $T \subset D$ a multiplicatively closed subset, $P \in r$-$\text{spec}(D)$ and $P \cap T = \emptyset$.

(a) If $P \in \text{Ass}(J)$, then $T^{-1} P \in \text{Ass}(T^{-1} J)$.

(b) If $P \in \mathcal{I}_{r,T}(D)$ and $T^{-1} P \in \text{Ass}(T^{-1} J)$, then $P \in \text{Ass}(J)$.

**Proof.** 1. Let $c \in D \setminus J$ be such that $(J : c)$ is maximal in the set $\{(J : z) \mid z \in D \setminus J\}$. Let $a, b \in D$ be such that $ab \in (J : c)$ and $a \notin (J : c)$. Then it follows that $ac \notin J$, $b \in (J : ac)$. Since obviously $(J : c) \subset (J : ac)$, equality holds by the maximal choice of $(J : c)$, and thus $b \in (J : c)$. Therefore $(J : c)$ is a prime ideal and belongs to $\text{Ass}(J)$.

2. (a) If $P = (J : z) \in \text{Ass}(J)$, then $T^{-1} P = (T^{-1} J : T^{-1} D j_P(z))$ is a prime ideal of $T^{-1} D$ and thus it belongs to $\text{Ass}(T^{-1} J)$.

(b) Suppose that $P = \{a_1, \ldots, a_n\}_r$, where $n \in \mathbb{N}_0$ and $a_1, \ldots, a_n \in P$, and $T^{-1} P = (T^{-1} J : T^{-1} D \frac{Z}{T})$, where $z \in D$ and $t \in T$. For $i \in [1, n]$, we obtain $\frac{a_i}{s_i} \frac{z}{t} = \frac{c_i}{s_i}$, where $c_i \in J$ and $s_i \in T$, and therefore $w_i s_i a_i z = w_i t c_i \in J$ for some $w_i \in T$.

If $v = (w_1 s_1) \cdots (w_n s_n)$, then $v \in T$ and $v z a_i \in J$ for all $i \in [1, n]$. Hence it follows that $v z P \subset J$ and $P \subset (J : v z)$. We assert that equality holds (which implies $P \in \text{Ass}(J)$). Thus let $x \in (J : v z)$. Then $x v z \in J$, and $\frac{x v}{t} \frac{z}{t} \in T^{-1} J$, which implies $\frac{x v}{t} \in T^{-1} P$.

Hence $x v \in P$ and finally $x \in P$, since $v \in T \subset D \setminus P$. \hfill \Box
Theorem 3.2.6. Let \( r \) be a weak ideal system of \( D \) such that \( D \) is \( r \)-noetherian and \( J \in \mathcal{I}_r(D) \).

1. \( \mathcal{P}(J) \) is finite, and \( \mathcal{P}(J) \subset \text{Ass}(J) \).
2. If \( \Omega \) is a reduced \( r \)-primary decomposition of \( J \), then \( \text{Ass}(J) = \{ \sqrt{Q} \mid Q \in \Omega \} \).
3. \( J \) possesses a representation \( J = Q_1 \cap \ldots \cap Q_n \), where \( n \in \mathbb{N}_0 \) and \( Q_1, \ldots, Q_n \in \mathcal{I}_r(D) \) are \( r \)-irreducible.
4. \( D \) is \( r \)-laskerian if and only if every \( r \)-irreducible \( r \)-ideal is primary.

Proof. 1. By Theorem 3.1.6.2 the set \( \mathcal{P}(J) \) is finite. Thus let \( P \in \mathcal{P}(J) \). By Theorem 3.2.5.2(b) it suffices to prove that \( P \in \text{Ass}(J) \). Since \( D \) is \( r \)-noetherian, the set \( \{ (J_P : z) \mid z \in D \setminus J \} \) has maximal elements, and thus \( \text{Ass}(J_P) \neq \emptyset \) by Theorem 3.2.5.1. If \( \sqrt{Q} \in \text{Ass}(J_P) \), then \( J_P \subset \sqrt{Q} \subset P \), and \( P \) is the only prime \( r_P \)-ideal of \( P \) containing \( P \) by Theorem 3.2.3.3(a). Hence \( P = \sqrt{Q} \in \text{Ass}(J_P) \).

2. If \( P = (J : z) \in \text{Ass}(J) \), where \( z \in D \setminus P \), then \( P = \sqrt{Q} \) for some \( Q \in \Omega \) by Theorem 1.3.5.2. To prove the converse, let \( P = \sqrt{Q} \) for some \( Q \in \Omega \). Then \( \Omega_P = \{ Q^P \mid Q \in \Omega, Q \subset P \} \) is the reduced primary decomposition of \( J_P \) and \( P = \sqrt{Q_P} = \sqrt{(J_P : z)} \) for some \( z \in D \setminus J_P \). As \( D \) is \( r \)-noetherian, it follows that \( P = \sqrt{Q} \) is \( r \)-finitely generated, and by Theorem 3.1.6.4 there is some \( k \in \mathbb{N} \) such that \( P^k \subset (J : z) \). If \( k \) is minimal with this property, then there exists some \( y \in P^{k-1} \) such that \( yz \notin J_P \). It follows that \( P \) contains \( (J : yz) \subset \text{Ass}(J_P) \), hence \( P = (J : yz) \in \text{Ass}(J_P) \). Hence we obtain \( P \in \mathcal{P}_r(J) \) by Theorem 3.2.5.2(b).

3. We assume that the set \( \Omega \) of all \( I \in \mathcal{I}_r(D) \), which are not intersections of finitely many \( r \)-irreducible \( r \)-ideals, is not empty. Then \( \Omega \) possesses a maximal element \( I \). Since \( I \) is not \( r \)-irreducible, there exist \( I_1, I_2 \in \mathcal{I}_r(D) \) such that \( I = I_1 \cap I_2 \), \( I_1 \neq I \) and \( I_2 \neq I \). Since \( I \nsubseteq I_1 \) and \( I \nsubseteq I_2 \), it follows that \( I_1, I_2 \notin \Omega \). Since both \( I_1 \) and \( I_2 \) are intersections of finitely many \( r \)-irreducible \( r \)-ideals, the same is true for \( I \), a contradiction.

4. If every \( r \)-irreducible \( r \)-ideal is primary, then \( D \) is \( r \)-laskerian by 3. If \( D \) is \( r \)-laskerian and \( Q \in \mathcal{I}_r(D) \) is reducible by \( r \)-primary decomposition of \( Q \), then \( Q = \{ Q \} \) and thus \( Q \) is primary.

\( \square \)

Theorem 3.2.7. Let \( D \) be a Mori monoid.

1. If \( I \in \mathcal{I}_r(D)^* \) is \( v \)-irreducible, then \( I = zD \cap D \) for some \( z \in K^\times \).
2. If \( a \in D^* \), then \( \text{Ass}(aD) = \{ P \in v\text{-spec}(D) \mid a \in P \} \) is a finite set.

In particular, if \( X \subset D \) and \( X^\times \neq \emptyset \), then the set \( \{ P \in v\text{-spec}(D) \mid X \subset P \} \) is finite.

Proof. 1. Let \( I \in \mathcal{I}_r(D)^* \) be \( v \)-irreducible. By Theorem 2.6.6, the set \( \Omega = \{ J \in \mathcal{I}_r(D) \mid J \supseteq I \} \) has minimal elements, and we assert that it even has a smallest element. Indeed, if \( J_1, J_2 \in \Omega \) are minimal elements, then \( J_1 \cap J_2 \supseteq I \), since \( I \) is \( v \)-irreducible, hence \( J_1 \cap J_2 \in \Omega \) and therefore \( J_1 = J_2 \).

Let \( I^* \) be the smallest element of \( \Omega \). Since
\[
I = I_v = \bigcap_{z \in K^\times} zD \subseteq I^*,
\]
there is some \( z \in K^\times \) such that \( I \subset zD \) and \( I^* \nsubseteq zD \). Since \( zD \cap D \in \mathcal{I}_r(D) \), \( I \subset zD \cap D \) and \( I^* \nsubseteq zD \cap D \), we obtain \( I = zD \cap D \).

2. Let \( a \in D^* \). If \( P \in \text{Ass}(aD) \), then clearly \( P \in v\text{-spec}(D) \) and \( a \in P \). Conversely, suppose that \( P \in v\text{-spec}(D) \) and \( a \in P \). As \( P \) is \( v \)-irreducible, we obtain \( P = zD \cap D \) for some \( z \in K^\times \) by 1. Hence \( z^{-1}a \in D \), and \( P = zD \cap D = (z^{-1}a)^{-1}aD \cap D = (aD : z^{-1}a) \in \text{Ass}(aD) \).

It remains to prove finiteness. To this end, suppose that the set \( \Omega = \{ P \in v\text{-spec}(D) \mid a \in P \} \) is infinite. Since \( D \) is \( v \)-noetherian, there exists a sequence \( (P_n)_{n \geq 0} \) in \( \Omega \) such that, for every \( n \geq 0 \), \( P_n \) is maximal in \( \Omega \setminus \{ P_0, \ldots, P_{n-1} \} \). By Theorem 2.6.6, there exists some \( m \geq 0 \) such that
\[
P_0 \cap \ldots \cap P_m = P_0 \cap \ldots \cap P_{m+1} \subset P_{m+1}
\]
and therefore $P_j \subset P_{m+1}$ for some $j \in [1, m]$. However, $P_j$ is maximal in $\Omega \setminus \{P_0, \ldots, P_{j-1}\}$, and since $P_{m+1} \in \Omega \setminus \{P_0, \ldots, P_m\} \subset \Omega \setminus \{P_0, \ldots, P_{j-1}\}$, it follows that $P_{m+1} = P_j$, a contradiction. □

**Theorem 3.2.8.** Let $D$ be a Mori monoid and $I \in \mathcal{I}_v(D)^*$. 
1. If $P \in \text{Ass}(I)$ and $I = IP \cap D$, then $P$ is the greatest element of $\text{Ass}(I)$.
2. If $I$ is $v$-irreducible, then $\text{Ass}(I)$ has a greatest element $P$, and $I = IP \cap D$.
3. If $P \in v\text{-spec}(D)$, $a \in P^*$ and $I = aDP \cap D$, then $I$ is $v$-irreducible, and $P$ is the greatest element of $\text{Ass}(I)$.

**Proof.** 1. Assume that there is some $Q \in \text{Ass}(I)$ such that $Q \not\subset P$, and fix an element $s \in Q \setminus P$. Let $b \in D \setminus I$ be such that $Q = (I : b)$. Then $sb \in I$ and therefore $b \in IP \cap D = I$, a contradiction.

2. Let $\Omega$ be the (finite non-empty) set of all maximal elements of $\text{Ass}(I)$. We assert that

$$I = \bigcap_{P \in \Omega} IP \cap D.$$  

Once this is proved, it follows that $|\Omega| = 1$ since $I$ is $v$-irreducible, hence $\text{Ass}(I)$ has a greatest element $P$, and $I = IP \cap D$.

Clearly, $I \subset IP \cap D$ for all $P \in \Omega$. Thus suppose that $x \in D \setminus I$. By Theorem 3.2.5.1, every maximal element in the set $\{(I : y) \mid y \in D \setminus I\}$ belongs to $\text{Ass}(I)$. Hence there is some $Q \in \Omega$ such that $(I : x) \subset Q$, and we assert that $x \notin I_Q$. Indeed, if $x \in I_Q$, then there is some $s \in D \setminus Q$ such that $xs \in I$ and therefore $s \in (I : x) \subset Q$, a contradiction.

3. If $P \in v\text{-spec}(D)$, $a \in P^*$ and $I = aDP \cap D$, then $I \in \mathcal{I}_v(D)$, $IP \cap D = I$, $P \in \text{Ass}(aD)$ by Theorem 3.2.7.2, and therefore there exists some $b \in D$ such that $P = (aD : b) = b^{-1}aD \cap D \subset b^{-1}I \cap D$, and we assert that equality holds. Indeed, if $x \in b^{-1}I \cap D$, then $xb \in I = aDP \cap D$, hence $xbs \in aD$ for some $s \in D \setminus P$ and therefore $xs \in ab^{-1}D \cap D = P$, which implies $x \in P$.

Hence it follows that $P = (I : b) \in \text{Ass}(I)$, and by 1. $P$ is the greatest element of $\text{Ass}(I)$. It remains to show that $I$ is $t$-irreducible, and for this we prove:

**A.** If $J \in \mathcal{I}_v(D)$ and $J \ni I$, then $aJ^{-1} \subset P$ and $b \in J$.

Assume that **A** holds. If $I = J_1 \cap J_2$ for some $J_1, J_2 \in \mathcal{I}_v(D)$ such that $J_1 \ni I$ and $J_2 \ni I$, then $b \in J_1 \cap J_2 = I$ and therefore $P = (I : b) = D$, a contradiction.

**Proof of A.** Let $J \in \mathcal{I}_v(D)$ be such that $J \ni I$. If $aJ^{-1} \notin P$, then $DP = (aJ^{-1})P = aJ_P^{-1}$, and as $J_P \in \mathcal{I}_v(D_P)$, it follows that $J_P = (J_P^{-1})^{-1} = aD_P$ and $J \subset aD_P \cap D = I$, a contradiction. Hence $aJ^{-1} \subset P$ and $aJ^{-1}b \subset Pb \subset aD$, which implies $J^{-1} \subset b^{-1}D$ and therefore $b \in bD = (J^{-1})^{-1} = J$. □

**Theorem 3.2.9.**
1. A Mori monoid $D$ is $v$-laskerian if and only if $\mathcal{X}(D) = \{P \in v\text{-spec}(D) \mid P^* \neq \emptyset\}$.
2. Every s-Noetherian monoid is s-laskerian.

**Proof.** 1. Let $D$ be a Mori monoid.

Let first $D$ be $v$-laskerian and $P, Q \in v\text{-spec}(D)$ such that $Q^* \neq \emptyset$ and $Q \subset P$. We must prove that $Q = P$. If $a \in Q^*$, then $P, Q \in \text{Ass}(aD)$ by Theorem 3.2.7.2, and $I = aDP \cap D$ is $v$-irreducible by Theorem 3.2.8.3. By Theorem 3.2.6.4 $I$ is primary, and since $I = aDP \cap D \subset Q_P \cap D = Q$, it follows that $I = IQ \cap D$. By Theorem 3.2.8.1 $Q$ is the greatest element of $\text{Ass}(I)$, and therefore $Q = P$.

Assume now that $\mathcal{X}(D) = \{P \in v\text{-spec}(D) \mid P^* \neq \emptyset\}$. By Theorem 3.2.6.4 we must prove that every $v$-irreducible $v$-ideal of $D$ is primary. Let $Q \in \mathcal{I}_v(D)^*$ be $v$-irreducible. By Theorem 3.2.8.2 $\text{Ass}(Q)$ has a greatest element $P$, and as $P \in \mathcal{X}(D)$, it follows that $P \in v\text{-max}(D)$, and $\text{Ass}(Q) = \{P\} = \mathcal{P}(Q) = \{P\}$. In particular, $P = \sqrt{Q}$, and Theorem 3.2.3.2 implies that $Q$ is primary.
2. Let \( D \) be an \( s \)-noetherian monoid. By Theorem 3.2.6.4 we must prove that every \( s \)-irreducible ideal of \( D \) is primary. Let \( Q \subseteq D \) be an ideal which is not primary. Then there exist \( a, b \in D \) such that \( ab \in Q, \ a \notin Q \) and \( b \notin \sqrt{Q} \). For all \( n \in \mathbb{N} \), we have \( Q \subseteq (Q : b) \subset (Q : b^n) \subset (Q : b^{n+1}) \), and as \( D \) is \( s \)-noetherian, there exists some \( n \in \mathbb{N} \) such that \((Q : b^n) = (Q : b^{2n})\). We assert that \( Q = (Q : b^n) \cap (Q \cup b^n D) \), which shows that \( Q \) is not \( s \)-irreducible. Clearly, \( Q \subseteq (Q : b^n) \cap (Q \cup b^n D) \), and we assume that there is some \( x \in (Q : b^n) \cap (Q \cup b^n D) \setminus Q \). Then \( x = b^n u \) for some \( u \in D \) and \( b^n x = b^{2n} u \in Q \). Since \((Q : b^n) = (Q : b^{2n})\), it follows that \( b^n u \in Q \), a contradiction. \( \square \)

### 3.3. Laskerian rings

In this Section, we use the common terminology of commutative ring theory.

**Theorem 3.3.1.** Every noetherian ring is laskerian.

**Proof.** Let \( D \) be a noetherian ring. By Theorem 3.2.6.4 it suffices to prove that every \((d)\)-irreducible ideal of \( D \) is primary. Let \( Q \subseteq D \) be an ideal which is not primary. Then there exist \( a, b \in D \) such that \( ab \in Q, a \notin Q \) and \( b \notin \sqrt{Q} \). For all \( n \in \mathbb{N} \), we have \( Q \subseteq (Q : b) \subset (Q : b^n) \subset (Q : b^{n+1}) \), and as \( D \) is noetherian, there exists some \( n \in \mathbb{N} \) such that \((Q : b^n) = (Q : b^{2n})\). We assert that \( Q = (Q : b^n) \cap (Q + b^n D) \), which shows that \( Q \) is not irreducible. Clearly, \( Q \subset (Q : b^n) \cap (Q + b^n D) \), and we assume that there is some \( x \in (Q : b^n) \cap (Q + b^n D) \setminus Q \). Then \( x = q + b^n u \) for some \( q \in Q \) and \( u \in D \), and \( b^n x = b^n q + b^{2n} u \in Q \). Hence \( b^{2n} u \in Q \), and since \((Q : b^n) = (Q : b^{2n})\), it follows that \( b^n u \in Q \) and therefore also \( x \in Q \), a contradiction. \( \square \)

**Theorem 3.3.2.** Every laskerian ring satisfies the \( \text{ACC} \) for radical ideals.

**Proof.** Let \( D \) be a laskerian ring. Then \( D \) satisfies the \( \text{ACC} \) for radical ideals if and only if \( D \) is \( \sqrt{d(D)} \)-noetherian. By Theorem 3.1.7 we must prove:

1. For every ideal \( J \subset D \) the set \( \mathcal{P}(J) \) is finite.
2. \( D \) satisfies the \( \text{ACC} \) on prime ideals.

1. Let \( J \subset D \) be an ideal and \( \Omega = \{Q_1, \ldots, Q_m\} \) a primary decomposition of \( J \). If \( P \in \mathcal{P}(J) \), then \( P \supset J = Q_1 \cap \ldots \cap Q_m \), and there exists some \( j \in [1, m] \) such that \( Q_j \subset P \). Since \( J \subset \sqrt{Q_j} \subset P \), it follows that \( P = \sqrt{Q_j} \), and thus \( \mathcal{P}(J) \subset \{\sqrt{Q_1}, \ldots, \sqrt{Q_m}\} \).

2. Assume to the contrary that there exists a sequence \( (P_n)_{n \geq 0} \) of prime ideals such that \( P_n \subsetneq P_{n+1} \) for all \( n \geq 0 \). For every \( n \geq 1 \), we fix an element \( p_n \in P_n \setminus P_{n-1} \), and we consider the ideals

\[ J = \sum_{i \geq 0} p_1 \cdots p_i P_i \quad \text{and} \quad J_n = (J : p_1 \cdots p_n) \supset P_n. \]

Let \( \Omega = \{Q_1, \ldots, Q_m\} \) be a primary decomposition of \( J \). For \( n \geq 1 \), we obtain

\[ J_n = \left( \bigcap_{j=1}^{m} Q_j : p_1 \cdots p_n \right) = \bigcap_{j=1}^{m} (Q_j : p_1 \cdots p_n), \]

and we set \( \Omega_n = \{(Q_j : p_1 \cdots p_n) \mid j \in [1, m], p_1 \cdots p_n \notin Q_j \} \). If \( j \in [1, m] \) and \( p_1 \cdots p_n \notin Q_j \), then \((Q_j : p_1 \cdots p_n)\) is primary, and \( \sqrt{(Q_j : p_1 \cdots p_n)} = \sqrt{Q_j} \) by Theorem 1.3.3.3 (b). In particular, it follows that \( \{\sqrt{Q} \mid Q \in \Omega_n\} \subset \{\sqrt{Q_1}, \ldots, \sqrt{Q_m}\} \) for all \( n \geq 1 \). Now we prove the following assertion:

**A.** For all \( n \geq 1 \) and all \( j \in [1, n + 1] \), we have

\[ p_jp_{j+1} \cdots p_n J_n \subset P_{j-1} + \sum_{i \geq j} p_j p_{j+1} \cdots p_i P_i. \]
3.4. Valuation monoids and primary monoids

Remarks and Definition 3.4.1.

1. Let $\Gamma$ be a (multiplicative) abelian group.
   (a) Let $\leq$ be a partial ordering on $\Gamma$. Then $(\Gamma, \leq)$ is called a partially ordered abelian group if, for all $a, b, c \in \Gamma$, $a \leq b$ implies $ac \leq bc$. The set $\Gamma_+ = \{ x \in \Gamma \mid x \geq 1 \}$ is called the positive cone of $\Gamma$. If $\Gamma_+ \cdot 1 = \{ x^{-1} \mid x \in \Gamma_+ \}$, then $\Gamma_+ \cap \Gamma_+ \cdot 1 = \{ 1 \}$ (that means, $\Gamma_+$ is a reduced submonoid of $\Gamma$), and $\leq$ is a total order (and thus $(\Gamma, \leq)$ a totally ordered abelian group) if and only if $\Gamma = \Gamma_+ \cup \Gamma_+ \cdot 1$.

2. Let $\Delta \subset \Gamma$ be a reduced submonoid. Then there exists a unique partial ordering $\leq$ on $\Gamma$ such that $(\Gamma, \leq)$ is a partially ordered abelian group and $\Gamma_+ = \Delta$ [indeed, define $\leq$ by $a \leq b$ if and only if $a^{-1}b \in \Delta$].

3. Let $\Gamma$ be an additive abelian group and $\leq$ a total ordering on $\Gamma$ such that, for all $a, b, c \in \Gamma$, $a \leq b$ implies $a + c \leq b + c$. Then we call $\Gamma = (\Gamma, \leq)$ a totally ordered additive abelian group, and we set $\Gamma_+ = \{ x \in \Gamma \mid x \geq 0 \}$. Then $\Gamma = \Gamma_+ \cup -\Gamma_+$ and $\Gamma_+ \cap -\Gamma_+ = \{ 0 \}$.

4. Let $D$ be a cancellative monoid and $K = \mathfrak{q}(D)$. On $K^\times/D^\times$, we define a partial ordering $\leq$ by $aD^\times \leq bD^\times$ if $aD \supset bD$ (equivalently, if $a^{-1}b \in D$). Obviously, this definition is independent of the choice of representatives, and it makes $K^\times/D^\times$ into a partially ordered abelian group. $\mathcal{G}(D) = (K^\times/D^\times, \leq)$ is called the group of divisibility of $D$. By definition, $\mathcal{G}(D)_+ = D^\times/D^\times$.

Theorem und Definition 3.4.2. Let $D$ be cancellative.

1. The following assertions are equivalent:
   (a) For all $a, b \in D$, if $a \notin bD$, then $b \in aD$.
   (b) Every $s$-finitely generated $s$-ideal $J \in \mathcal{I}_{s,f}(D)^\times$ is principal.
   (c) For all $z \in K^\times$, if $z \notin D$, then $z^{-1} \in D$. 

The group of divisibility $G(D)$ is totally ordered.

There exists a surjective group homomorphism $w: K^\times \to \Gamma$ onto a totally ordered abelian group $\Gamma$ such that $D^* = w^{-1}(\Gamma_+)$ = \{x \in K^\times \mid w(x) \geq 0\}.

The set $M_s(D)(K)$ of all $D$-submodules of $K$ is a chain.

The set $I_s(D)$ of all ideals of $D$ is a chain.

If these conditions are fulfilled, then $D$ is called a valuation monoid (of $K$), and a group epimorphism $w: K^\times \to \Gamma$ onto a totally ordered abelian group $\Gamma$ such that $D^* = w^{-1}(\Gamma_+)$ is called a valuation morphism of $D$.

If $D$ is a valuation monoid and $r$ is a module system on $K$ such that $D = D_r$, then $D$ is called an $r$-valuation monoid.

In particular:

- Every valuation monoid is a GCD-monoid.
- A monoid $D$ is a valuation monoid if and only if $D^*/D^\times$ is a valuation monoid.
- Every divisible monoid is a valuation monoid.

2. Let $D$ be a valuation monoid and $w: K^\times \to \Gamma$ a valuation morphism of $D$.  
(a) Ker$(w) = D^\times$, and $w$ induces an order isomorphism $w^*: G(D) \to \Gamma$, given by $w^* (xD^\times) = w(x)$ for all $x \in K^\times$.

In particular, $w^*(D^*/D^\times) = \Gamma_+$.

(b) If $w_1: K^\times \to \Gamma_1$ is another valuation morphism of $D$, then there exists a unique order isomorphism $\varphi: \Gamma \to \Gamma_1$ such that $\varphi w = w_1$.

(c) If $E \in \mathcal{P}_t(K)$ and $E^* \neq \emptyset$, then there exists some $a \in E$ such that $ED = aD$, and for every such $a \in E$ we have $w(a) = \min w(E^*)$.

3. If $D$ is a valuation monoid and $V$ is a monoid such that $D \subset V \subset K$, then $V$ is a valuation monoid, $V \setminus D \subset V^\times$, $P = D \setminus V^\times \in \text{s-spec}(D)$, and $V = DP = (V^\times \cap D)^{-1}D$.

4. Let $(V_\lambda)_{\lambda \in \Lambda}$ be a chain of valuation monoids such that $q(V_\lambda) = K$ for all $\lambda \in \Lambda$. Then $V^* = \bigcup_{\lambda \in \Lambda} V_\lambda$ and $V_\ast = \bigcap_{\lambda \in \Lambda} V_\lambda$ are valuation monoids of $K$.

Proof. 1. (a) $\Rightarrow$ (b) Let $J \in I_s(D)^\bullet$. Then $J = E_\lambda$, where $\emptyset \neq E \in \mathcal{P}_t(D^\bullet)$, and we proceed by induction on $|E|$. If $|E| = 1$, there is nothing to do. Thus suppose that $E = E' \cup \{a\}$, where $a \in E \setminus E'$, and that $E'_a = bD$. Then $J = bD \cup aD$. If $a \in bD$, then $J = bD$. If $a \notin bD$, then $b \in aD$, and $J = aD$.

(b) $\Rightarrow$ (c) Let $z = a^{-1}b \in K \setminus D$, where $a, b \in D^\bullet$ and $b \notin aD$. By assumption, there exists some $u \in D$ such that $aD \cup bD = uD$, and thus $u \in aD$ or $u \in bD$. If $u \in aD$, then $aD = uD \cap bD$ and $a^{-1}b = z \in D$. If $u \in bD$, then $bD = uD \cap aD$ and $b^{-1}a = z^{-1} \in D$.

(c) $\Rightarrow$ (d) If $x, y \in K^\times$, then either $x^{-1}y \in D$ or $y^{-1}x \in D$, and therefore either $xD^\times \leq yD^\times$ or $yD^\times \leq xD^\times$. Hence $G(D)$ is totally ordered.

(d) $\Rightarrow$ (e) Let $w: K^\times \to G(D)$ be the canonical epimorphism.

(e) $\Rightarrow$ (f) Let $w: K^\times \to \Gamma$ be an epimorphism onto a totally ordered abelian group $\Gamma$ such that $D^* = w^{-1}(\Gamma_+)$. Let $M, N \in M_s(K)$, $M \not\subset N$, $a \in M \setminus N$, and let $b \in N^\bullet$ be arbitrary. Then $b^{-1}a \notin D^\bullet$, since otherwise $a = b^{-1}ab \in DN = N$. Hence $w(b^{-1}a) < 0$, $w(a^{-1}b) = -w(ab^{-1}) > 0$, hence $ab^{-1} \in D$ and therefore $b \in ab^{-1}bD = aD \subset M$. Thus it follows that $N \subset M$.

(f) $\Rightarrow$ (g) $\Rightarrow$ (a) Obvious.

2. (a) If $x \in K^\times$, then $x \in \text{Ker}(w)$ if and only if $w(x) \geq 0$ and $w(x^{-1}) = -w(x) \geq 0$, that is, if and only if $x \in D$ and $x^{-1} \in D$ and thus $x \in D^\times$.
As $w$ is an epimorphism, it induces an isomorphism $w^*: K^*/D^* = G(D) \rightarrow \Gamma$, given as asserted, and we must prove that $w^*$ is an order isomorphism. If $x, y \in K^*$ and $xD^* \leq yD^*$, then $x^{-1}y \in D$ and therefore $0 \leq w(x^{-1}y) = -w(x) + w(y)$, which implies $w(x) \leq w(y)$.

(b) Let $w^*: G(D) \rightarrow \Gamma$ and $w_1^*: G(D) \rightarrow \Gamma_1$ be the order isomorphisms induced by $w$ and $w_1$ according to (a). Then $\varphi = w_1^* \circ w^{-1}: \Gamma \rightarrow \Gamma_1$ is an order isomorphism, and it is obviously the only order isomorphism satisfying $\varphi \circ w = w_1$.

(c) The finite set $\{cD \mid c \in E^*\}$ is a chain. Hence there exists some $a \in E^*$ such that $cD \subset aD$ for all $c \in E$ and thus $ED = aD$. For every such $a \in E^*$ we have $a^{-1}c \in D^*$ for all $c \in D^*$, hence $0 \leq w(a^{-1}c) = -w(a) + w(c)$, and therefore $w(a) = \min w(E^*)$.

3. Let $D \subset V \subset K$ be a monoid. Then $K = q(V)$, and if $z \in K \setminus V$, then $z \notin D$ and $\gamma^{-1} \in D \subset V$. Hence $V$ is a valuation monoid of $K$. If $z \in V \setminus D$, then $\gamma^{-1} \in D \subset V$ and thus $z \in V^x$. Hence $V \setminus D \subset V^x$. If $z \in V \setminus D$, then $z \in V^x$, $\gamma^{-1} \subset D \cap V^x$, and therefore $z \in (D \cap V^x)^{-1} \subset (D \cap V^x)^{-1}D$. Hence it follows that $V = (V \setminus D) \cup D \subset (D \cap V^x)^{-1}D \subset V$, and equality holds.

4. Since $(V_\lambda)_{\lambda \in \Lambda}$ is a chain, it follows that $V^*$ and $V_*$ are submonoids of $K$, and by 2. $V^*$ is a valuation monoid. If $x \in K \setminus V_\mu$, then $x \in K \setminus V_\lambda$, for some $\mu \in \Lambda$, and consequently $x^{-1} \in V_\mu$. If $\lambda \in \Lambda$ and $V_\mu \subset V_\lambda$, then $x^{-1} \in V_\lambda$. If $\lambda \in \Lambda$ and $V_\mu \not\subset V_\lambda$, then $V_\lambda \subset V_\mu$, hence $x \notin V_\lambda$, and consequently $x^{-1} \in V_\lambda$. Thus we have proved that $x^{-1} \in V_\lambda$ for all $\lambda \in \Lambda$ and therefore $x^{-1} \in V_*$. Consequently, also $V_*$ is a valuation monoid of $K$. \qed

**Theorem 3.4.3.** Let $\Gamma$ be an additive abelian group. Then the following assertions are equivalent:

(a) There exists an ordering $\leq$ on $\Gamma$ such that $(\Gamma, \leq)$ is a totally ordered additive abelian group.

(b) $\Gamma$ is torsion-free.

(c) There exists a subset $P \subset \Gamma$ such that $P + P \subset P$, $P \cap -P = \{0\}$ and $\Gamma = P \cup -P$.

**Proof.** (a) $\Rightarrow$ (b) Let $\alpha \in \Gamma$ and $n \in \mathbb{N}$ be such that $n\alpha = 0$. Then $n(-\alpha) = 0$, and thus we may assume that $\alpha \geq 0$. If $\alpha > 0$, then it follows that $n\alpha \geq \alpha > 0$, a contradiction. Hence $\alpha = 0$ and $\Gamma$ is torsion-free.

(b) $\Rightarrow$ (c) Let $\Omega$ be the set of all subsets $R \subset \Gamma$ such that $R + R \subset R$ and $R \cap -R = \{0\}$. Then $\{0\} \in \Omega$, and the union of every chain in $\Omega$ again belongs to $\Omega$. By Zorn’s Lemma, $\Omega$ contains a maximal element $E$, and we must prove that $\Gamma = E \cup -E$. Assume to the contrary that there is an element $\gamma \in \Gamma \setminus (E \cup -E)$. Then $\gamma \neq 0$, and we assert that either $E^+ = E \cup \{0\} \gamma \in \Omega$ or $E^+ \cap \{0\} \gamma \in \Omega$ (which gives the desired contradiction). Assume the contrary. Then $E^+ \cap -E^+ \not\supset \{0\}$ and $E^+ \cap -E^+ \not\supset \{0\}$, and there exist $p_1, p_1', p_2, p_2' \in E$ and $n_1, n_1', n_2, n_2' \in \mathbb{N}_0$, such that $p_1 + n_1 \gamma = -p_1' + n_1' \gamma \neq 0$ and $p_2 - n_2 \gamma = -(p_2' - n_2' \gamma) \neq 0$. Since $E \cap -E = \{0\}$, we have $n_1 + n_1' > 0$ and $n_2 + n_2' > 0$, and since $(n_1 + n_1') \gamma = -(p_1 + p_1')$ and $(n_2 + n_2') \gamma = p_2 + p_2' \in E$, we obtain $(n_1 + n_1')(n_2 + n_2') \gamma \in E \cap -E$, a contradiction.

(c) $\Rightarrow$ (a) For $\alpha, \beta \in \Gamma$, we define $\alpha \leq \beta$ if and only if $\beta - \alpha \in P$. Then $(\Gamma, \leq)$ is a totally ordered additive abelian group and $\Gamma_+ = P$. \qed

**Theorem 3.4.4.** Let $D$ be a valuation monoid, $P \subset D$ a prime ideal, $Q \subset D$ an ideal and

$$Q_0 = \bigcap_{n \in \mathbb{N}} Q^n.$$  

1. $Q_0$ and $\sqrt{Q}$ are prime ideals.

2. If $Q$ is $P$-primary and $a \in D \setminus P$, then $Q = Qa$. In particular, if $Q$ is $P$-primary and principal, then $P = D \setminus D^\times$. \qed
3. If $Q_1, Q_2 \subset D$ are $P$-primary ideals, then $Q_1Q_2$ is $P$-primary. In particular, $P^m$ is $P$-primary for all $m \in \mathbb{N}.$
4. If $Q$ is $P$-primary and $P \neq P^2,$ then $Q = P^n$ for some $n \in \mathbb{N}.$
5. If $P = D \setminus D^\times$ and $P \neq P^2,$ then $P = pD$ for some $p \in D^\times.$

PROOF. 1. If $a, b \in D \setminus Q_0,$ then there exist $m, n \in \mathbb{N}$ such that $a \notin Q^m$ and $b \notin Q^n.$ Hence it follows that $Q^m \subset Da,$ $Q^n \subset Db,$ $Q^m b \subset Dab,$ and therefore $ab \notin Q^{m+n}.$ Hence $ab \notin Q_0,$ and thus $Q_0$ is a prime ideal.

Since $\mathcal{P}(Q)$ is a chain, it follows that $|\mathcal{P}(Q)| = 1,$ and if $\mathcal{P}(Q) = \{P_0\},$ then $\sqrt{Q} = P_0.$

2. Since $a \notin P,$ we obtain $Q \subset P \subset aD,$ hence $A = a^{-1}Q \subset D$ and $Q = aA.$ Since $a \notin P,$ it follows that $A \subset Q = aA \subset A,$ hence $A = Q$ and $Q = aQ.$

Assume now that $Q = qD$ for some $q \in D.$ If $a \in D \setminus D^\times,$ then $qD = aqD$ implies $a \in D^\times,$ and therefore we obtain $P = D \setminus D^\times.$

3. By Theorem 1.3.2 we have $\sqrt{Q_1Q_2} = P.$ Suppose that $a, b \in D,$ $ab \in Q_1Q_2$ and $a \notin P.$ Then $Q_1 = Q_1a$ by 1., hence $a \notin qQ_2$ and therefore $b \in Q_1Q_2.$ Hence $Q_1Q_2$ is $P$-primary.

4. We prove first that $P^m \subset Q$ for some $m \in \mathbb{N}.$ Assume the contrary. Then $Q \subset P^m$ for all $m \in \mathbb{N},$ hence

$$Q \subset P_0 = \bigcap_{m \in \mathbb{N}} P^m.$$ Since $P_0$ is a prime ideal by 1., we obtain $P = \sqrt{Q} \subset P_0 \subset P^2 \subset P,$ a contradiction. Let now $n \in \mathbb{N}$ be minimal such that $P^n \subset Q,$ and let $y \in P^{n-1} \setminus Q.$ Then $Q \subset yD$ and $A = y^{-1}Q \in \mathcal{I}(D).$ Since $Q = yA$ and $y \notin Q,$ we obtain $A \subset P$ and therefore $Q = yA \subset yP \subset P^n.$ Hence $Q = P^n.$

5. If $p \in P \setminus P^2,$ then $P^2 \subset pD \subset P,$ hence $\sqrt{pD} = P$ and thus $P \subset D$ is $P$-primary by Theorem 3.2.3.2. Hence $pD = P$ by 4. 

\begin{theorem}
Let $D$ be a valuation monoid, $K \neq D$ and $M = D \setminus D^\times.$
1. If $M$ is not a principal ideal of $D,$ then $M^{-1} = M_v = D.$
2. If $\emptyset \neq X \subset D,$ then

$$X_v = \begin{cases} aD & \text{if } X_s = aM \text{ and } M \text{ is not principal}, \\ X_s & \text{otherwise}. \end{cases}$$

3. If $M$ is principal, then $v = s$ is the only ideal system of $D.$ If $M$ is not principal, then $v \neq s,$ $v$ and $s$ are the only ideal systems of $D$ defined on $K,$ and $s = t.$ In any case, $t = s$ is the only finitary ideal system of $D$ defined on $K.$

PROOF. 1. Suppose that there is some $z \in M^{-1} \setminus D.$ Then it follows that $z^{-1} \in D \setminus D^\times = M,$ hence $Mz \subset D$ and $M \subset Dz^{-1} \subset M,$ which implies that $M = Dz^{-1}$ is principal. Consequently, if $M$ is not principal, then $M^{-1} = D$ and thus $M_v = (M^{-1})^{-1} = D.$

2. If $\emptyset \neq X \subset D,$ $a \in D,$ $X_s = aM$ and $M$ is not principal, then $X_v = (X_s)_v = aM_v = aD$ by 1.

Assume now that $X_s \neq X_v.$ Then $X_s \subset X_v,$ we fix an element $a \in X_v \setminus X_s,$ and we assert that $X_s = aM$ and $M$ is not principal.

As $aD \not\subset X_s,$ we obtain $X_s \not\subset aD,$ hence $a^{-1}X_s \not\subset D,$ and as $a^{-1}X_s \subset D$ is an ideal, it follows that $a^{-1}X_s \subset aM.$ If $X_s \not\subset aM,$ then there is some $c \in M$ such that $ac \not\in X_s,$ hence $X_s \subset acD$ and therefore $aD \subset X_s = (X_s)_v \subset acD,$ which implies $c \in D^\times,$ a contradiction. Therefore we obtain $X_s = aM.$ If $M$ were principal, say $M = pD$ for some $p \in D,$ then $X_s = aM = apD$ and therefore $X_v = (X_s)_v = apD = X_s.$

3. Let $r : \mathbb{P}(K) \rightarrow \mathbb{P}(K)$ be an ideal system of $D.$ Then $s \leq r \leq v,$ and $M_r \in (M, D).$ We assert that $r = s$ if $M_r = M,$ and $r = v$ if $M_r = D.$ Indeed, let $X \subset D$ be any subset such that $X_s \neq X_v.$ Then
$X_s = aM$, hence $X_r = aM_r$, and the assertion follows by 2. Consequently, if $M$ is a principal ideal, then $M_r = M$ and therefore $r = s$. If $M$ is not principal, then $v \neq s$ and $r \in \{s, v\}$. □

**Theorem und Definition 3.4.6.** Let $D$ be cancellative, $K \neq D$ and $r : \mathbb{P}(K) \to \mathbb{P}(K)$ a finitary ideal system of $D$.

1. The following assertions are equivalent:
   (a) Every $q \in D \setminus D^\times$ the principal ideal $qD$ is primary.
   (b) For all $a \in D \setminus D^\times$ and $b \in D^\bullet$ there is some $n \in \mathbb{N}$ such that $b | a^n$.
   (c) $D \setminus D^\times$ is the only non-zero prime ideal of $D$.
   (d) Every ideal $J \subseteq D$ is primary.

   If these conditions are fulfilled, then $D$ is called primary.

2. If $D$ is primary, then $D$ is $r$-local.
3. If $P \in r$-spec$(D)$ and $P^\bullet \neq \emptyset$, then $D_P$ is primary if and only if $P \in \mathbb{X}(D)$.
4. Let $T \subset D^\bullet$ be a multiplicatively closed subset such that $T^{-1}D$ is primary. Then $T^{-1}D = D_P$ for some $P \in \mathbb{X}(D)$.

**Proof.** 1. (a) $\Rightarrow$ (b) If $a \in D \setminus D^\times$ and $b \in D^\bullet$, then $ab \notin D^\times$, hence $abD$ is a primary ideal, $ab \in abD$ and $b \notin abD$. Hence there is some $n \in \mathbb{N}$ such that $a^{n+1} \in abD$, which implies $b | a^n$.

   (b) $\Rightarrow$ (c) Let $b \in D^\bullet \setminus D^\times$ and $P \in r$-spec$(D)$ be such that $b \in P$. Then $P \subset D \setminus D^\times$, and we assert that equality holds. If $a \in D \setminus D^\times$, then there exists some $n \in \mathbb{N}$ such that $b | a^n$, hence $a^n \in P$ and thus $a \in P$. Hence $P = D \setminus D^\times$.

   (c) $\Rightarrow$ (d) If $J \subseteq D$ is an ideal, then Theorem 1.3.2.3 implies
   \[
   \sqrt{J} = \bigcap_{P \in \mathbb{P}(J)} P = D \setminus D^\times \in s$-max$(D),
   \]
   and thus $J$ is primary by Theorem 3.2.3.2.

   (d) $\Rightarrow$ (a) Obvious.

2. Obvious by 1.

3. By Theorem 1.3.6.2.

4. Let $\overline{T}$ be the saturation of $T$. Then $P = D \setminus \overline{T} \in s$-spec$(D)$, $T^{-1}D = D_P$ and the assertion follows by 3. □

**Theorem 3.4.7.** Let $D$ be a valuation monoid and $K \neq D$. Then the following assertions are equivalent:

(a) $D$ is primary.

(b) There is an additive subgroup $\Gamma \subset \mathbb{R}$ such that $D^\bullet/D^\times \cong \Gamma_+$. 

(c) There is no monoid $B$ such that $D \subseteq B \subseteq K$.

**Proof.** (a) $\Rightarrow$ (b) If $D$ is primary, then $D/D^\times$ is also primary. Hence we may assume that $D$ is reduced, and it suffices to prove that there is an additive subgroup $\Gamma \subset \mathbb{R}$ and an isomorphism $\Phi : K^\times \to \Gamma$ such that $\Phi(D^\bullet) = \Gamma_+$.

We fix an element $a_0 \in D' = D^\bullet \setminus \{1\}$, and for $a \in D^\bullet$, we define
\[
M(a) = \left\{ \frac{m}{n} \mid m \in \mathbb{N}_0, \ n \in \mathbb{N}, \ a_0^m | a^n \right\} \subset \mathbb{Q}_{\geq 0}.
\]
We assert that, for every $a \in D^\bullet$, the set $M(a)$ is bounded, $0 \in M(a)$, and $M(a) = \{0\}$ if and only if $a = 1$. Indeed, we obviously have $M(1) = \{0\}$, and $0 \in M(a)$ for all $a \in D$. Thus let $a \in D'$. As $D$ is
primary, there exist \( k, l \in \mathbb{N} \) such that \( a_0 | a^k \) and \( a | a_0^l \). Hence \( \frac{1}{k} \in M(a) \), and if \( \frac{m}{n} \in M(a) \) for some \( m, n \in \mathbb{N} \), then \( a_0^m | a^m | a_0^nl \), hence \( m \leq nl \) and therefore \( 0 < \sup M(a) \leq l \). Now we define

\[
\Phi: D^* \to \mathbb{R}_{\geq 0} \quad \text{by} \quad \Phi(a) = \sup M(a).
\]

Then \( \Phi(a) = 0 \) if and only if \( a = 1 \). We prove first that \( \Phi \) is a homomorphism. Let \( a_1, a_2 \in D \), \( n \in \mathbb{N} \), and for \( i \in \{1, 2\} \), let \( m_i \in \mathbb{N}_0 \) be such that \( a_0^{m_1} | a^n | a_0^{m_1+1} \). Then \( a_0^{m_1+m_2} | (a_1a_2)^n | a_0^{m_1+m_2+2} \), hence

\[
\frac{m_1}{n} \leq \Phi(a_1) \leq \frac{m_1+1}{n}, \quad \frac{m_2}{n} \leq \Phi(a_2) \leq \frac{m_2+1}{n} \quad \text{and} \quad \frac{m_1+m_2}{n} \leq \Phi(a_1a_2) \leq \frac{m_1+m_2+2}{n},
\]

and therefore

\[
|\Phi(a_1) + \Phi(a_2) - \Phi(a_1a_2)| \leq \frac{2}{n}.
\]

As \( n \to \infty \), we obtain \( \Phi(a_1a_2) = \Phi(a_1) + \Phi(a_2) \). If \( a_1, a_2 \in D \) and \( a_2 | a_1 \), then \( a_1a_2^{-1} \in D \), and \( \Phi(a_1) = \Phi(a_1a_2^{-1}) + \Phi(a_2) \geq \Phi(a_2) \).

Let \( \tilde{\Phi}: K^* \to \mathbb{R} \) be the extension of \( \Phi \) to a homomorphism of the quotient groups, given by \( \tilde{\Phi}(a_1a_2^{-1}) = \Phi(a_1) - \Phi(a_2) \) for all \( a_1, a_2 \in D^* \). If \( a \in \text{Ker}(\tilde{\Phi}) \cap D^* \), then \( \Phi(a) = 0 \) and thus \( a = 0 \). If \( a \in \text{Ker}(\tilde{\Phi}) \cap D^* \), then \( a^{-1} \in \text{Ker}(\tilde{\Phi}) \cap D^* \) and thus again \( a = 0 \). Hence \( \tilde{\Phi} \) is a homomorphism, \( \tilde{\Phi}(K^*) \subset \mathbb{R} \) is a subgroup, \( \tilde{\Phi}: K^* \to \Gamma = \tilde{\Phi}(K^*) \) is an isomorphism, \( \tilde{\Phi}(D^*) \subset \Gamma_+ \), and it remains to prove equality. Let \( a = a_1a_2^{-1} \in K^* \) be such that \( \Phi(a_1) - \Phi(a_2) = \tilde{\Phi}(a) > 0 \). Then \( a_2 | a_1 \) and therefore \( a = a_1a_2^{-1} \in D^* \).

(b) \( \Rightarrow \) (c) Let \( \Gamma \subset \mathbb{R} \) be a subgroup, \( \Phi: D^*/D^* \to \Gamma \) an isomorphism and \( \tilde{\Phi}: K^*/D^* \to \Gamma \) its extension to an isomorphism of the quotient groups. Let \( B \) be a monoid such that \( D \subseteq B \subseteq K \). Then \( D^*/D^* \leq B^*/D^* \leq K^*/D^* \), and if \( \Delta = \Phi(B^*/D^*) \), then \( \Gamma_+ \subseteq \Delta \subseteq \Gamma \). It is now sufficient to prove that \( \Gamma = \Delta \), for then it follows that \( B^*/D^* = K^*/D^* \) and therefore \( B = K \).

We fix an element \( a \in \Delta \setminus \Gamma_+ \). If \( c \in \Gamma_+ \), then \( -a > 0 \) implies that there is some \( n \in \mathbb{N} \) such that \( -c \leq n(-a) \), hence \( c - na \in \Gamma_+ \) and \( c = (c - na) + na \in \Delta \).

(c) \( \Rightarrow \) (a) Let \( P \subset D \) be a prime ideal such that \( P^* \neq \emptyset \). If \( a \in P^* \), then \( a^{-1} \notin D_P \). Hence \( D \subset D_P \subset K \), which implies \( D = D_P \) and therefore \( P = D \setminus D^* \). Consequently, \( D \setminus D^* \) is the only non-zero prime ideal of \( D \), and thus \( D \) is primary. \( \square \)

**Theorem und Definition 3.4.8.** Let \( D \) be cancellative, \( K \neq D \) and \( P = D \setminus D^* \). Then the following assertions are equivalent:

- (a) \( D \) is factorial, and there is some \( p \in D \) such that \( \{p\} \) is a complete set of primes \( [ \text{equivalently: There is some } p \in D \text{ such that every } a \in D^* \text{ has a unique representation } a = p^n u, \text{ where } n \in \mathbb{N}_0 \text{ and } u \in D^* ] \). \( \text{[and ]} \)

- (b) \( D \) is an atomic valuation monoid.

- (c) \( D \) is atomic and \( P \) is a principal ideal.

- (d) \( D \) is primary and contains a prime element.

- (e) \( D \) is a valuation monoid, and

\[
\bigcap_{n \in \mathbb{N}} P^n = \{0\}.
\]

- (f) \( D \) is an s-noetherian valuation monoid.

- (g) \( D \) is a \( v \)-noetherian valuation monoid.

If these conditions are fulfilled, then \( D \) is called a **discrete valuation monoid** or a **dv-monoid**.
3.4. VALUATION MONOIDS AND PRIMARY MONOIDS

Let $D$ be a valuation monoid of $K$ and $r$ a finitary module system on $K$. Then the following assertions are equivalent:

(a) $V = V_r$.

(b) $id_K$ is an $(r, t(V))$-homomorphism.

(c) $X_r \subseteq XV$ for all $X \subseteq K$.

The following assertions are equivalent:

(a) $V$ is a $t$-valuation monoid.

(b) $V$ is a valuation monoid, $D \subseteq V$, and the inclusion map $D \hookrightarrow V$ is a GCD-homomorphism.

(c) $V = D_p$ for some $P \in t$-spec($D$).

For every subset $X \subseteq K$ we have

\[ X_r = \bigcap_{P \in t\text{-spec}(D)} XD_P = \bigcap_{P \in t\text{-max}(D)} XD_P = \bigcap_{V \in \mathcal{V}} XV, \]

where $\mathcal{V}$ is the set of all $t$-valuation monoids of $K$.

**Proof.** (a) $\Rightarrow$ (b) We must prove that $E_r \subseteq E_{t(V)}$ for all $E \in \mathbb{P}_t(K)$. If $E^* = \emptyset$, this is obvious. If $E \in \mathbb{P}_t(K)$ and $E^* \neq \emptyset$, then $E_{t(V)} = E_{s(V)} = EV = aV$ for some $a \in E$ by the Theorems 3.4.2.2 (c) and 3.4.5. Now $E \subseteq aV$ implies $E_r \subseteq (aV)_r = aV$.

(b) $\Rightarrow$ (c) If $X \subseteq V$, then $X_r \subseteq X_{t(V)} = X_{s(V)} = XV$ by Theorem 3.4.5.

(c) $\Rightarrow$ (a) $V_r \subseteq XV$ implies $V_r = V$.

2. (a) $\Rightarrow$ (b) By 1., $id_K$ is a $(t, t(V))$-homomorphism, and thus Theorem 2.6.5 implies that $D \subseteq V$ and $D \hookrightarrow V$ is a GCD-homomorphism.

---

**Theorem 3.4.9.** Let $D$ be a GCD-monoid, $t = t(D)$ and $V \subseteq K$ a submonoid.

1. Let $V$ be a valuation monoid of $K$ and $r$ a finitary module system on $K$. Then the following assertions are equivalent:

(a) $V = V_r$.

(b) $id_K$ is an $(r, t(V))$-homomorphism.

(c) $X_r \subseteq XV$ for all $X \subseteq K$.

2. The following assertions are equivalent:

(a) $V$ is a $t$-valuation monoid.

(b) $V$ is a valuation monoid, $D \subseteq V$, and the inclusion map $D \hookrightarrow V$ is a GCD-homomorphism.

(c) $V = D_p$ for some $P \in t$-spec($D$).

3. For every subset $X \subseteq K$ we have

\[ X_r = \bigcap_{P \in t\text{-spec}(D)} XD_P = \bigcap_{P \in t\text{-max}(D)} XD_P = \bigcap_{V \in \mathcal{V}} XV, \]

where $\mathcal{V}$ is the set of all $t$-valuation monoids of $K$. 

**Proof.** 1. (a) $\Rightarrow$ (b) We must prove that $E_r \subseteq E_{t(V)}$ for all $E \in \mathbb{P}_t(K)$. If $E^* = \emptyset$, this is obvious. If $E \in \mathbb{P}_t(K)$ and $E^* \neq \emptyset$, then $E_{t(V)} = E_{s(V)} = EV = aV$ for some $a \in E$ by the Theorems 3.4.2.2 (c) and 3.4.5. Now $E \subseteq aV$ implies $E_r \subseteq (aV)_r = aV$.

(b) $\Rightarrow$ (c) If $X \subseteq V$, then $X_r \subseteq X_{t(V)} = X_{s(V)} = XV$ by Theorem 3.4.5.

(c) $\Rightarrow$ (a) $V_r \subseteq XV$ implies $V_r = V$. 

2. (a) $\Rightarrow$ (b) By 1., $id_K$ is a $(t, t(V))$-homomorphism, and thus Theorem 2.6.5 implies that $D \subseteq V$ and $D \hookrightarrow V$ is a GCD-homomorphism.
(b) ⇒ (c) By Theorem 2.6.5, \( id_K \) is a \((t,t(V))\)-homomorphism. Since \( s(V) = t(V) \), it follows that \( V \setminus V^x \in M_t(K) \), hence \( P = D \cap (V \setminus V^x) \in t\text{-}\text{spec}(D) \), and obviously \( D \ni P \subset V \). To prove the reverse inclusion, let \( z = a^{-1}b \in V \), where \( a, b \in D \) and \( \text{GCD}(a,b) = D^x \). Since \( D \twoheadrightarrow V \) is a \text{GCD}\text{-}homomorphism, we obtain \( \text{GCD}_V(a,b) = V^x \) and thus either \( a \in V^x \) or \( b \in V^x \). If \( a \in V^x \), then \( a \notin P \) and \( z \in D \). If \( b \in V^x \), then \( b \in aV \) implies \( a \in V^x \) and again \( z \in D \).

(c) ⇒ (a) By Theorem 2.5.4.1 we have \( (D_p)_t = (D_t)p = D_p \). It remains to prove that \( D_p \) is a valuation monoid. Thus let \( z \in K \), say \( z = a^{-1}b \), where \( a, b \in D \) and \( \text{GCD}(a,b) = D^x \). Then \( \{a,b\}_t = D \), hence \( \{a,b\} \notin P \). If \( a \notin P \), then \( z \in D_p \), and if \( b \notin P \), then \( z^{-1} \notin P \).

3. If \( P \in t\text{-}\text{spec}(D) \), then \( tp \) is a finitary \( D_p \)-module system on \( K \), hence \( tp = s(Dp) = sp \) by Theorem 3.4.5, and for every subset \( X \subset K \) we have \( X_t \subset (X_t)_p = X_{tp} = XDp \). By Theorem 3.2.2 we obtain

\[
X_t \subset \bigcap_{P \in t\text{-}\text{spec}(D)} XDp \subset \bigcap_{P \in t\text{-}\text{max}(D)} XDp = \bigcup_{P \in t\text{-}\text{max}(D)} (X_t)_p = X_t,
\]

and the assertion follows.

\[\square\]

**Theorem 3.5.10.** Let \( \varepsilon : K \to K' \) be a homomorphism of divisible monoids, \( r' \) a module system on \( K' \) and \( r = \varepsilon r' \).

1. If \( V' \) is an \( r' \)-valuation monoid of \( K' \), then \( \varepsilon^{-1}(V') \) is an \( r \)-valuation monoid of \( K \).

2. Let \( \varepsilon \) be surjective. Then the assignment \( V \mapsto \varepsilon(V) \) defines a bijective map from the set of all \( r \)-valuation monoids of \( K \) onto the set of all \( r' \)-valuation monoids of \( K' \).

**Proof.**

1. Let \( V' \) be an \( r' \)-homomorphism of \( K' \). If \( x \in K \setminus \varepsilon^{-1}(V') \), then \( \varepsilon(x) \in K' \setminus V' \), hence \( \varepsilon(x) = \varepsilon(x)^{-1} \in V' \) and \( x^{-1} \in \varepsilon^{-1}(V') \). Hence \( \varepsilon^{-1}(V') \) is a valuation monoid of \( K \) and, by Theorem 2.3.6, it is an \( r \)-valuation monoid.

2. Let \( V \subset K \) be an \( r \)-valuation monoid and \( x' \in K' \setminus \varepsilon(V) \). Then \( x' = \varepsilon(x) \) for some \( x \in K \setminus V \). Hence we obtain \( x^{-1} \in V \) and \( x'^{-1} = \varepsilon(x)^{-1} = \varepsilon(x^{-1}) \in \varepsilon(V) \). Hence \( \varepsilon^{-1}(V') \) is a valuation monoid of \( K' \), and since \( V = V_e = \varepsilon^{-1}(\varepsilon(V)_{r'}) \), it follows that \( \varepsilon(V) = \varepsilon(V)_{r'} \) and \( V = \varepsilon^{-1}(\varepsilon(V)) \).

Conversely, if \( V' \) is an \( r' \)-valuation monoid of \( K' \), then \( \varepsilon^{-1}(V') \) is an \( r \)-valuation monoid of \( K \) by 1., and \( V' = \varepsilon(\varepsilon^{-1}(V')) \).

\[\square\]

### 3.5. Valuation domains

In this Section, we use the common terminology of commutative ring theory.

A domain \( D \) is called a valuation domain if its multiplicative monoid is a valuation monoid, and if \( K = q(D) \), then \( D \) is called a valuation domain of \( K \). In this case, the totally ordered abelian group \( \mathcal{G}(D) = K^x / D^x \) is called the value group of \( D \).

**Theorem 3.5.1.** A domain \( D \) is a valuation domain if and only if \( d(D) = s(D) \).

**Proof.** If \( D \) is a valuation domain, then \( s(D) = t(D) \), and as \( s(D) \leq d(D) \leq t(D) \), we obtain \( s(D) = d(D) \). Conversely, assume that \( s(D) = d(D) \), and let \( a, b \in D \). Then it follows that

\[
a + b \in I_{s(D)}(D) = I_{s(D)}(D) = aD \cup bD,
\]

say \( a + b \in aD \). Consequently, \( a + b = ax \) for some \( x \in D \), and \( b = a(x - 1) \in aD \). Hence \( D \) is a valuation domain.

\[\square\]

**Remarks and Definition 3.5.2.** Let \( (\Gamma, \leq) \) be a totally ordered additive abelian group. We consider the extension \( \Gamma \uplus \{\infty\} \), where \( \alpha \leq \infty = \alpha + \infty \) for all \( \alpha \in \Gamma \).
1. Let $K$ be a field. A valuation of $K$ (with value group $\Gamma$) is a surjective map $v: K \to \Gamma \cup \{\infty\}$, such that for all $a, b \in K$ the following assertions hold:

V1. $v(a) = \infty$ if and only if $a = 0$.

V2. $v(ab) = v(a) + v(b)$.

V3. $v(a + b) \geq \min\{v(a), v(b)\}$.

Consequences: If $a, b \in K$, then $v(a - b) = v(a) - v(b)$, and if $v(a) < v(b)$, then $v(a + b) = v(a)$.

Proof: Since $v : K^\times \to \Gamma$ is a homomorphism. Hence $2v(-1) = v((-1)^2) = v(1) = 0$, $v(-1) = 0$, $v(-a) = v(-1) + v(a) = v(a)$, and $v(a - b) \geq \min\{v(a), v(b)\} = \min\{v(a), v(b)\}$. If $v(a) < v(b)$, then $v(a) = v(a + b - b) \geq \min\{v(a + b), v(b)\} = v(a + b)$.

If $v$ is a valuation of $K$, then $O_v = \{a \in K \mid v(a) \geq 0\}$ is a valuation domain with maximal ideal $p_v = \{a \in K \mid v(a) > 0\} = O_v \setminus O_v^\times$, and $v$ induces an isomorphism $K^\times/O_v^\times \cong \Gamma$.

We call $(K, v)$ a valued field, $O_v$ the valuation domain, $p_v$ the valuation ideal and $O_v/p_v$ the residue field of $(K, v)$.

2. Let $D$ be a valuation domain, $K = q(D)$ and $w : K^\times \to \Gamma$ a valuation morphism of $D$. We set $w(0) = \infty$. Then $w : K \to \Gamma \cup \{\infty\}$ is a valuation of $K$, and $O_w = D$.

Proof: Since $D = \{x \in K \mid w(x) \geq 0\}$, it suffices to prove that $w(x + y) \geq \min\{w(x), w(y)\}$ for all $x, y \in K$. Thus let $x, y \in K$, and assume that $w(x) \geq w(y)$. If $y = 0$, then $x = 0$, and there is nothing to do. If $y \neq 0$, then $w(y^{-1}x) = w(y) + w(x) \geq 0$, hence $y^{-1}x \in D$ and therefore also $1 + y^{-1}x \in D$. But this implies $w(x + y) = w(y(1 + y^{-1}x)) = w(y) + w(1 + y^{-1}x) \geq w(y)$. □

3. Let $D$ be a ring and $v_0 : D \to \Gamma \cup \{\infty\}$ a surjective map satisfying V1, V2, V3 for all $a, b \in D$.

Then $D$ is a domain. If $K = q(D)$, then there exists a unique valuation $v : K \to \Gamma \cup \{\infty\}$ such that $v | D = v_0$. It is given by $v(a^{-1}b) = v_0(b) - v_0(a)$ for all $a \in D^*$ and $b \in D$.

**Theorem und Definition 3.5.3.** Let $K$ be a field, $K[X]$ a polynomial domain and $v : K \to \Gamma \cup \{\infty\}$ a valuation. Then there is a unique valuation $v^* : K[X] \to \Gamma \cup \{\infty\}$ such that, for all $f \in K[X]$,

$$f = \sum_{i \geq 0} a_i X^i \quad \text{(where } a_i \in K, \ a_i = 0 \text{ for almost all } i \geq 0) \quad \text{implies } v^*(f) = \min\{v(a_i) \mid i \geq 0\}.$$

$v^*$ is called the trivial extension of $v$.

**Proof.** It suffices to prove that $v^* | K[X]$ satisfies V1, V2, V3 for all $f, g \in K[X]$. V1 is obvious. Suppose that

$$f = \sum_{i \geq 0} a_i X^i \quad \text{and} \quad g = \sum_{i \geq 0} b_i X^i, \quad \text{where } a_i, b_i \in K, \ a_i = b_i = 0 \text{ for almost all } i \geq 0.$$

V2. By definition,

$$v^*(f + g) = \min\{v(a_i + b_i) \mid i \geq 0\} \geq \min\{\min\{v(a_i), v(b_i)\} \mid i \geq 0\}$$

$$= \min\{\min\{v(a_i) \mid i \geq 0\}, \min\{v(b_i) \mid i \geq 0\}\} = \min\{v^*(f), v^*(g)\}.$$

V3. We may assume that $fg \neq 0$ and $k, l \in \mathbb{N}_0$ are such that $v^*(f) = v(a_k) < v(a_i)$ for all $i > k$, and $v^*(g) = v(b_l) < v(b_i)$ for all $i > l$. Then we have $v(a_i) \geq v(a_k)$ for all $i \geq 0$ and $v(b_i) \geq v(b_l)$ for all $i \geq 0$. We set

$$fg = \sum_{i \geq 0} c_i X^i, \quad \text{where } c_i = \sum_{i \geq 0} a_{\nu} b_{i-\nu}, \quad \text{and in particular } c_{k+l} = a_k b_l + \sum_{\nu=1}^{k+l} a_{\nu} b_{k+l-\nu}.$$
Hence \( v(c_i) \geq \min\{v(a_\nu) + v(b_{i-\nu}) \mid \nu \in [0, i] \} \geq v(a_k) + v(b_l) \) for all \( i \geq 0 \), and \( v(c_{k+l}) = v(a_k) + v(b_l) \), since \( v(a_kb_{k+l-\nu}) = v(a_k) + v(b_{k+l-\nu}) > v(a_k) + v(b_l) \) for all \( \nu \in [1, k + l] \). Therefore we obtain \( v^*(fg) = v(a_k) + v(b_l) = v^*(f) + v^*(g) \). \( \square \)

**Theorem 3.5.4.** Let \( k \) be a field and \( (\Gamma, \leq) \) an ordered additive abelian group. Then there exists a valued field \((K, v)\) with value group \( \Gamma \) and residue field \( k \).

**Proof.** We consider the semigroup ring \( D = k[\Gamma, X] \), consisting of all sums

\[
a = \sum_{\gamma \in \Gamma} a_\gamma X^\gamma, \quad \text{where} \quad a_\gamma \in k, \ a_\gamma = 0 \quad \text{for almost all} \quad \gamma \in \Gamma_+,
\]

and we set

\[
v_0(a) = \min\{\gamma \in \Gamma_+ \mid a_\gamma \neq 0\} \in \Gamma_+ \quad \text{if} \quad a \neq 0, \quad \text{and} \quad v_0(0) = \infty.
\]

Then \( v_0 : D \to \Gamma_+ \cup \{\infty\} \) is a surjective map satisfying \( \textbf{V1, V2, V3} \) for all \( a, b \in D \). By 3.4.2.3, \( D \) is a domain. If \( K = q(D) \), then there exists a unique valuation \( v : K \to \Gamma \cup \{\infty\} \) such that \( v | D = v_0 \). It remains to prove that \( k \) is the residue field of \((K, v)\).

If \( p = \{a \in D \mid v(a) > 0\} \), then \( p \in \text{spec}(D) \) and \( D = k + p \). Every \( z \in K^\times \) has a representation

\[
z = X^\gamma \frac{a + p}{1 + q}, \quad \text{where} \quad \gamma \in \Gamma, \ a \in k, \ p, q \in p, \quad \text{and then} \quad v(z) = \gamma.
\]

In particular, we have \( z \in \mathcal{O}_v \) if and only if \( \gamma \geq 0 \), and therefore \( \mathcal{O}_v = D_p \). Hence \( p_v = pD_p \), and \( \mathcal{O}_v/p_v = D/p = k \). \( \square \)

**Theorem 3.5.5.** Let \( K \) be a field, \( D \subset k \) a subring and \( P \subset D \) a prime ideal. Then there exists a valuation domain \( V \) of \( K \) such that \( D \subset V \) and \( P = D \setminus V^\times \).

The proof requires the following Lemma from Commutative Algebra.

**Lemma 3.5.6 (The \((u, u^{-1})\)-Lemma).** Let \( R \subset S \) be rings, \( u \in S^\times \), \( I \subset R \) and \( b \in IR[u] \cap IR[u^{-1}] \). Then there exist some \( k \in \mathbb{N} \) and \( r_0, \ldots, r_k-1 \in I \) such that \( b^k + r_{k-1}b^{k-1} + \ldots + r_1b + r_0 = 0 \). In particular, if \( I \neq R \), then \( IR[u] \neq R[u] \) or \( IR[u^{-1}] \neq R[u^{-1}] \).

**Proof of the Lemma.** Suppose that \( b = a_0 + a_1u + \ldots + a_nu^n = c_0 + c_1u^{-1} + \ldots + c_mu^{-m} \), where \( m, n \in \mathbb{N} \) and \( a_0, \ldots, a_n, c_0, \ldots, c_m \in I \). We set \( M = R + Ru + \ldots + Ru^{m-1} \), and we assert that \( bM \subset IM \). Indeed,

\[
bu^l = \sum_{i=0}^{n} a_iu^{i+l} \quad \text{for} \quad l \in [0, m-1], \quad \text{and} \quad bu^l = \sum_{j=0}^{m} c_ju^{-j+l} \quad \text{for} \quad l \in [m, m+n-1].
\]

In particular, for every \( i \in [0, m+n-1] \), there is a relation of the form

\[
b u^l = \sum_{j=0}^{m+n-1} d_{i,j}u^l, \quad \text{where} \quad d_{i,j} \in I, \quad \text{and therefore} \quad \sum_{j=0}^{m+n-1} (b d_{i,j} - d_{i,j})u^l = 0,
\]

which implies \( \det(b d_{i,j} - d_{i,j})_{i,j \in [0, m+n-1]}u^l = 0 \) for all \( l \in [0, m+n-1] \), and therefore, as \( u \in S^\times \),

\[
0 = \det(b d_{i,j} - d_{i,j})_{i,j \in [0, m+n-1]} = b^{m+n-1} + r_{m+n-2}b^{m+n-2} + \ldots + r_1b + r_0, \quad \text{where} \quad r_i \in I \quad \text{for all} \quad i \in [0, m+n-2].
\]

If \( IR[u] = R[u] \) and \( IR[u^{-1}] = R[u^{-1}] \), then \( 1 \in IR[u] \cap IR[u^{-1}] \), and the above relation implies \( 1 \in I \) and thus \( I = R \). \( \square \)
Proof of the Theorem. Let $\Omega$ be the set of all domains $W$ satisfying $D_P \subset W \subset K$ such that $PW \neq W$. Then $D_p \in \Omega$ and the union of every chain in $\Omega$ belongs to $\Omega$. Indeed, let $(W_\lambda)_{\lambda \in \Lambda}$ be a chain in $\Omega$,

$$W = \bigcup_{\lambda \in \Lambda} W_\lambda,$$

and assume that $1 \in PW$.

Then $1 = p_1 w_1 + \ldots + p_n w_n$ for some $n \in \mathbb{N}$, $p_1, \ldots, p_n \in P$ and $w_1, \ldots, w_n \in W$. Hence there is some $\lambda \in \Lambda$ such that $\{w_1, \ldots, w_n\} \subset W_\lambda$ and $1 \in PW_\lambda$, a contradiction.

By Zorn’s Lemma, $\Omega$ contains a maximal element $V$, and we assert that $V$ is a valuation domain of $K$ such that $D \setminus P = D \cap V^\times$. Thus suppose that $z \in K \setminus V$. Then $V[z] \supset V$, and as $V$ is maximal in $\Omega$ it follows that $PV[z] = V[z]$. By the $(u, u^{-1})$-Lemma we obtain $PV[z^{-1}] \neq V[z^{-1}]$ and therefore $z^{-1} \in V$. Hence $V$ is a valuation domain of $K$, and

$$P = PD_P \cap D \subset PV \cap D \subset D \setminus V^\times = (V \setminus V^\times) \cap D_P \cap D \subset PD_P \cap D = P.$$

Hence $P = D \setminus V^\times$. □
CHAPTER 4

Invertibility, Cancellation and Integrality

4.1. Invertibility and class groups

Definition 4.1.1. Let $D$ be a cancellative monoid, $K = q(D)$ and $r : \mathbb{P}(K) \to \mathbb{P}(K)$ an ideal system of $D$. A fractional $r$-ideal $J \in \mathcal{F}_r(D)$ is called $r$-invertible if $J \in \mathcal{F}_r(D)^\times$ (equivalently, $J \cdot_r J' = D$ for some $J' \in \mathcal{F}_r(D)$).

If $D$ is a domain, we use the common terminology of Commutative Algebra. In particular, we set $\mathcal{F}(D) = \mathcal{F}_d(D)(D)$ and $\mathcal{I}(D) = \mathcal{I}_d(D)(D)$. In this case, (fractional) $d(D)$-ideals are called (fractional) ideals, and they are called invertible if they are $d(D)$-invertible.

Theorem 4.1.2. Let $D$ be a cancellative monoid, $K = q(D) \neq D$, $r : \mathbb{P}(K) \to \mathbb{P}(K)$ an ideal system of $D$, $v = v(D)$, $t = t(D)$, and for $X \subset K$, let $X^{-1} = (D : X)$.

1. Let $X, Y \subset K$ be such that $(XY)_r = D$. Then $Y_r = X^{-1} = X^{-1}$.
2. If $J \in \mathcal{F}_r(D)^\times$, then $J \cdot_r J^{-1} = D$ (hence $J^{-1}$ is the inverse of $J$ in $\mathcal{F}_r(D)$).
3. If $q$ is an ideal system of $D$ defined on $K$ such that $r \leq q$, then $\mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)^\times$ is a subgroup.
   In particular, every $r$-invertible fractional $r$-ideal is $v$-invertible, and $\mathcal{F}_r(D)^\times \subset \mathcal{F}_v(D)^\times$ is a subgroup.
4. If $r$ is finitary, then $\mathcal{F}_r(D)^\times = \mathcal{F}_{r|t}(D)^\times$, and $\mathcal{F}_r(D)^\times \subset \mathcal{F}_{t}(D)^\times$ is a subgroup. In particular, if $J$ is $r$-invertible, then both $J$ and $J^{-1}$ are $r$-finitely generated.
5. $\mathcal{F}_{r|t}(D)^\times = \mathcal{F}_{r}(D)^\times$.
6. $\mathcal{F}_v(D)^\times = \{ J \in \mathcal{F}_v(D)^\times : (J : J) = D \}$.

Proof. 1. Clearly, $X^{-1} = (D : X) = (D : X_r) = X^{-1}$. Since $XY \subset (XY)_r = D$, it follows that $Y \subset X^{-1}$ and therefore $Y_r \subset X^{-1}$, since $X^{-1} \in \mathcal{M}_v(K) \subset \mathcal{M}_r(K)$. On the other hand, we have $X^{-1} = X^{-1}(XY)_r \subset (X^{-1}XY)_r \subset (DY)_r = Y_r$.

2. Let $J' \in \mathcal{F}_r(D)$ be such that $J \cdot_r J' = (JJ')_r = D$. Then $J' = J' = J^{-1}$ by 1.

3. Let $q$ be an ideal system of $D$ such that $r \leq q$. If $J \in \mathcal{F}_r(D)^\times$, then $J = (J^{-1})^{-1} = J_v$ and thus $J \in \mathcal{F}_v(D) \subset \mathcal{F}_q(D)$. As $JJ^{-1} \subset D$ and $D = (JJ^{-1})_r \subset (JJ^{-1})_q \subset D$, it follows that $(JJ^{-1})_q = D$ whence $J \in \mathcal{F}_q(D)^\times$. Hence $\mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)^\times$, and it remains to prove that it is a subgroup. Thus let $I, J \in \mathcal{F}_r(D)^\times$. Then $I \cdot_r J = (JJ)_r = (IJ)_r \in \mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)^\times$, and therefore $I \cdot_r J = (IJ)_r = (IJ)_r = I \cdot_r J$.

4. Let $r$ be finitary. Then $r \leq t$, and thus $\mathcal{F}_r(D)^\times \subset \mathcal{F}_t(D)^\times$ is a subgroup by 3. As $\mathcal{F}_{r|t}(D) \subset \mathcal{F}_r(D)$ is a submonoid, it follows that $\mathcal{F}_{r|t}(D)^\times \subset \mathcal{F}_r(D)^\times$. Thus let $J \in \mathcal{F}_r(D)^\times$. Then $1 \in D = J \cdot_r J^{-1} = \left( \bigcup_{E \in \mathcal{P}_r(J)} E_r \right) \cdot_r J^{-1} = \left( \bigcup_{E \in \mathcal{P}_r(J)} E_r \cdot_r J^{-1} \right) \cdot_r J^{-1} = \bigcup_{E \in \mathcal{P}_r(J)} E_r \cdot_r J^{-1}$, since $\{ E_r \cdot_r J^{-1} : E \in \mathcal{P}_r(J) \}$ is directed. Hence there exists some $E \in \mathcal{P}_r(J)$ such that $1 \in E \cdot_r J^{-1} \subset D$ and therefore $E \cdot_r J^{-1} = D$, which implies $E_r = (J^{-1})^{-1} = J_v = J \in \mathcal{F}_{r|t}(D)$. The same argument, applied for $J^{-1}$ instead of $J$, shows that $J^{-1} \in \mathcal{F}_{r|t}(D)$, and consequently $J \in \mathcal{F}_{r|t}(D)^\times$. 

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5. $\mathcal{F}_r(D)^\times = \mathcal{F}_{r,1}(D)^\times = \mathcal{F}_{r,t}(D)^\times$.

6. If $J \in \mathcal{F}_o(D)$, then $J = X^{-1}$ for some $X \subset K$, and $(J : J) = (XX^{-1})^{-1} = (X \cdot_v X^{-1})^{-1}$ by Theorem 2.6.2. Hence $(J : J) = D$ if and only if $X \cdot_v X^{-1} = D$. □

**Theorem 4.1.3.** Let $D$ be a cancellative monoid, $K = q(D) \neq D$, $r : \mathbb{P}(K) \to \mathbb{P}(K)$ an ideal system of $D$, and for $X \subset K$, let $X^{-1} = (D : X)$.

For $I \in \mathcal{F}_r(D)^*$, the following assertions are equivalent:

(a) $I \in \mathcal{F}_r(D)^\times$.
(b) $I \cdot_r J = (J : I^{-1})$ for all $J \in \mathcal{F}_r(D)$.
(c) For all $J \in \mathcal{F}_r(D)$ satisfying $J \subset I$ there exists some $C \in \mathcal{I}_r(D)$ such that $J = I \cdot_r C$.

**Proof.** (a) $\Rightarrow$ (b) Let $J \in \mathcal{F}_r(D)$. From $I^{-1}(I \cdot_r J) \subset (I^{-1} I)_r = ((I^{-1} I)_r J)_r = J$ we obtain $I \cdot_r J \subset (J : I^{-1})$. Conversely, if $z \in (J : I^{-1})$, then $z \in zD = I \cdot_r zI^{-1} \subset I \cdot_r J$.

(b) $\Rightarrow$ (a) With $J = I^{-1}$, we obtain $1 \in (I^{-1} : I^{-1}) = I \cdot_r I^{-1} \subset D$ and therefore $I \cdot_r I^{-1} = D$.

(a) $\Rightarrow$ (c) Set $C = I^{-1} \cdot_r J \in \mathcal{F}_r(D)$. Then $I \cdot_r C = I \cdot_r I^{-1} \cdot_r J = J$, and since $C \subset I^{-1} \cdot I = D$, we obtain $C \in \mathcal{I}_r(D)$.

(c) $\Rightarrow$ (a) If $a \in I^*$, then $aD \subset I$, and there exists some $C \in \mathcal{I}_r(D)$ such that $aD = I \cdot_r C$. Then $a^{-1} C \in \mathcal{F}_r(D)$, and $I \cdot_r (a^{-1} C) = D$, whence $I \in \mathcal{F}_r(D)^\times$. □

**Theorem 4.1.4.** Let $D$ be a cancellative monoid, $K = q(D) \neq D$, $r : \mathbb{P}(K) \to \mathbb{P}(K)$ a finitary ideal system of $D$ and $t = t(D)$.

1. Let $D$ be $r$-local and $X \subset K$ a $D$-fractional subset such that $X_r$ is $r$-invertible. Then there exists some $a \in X$ such that $X_r = aD$. In particular, every $r$-invertible fractional $r$-ideal is principal.
2. If $J \in \mathcal{F}_r(D)^\times$ and $T \subset D^*$ is a multiplicatively closed subset, then $T^{-1} J \in \mathcal{F}_{r^{-1},r}(T^{-1} D)^\times$.
3. For $J \in \mathcal{F}_r(D)^*$, the following assertions are equivalent:

(a) $J$ is $r$-invertible.
(b) $J \in \mathcal{F}_{r,t}(D)$ and $J_P$ is principal for all $P \in r$-max$(D)$.
(c) $J_t \in \mathcal{F}_{r,t}(D)$ and $J_P$ is principal for all $P \in r$-max$(D)$.

**Proof.** 1. By Corollary 3.1.5 $M = D \setminus D^\times$ is he only $r$-maximal $r$-ideal of $D$. Let $X \subset K$ be a $D$-fractional subset such that $X_r$ is $r$-invertible. Then $X \not\subseteq (XM)_r$. Indeed, otherwise it follows that $X_r \subset X_r \cdot_r M$ and therefore $D = X_r^{-1} \cdot_r X_r \subset X_r^{-1} \cdot_r X_r \cdot_r M = M$, a contradiction. If $a \in X \setminus (XM)_r$, then $aX^{-1} \in \mathcal{I}_r(D)$, and we assert that $aX^{-1} \not\subseteq M$. Indeed, otherwise $a \in aD = a(X^{-1}X)_r \subset (XM)_r$, a contradiction. Hence $aX^{-1} = D$, and $X_r = aX^{-1} \cdot_r X_r = a(X^{-1}X)_r = aD$.

2. Obvious, since the map $\mathcal{F}_r(D) \to \mathcal{F}_{r^{-1},r}(T^{-1} D), \ J \mapsto T^{-1} J$, is a monoid homomorphism.

3. (a) $\Rightarrow$ (b) If $J$ is $r$-invertible, then $J$ is $r$-finitely generated by Theorem 4.1.2.4, $J_P$ is $r_P$-invertible by 2. and thus $J_P$ principal by 1.

(b) $\Rightarrow$ (c) If $J = E_r$ for some $E \in \mathbb{P}(D)$, then $J_t = E_t$.

(c) $\Rightarrow$ (a) Assume that $J \in \mathcal{F}_{r,t}(D)$ and that for all $P \in r$-max$(D)$ there is some $a_P \in D_P^*$ such that $J_P = a_P D_P$. Since $J \in \mathcal{F}_{r,t}(D)$, we obtain $(J^{-1})_P = (J_P)^{-1} = a_P^{-1} D_P$, and therefore $(J \cdot_r J^{-1})_P = J_P \cdot_r P, J_P^{-1} = (a_P D_P) \cdot_r (a_P^{-1} D_P) = D_P$. Hence $J \cdot_r J^{-1} = D$ by Theorem 3.2.2. □

**Remarks and Definition 4.1.5.** Let $D$ be a cancellative monoid, $K = q(D)$, $r : \mathbb{P}(K) \to \mathbb{P}(K)$ an ideal system of $D$, $v = v(D)$ and $t = t(D)$.
1. The map $\partial_r: K^\times \to \mathcal{F}_r(D)^\times$, defined by $\partial_r(a) = aD$, is a group homomorphism with kernel $\ker(\partial_r) = D^\times$. Its cokernel

$$\mathcal{C}_r(D) = \mathcal{F}_r(D)^\times / \partial_r(K^\times)$$

is called the $r$-class group of $D$, and it is usually written additively. It gives rise to an exact sequence

$$1 \to K^\times/D^\times \to \mathcal{F}_r(D)^\times \to \mathcal{C}_r(D) \to 0.$$

2. Let $q: \mathbb{P}(K) \to \mathbb{P}(K)$ be an ideal system of $D$ such that $r \leq q$. Then $\mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)^\times$ by Theorem 4.1.2.3, and thus also $\mathcal{C}_r(D) \subset \mathcal{C}_q(D)$. In particular, it follows that $\mathcal{C}_r(D) \subset \mathcal{C}_v(D)$, and if $r$ is finitary, then $\mathcal{C}_r(D) \subset \mathcal{C}_t(D)$.

3. Let $D$ be a domain and $d = d(D)$. Then $\text{Pic}(D) = \mathcal{C}_d(D)$ is called the Picard group and $\mathcal{C}(D) = \mathcal{C}_t(D)$ is called the divisor class group of $D$.

By 2. we have $\text{Pic}(D) \subset \mathcal{C}(D)$. The factor group $\mathcal{G}(D) = \mathcal{C}(D)/\text{Pic}(D)$ is called the local class group of $D$. By definition, $\mathcal{G}(D) \cong \mathcal{F}_t(D)^\times / \mathcal{F}(D)^\times$.

**Theorem 4.1.6.** Let $D$ be a domain.

1. If $D$ is semilocal, then $\text{Pic}(D) = 0$ [every invertible ideal is principal].
2. Suppose that $\mathcal{C}(D_M) = 0$ for all $M \in \max(D)$. Then $\mathcal{G}(D) = 0$.

**Proof.** 1. Let $\max(D) = \{M_1, \ldots, M_r\}$, and for $i \in [1, r]$, let

$$M_i^* = \bigcap_{j \neq i} M_j,$$

whence $M_i^* \triangleleft D$ and $M_i^* \not\subset M_i$. If $J \in \mathcal{F}(D)^\times$ and $i \in [1, r]$, then $JM_i^* \not\subset JM_i$, we fix an element $a_i \in JM_i^* \setminus JM_i$, and we set $a = a_1 + \ldots + a_r$. Then $a \in J \setminus JM_i$ for all $i \in [1, r]$, hence $aJ^{-1} \triangleleft D$ and $aJ^{-1} \not\subset JM_i$ for all $i \in [1, r]$, which implies $aJ^{-1} = D$ and $J = aD$.

2. Let $J \in \mathcal{F}_t(D)^\times$. If $M \in \max(D)$, then $J_M \in \mathcal{F}_{tM}(D_M)^\times \subset \mathcal{F}_{t(D_M)}(D_M)^\times$ and thus $J_M$ is principal. Since $J \in \mathcal{F}_{t,t}(D)$, it follows that $J \in \mathcal{F}(D)^\times$ by Theorem 4.1.4.2. 

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### 4.2. Cancellation

*Throughout this section, let $K$ be a monoid, and $\mathbb{P}_r^\times(K) = \{X \in \mathbb{P}(K) \mid X \cap K^\times \neq \emptyset\}$.*

**Definition 4.2.1.** Let $r$ be a weak module system on $K$.

1. An $r$-module $A \in \mathcal{M}_r(K)$ is called (r-finitely) $r$-cancellative if, for all (r-finitely generated) $r$-modules $M, N \in \mathcal{M}_r(K)$, $A \cdot_r M = A \cdot_r N$ implies $M = N$.

   In particular, $A \in \mathcal{M}_r(K)$ is $r$-cancellative if and only if $A \in \mathcal{M}_r(K)^*$, and then $A$ is r-finitely r-cancellative. If $A \in \mathcal{M}_{r,t}(K)$, then $A$ is r-finitely r-cancellative if and only if $A \in \mathcal{M}_{r,t}(K)^*$.

2. $r$ is called cancellative or arithmetisch brauchbar if every $A \in \mathcal{M}_r(K) \cap \mathbb{P}_r^\times(K)$ is $r$-cancellative.

   If $\mathcal{M}_r(K)$ is a cancellative monoid, then $r$ is cancellative, and the converse is true if $K$ itself is cancellative.

3. $r$ is called finitely cancellative or endlich arithmetisch brauchbar if every $A \in \mathcal{M}_{r,t}(K) \cap \mathbb{P}_r^\times(K)$ is r-finitely r-cancellative.

   If $\mathcal{M}_{r,t}(K)$ is a cancellative monoid, then $r$ is finitely cancellative, and the converse is true if $K$ itself is cancellative.
Theorem 4.2.2. Let \( r \) be a weak module system on \( K \) and \( A \in M_r(K) \).

1. The following assertions are equivalent:
   (a) \( A \) is \((r, \text{finitely}) \) \( r \)-cancellative.
   (b) For all \((r, \text{finitely generated}) \) \( r \)-modules \( M, N \in M_r(K) \), \( A \cdot M \subset A \cdot N \) implies \( M \subset N \).
   (c) For all \( (\text{finite}) \) subsets \( M, N \subset K \), \( \text{AM} \subset (AN)_r \) implies \( M \subset N_r \).
   (d) For all \((r, \text{finitely generated}) \) \( r \)-modules \( N \in M_r(K) \) and all \( c \in K \), \( cA \subset A \cdot N \) implies \( c \in N \).
   (e) For all \((r, \text{finitely generated}) \) \( r \)-modules \( N \in M_r(K) \) we have \( (A \cdot N : A) \subset N \).

2. Let \( r \) be finitary, and let \( A \) be \((r, \text{finitely generated}) \) \( r \)-finitely cancellative.
   (a) \( A \) is \( r \)-cancellative.
   (b) If \( T \subset K \) is a multiplicatively closed subset, then \( T^{-1}A \) is \( T^{-1}r \)-cancellative.

3. If \( A \) is \((r, \text{finitely}) \) \( r \)-cancellative, then \((A : A) \subset \{1\}_r \).

4. \( r \) is finitely cancellative if and only if \((EF)_r : E \subset F_r \) for all \( E \in \mathbb{P}_r(K) \) and \( F \in \mathbb{P}_r(K) \).

Proof. 1. We prove the equivalence under the additional specification of \( r \)-finiteness.

(a) \( \Rightarrow \) (b) If \( M, N \in M_{r,t}(K) \) and \( A \cdot r M \subset A \cdot r N \), then \( A \cdot r (M \cup N)_r = (A \cdot r M) \cup (A \cdot r N)_r = A \cdot r N \), and as \( (M \cup N)_r \in M_{r,t}(DK) \), it follows that \( M \subset (M \cup N)_r = N \).

(b) \( \Rightarrow \) (c) If \( M, N \in \mathbb{P}_r(K) \) and \( AM \subset (AN)_r \), then \( A \cdot r M_r = (AM)_r \subset (AN)_r = A \cdot r N_r \) and \( M_r, N_r \in M_{r,t}(K) \). Hence it follows that \( M \subset M_r \subset N_r \).

(c) \( \Rightarrow \) (d) Obvious, setting \( M = \{c\} \).

(d) \( \Rightarrow \) (e) Obvious.

(e) \( \Rightarrow \) (a) Let \( A \cdot r M_A = A \cdot r N \). If \( x \in M \), then \( Ax \subset A \cdot r M = A \cdot r N \) and therefore \( x \in (A \cdot N : A) \subset N \). Hence \( M \subset N \), and by symmetry equality follows.

2. Suppose that \( A = E_r \), where \( E \in \mathbb{P}_r(K) \).

(a) By 1. we must prove that, for all subsets \( N \subset K \) and \( c \in K \), \( cE \subset (EN)_r \) implies \( c \in N_r \).

(b) By 1. we must prove that \( (T^{-1}A \cdot r T^{-1}r N : T^{-1}A) \subset N \) for every \( N \in M_{T^{-1},r}(T^{-1}K) \). If \( \overline{N} \in M_{T^{-1},r}(T^{-1}K) \), then \( T^{-1}N \subset N \) for some \( N \in M_{T^{-1},r}(K) \), and \( (T^{-1}A \cdot r T^{-1}r T^{-1}N : T^{-1}A) = (T^{-1}(A \cdot r N : T^{-1}E) = T^{-1}(A \cdot r N : E) \subset T^{-1}A \cdot r N : A) \subset T^{-1}N \).

3. If \( A \) is \((r, \text{finitely}) \) \( r \)-cancellative, then \( A \subset A \cdot r \{1\}_r \) implies \((A : A) \subset (A \cdot r \{1\}_r : A) \subset \{1\}_r \) by 1.(d).

4. Let \( r \) be finitely cancellative, \( E \in \mathbb{P}_r(K) \) and \( F \in \mathbb{P}_r(K) \). Then \( E_r \) is \((r, \text{finitely}) \) \( r \)-cancellative, and as \( F \in M_{r,t}(K) \), it follows that \((EF)_r : E) = (E_r \cdot r F_r : E_r) 

Conversely, assume that \((EF)_r : E) \subset F_r \) for all \( E \in \mathbb{P}_r(K) \) and \( F \in \mathbb{P}_r(K) \). If \( A \in M_{r,t}(K) \cap \mathbb{P}_r(K) \), then \( A = E_r \) for some \( E \in \mathbb{P}_r(K) \), and \((A : A) \subset N \) for all \( N \in M_{r,t}(K) \). Indeed, if \( N \in M_{r,t}(K) \), then \( N = F_r \) for some \( F \in \mathbb{P}_r(K) \), and \((A : A) = ((EF)_r : E_r) = (EF)_r : E) \subset F_r = N \).

Theorem 4.2.3. Let \( D \) be a cancellative monoid, \( K = q(D), r : \mathbb{P}(K) \to \mathbb{P}(K) \) an ideal system of \( D \) and \( J \in \mathcal{F}_r(D) \).

1. If \( J \) is \((r, \text{finitely}) \) \( r \)-cancellative, then \((J : J) = D \).

2. If \( J \) is \( r \)-invertible, then \( J \) is \( r \)-cancellative.
PROOF. 1. If \( J \) is \( r \)-finitely \( r \)-cancellable, then \( D \subseteq (J : J) \subseteq \{ 1 \}_r = D \) by Theorem 4.2.2.3, and therefore \((J : J) = D\).

2. Let \( J \) be \( r \)-invertible and \( M, N \in \mathcal{M}_r(K) \) such that \( J \cdot_r M = J \cdot_r N \). Then \( M = J^{-1} \cdot_r J \cdot_r M = J^{-1} \cdot_r J \cdot_r N = N \). \( \square \)

**Theorem 4.2.4.** Let \( D \) be a ring and \( I \triangleleft D \).

1. Then the following assertions are equivalent:
   - (a) \( I \) is (d-)cancellable.
   - (b) For every \( M \in \max(D) \) there exists some \( a_M \in D_M^* \) such that \( I_M = a_M D_M \).
     If \( I \) is finitely generated, then there is also equivalent:
   - (a') \( I \) is (d-)finitely (d-)cancellable.

2. Let \( D \) be a domain, and let \( I \) be finitely generated. Then \( I \) is cancellative if and only if \( I \) is invertible.

**Proof.** 1. (a) \( \Rightarrow \) (a') Obvious.
   (a') \( \Rightarrow \) (a) By Theorem 4.2.2.2(a).

   (b) \( \Rightarrow \) (a) Let \( B, C \triangleleft D \) be such that \( IB = IC \). For \( M \in \max(D) \), this implies \( I_M B_M = I_M C_M \), hence \( a_M B_M = a_M C_M \) and therefore \( A_M = B_M \), since \( a_M \in D_M^* \). Now \( B = C \) follows by Theorem 3.2.2.

   (a) \( \Rightarrow \) (b) We prove first: If \( I = (a, b, A) \), where \( a, b \in D \), \( A \triangleleft D \), \( M \in \max(D) \) and \( MI \subseteq A \), then \( I = (a, A) \) or \( I = (b, A) \).

   We consider the ideal \( J = (A^2, a^2 + b^2, ab) \triangleleft D \) and calculate
   \[
   I^2 J = (a^2, b^2, ab, aA, bA, A^2)(a^2, a^2 + b^2, ab) = (a^2 A^2, b^2 A^2, abA^2, aA^3, bA^3, A^4, a^2 + a^2 b^2, a^2 b^2 + b^4, a^2 b^2 + b^4, a^3 b + ab^3, \
   (a^3 + ab^2) A, (a^2 b + b^3) A, (a^2 + b^2) A, a^3 b, a^2 b^2, a^2 b^2, a^2 b^2, a^2 b^2, a^2 b^2, ab A, ab b A, ab^2 A) = I^4.
   \]

   Hence it follows that \( I^2 = J \) and therefore \( a^2 \in J \), say \( a^2 = \lambda (a^2 + b^2) + z \), where \( \lambda \in D \) and \( z \in (A^2, ab) \).

   If \( \lambda \in M \), then \( \lambda a \in MI \subseteq A \), and \( a^2 = (\lambda a) a + \lambda b + z \in (A^2, ab, aA) \), and therefore
   \[
   I(b, A) = (b^2, ab, aA, bA, A^2) = (a^2, b^2, ab, ab, ab, A^2) = I^2, \quad \text{which implies } I = (b, A).
   \]

   If \( \lambda \notin M \), then \( D = (M, \lambda) \), say \( 1 = m + \lambda u \) for some \( m \in M \) and \( u \in D \). Since \( mb^2 = (mb)b \subseteq MIb \subseteq bA \) and \( \lambda b^2 = (1 - \lambda) a^2 - z \in (a^2, ab, A^2) \), we obtain \( b^2 = mb^2 + \lambda b^2 u \in (a^2, ab, A^2) \), and therefore
   \[
   I(a, A) = (a^2, ab, aA, bA, A^2) = (a^2, b^2, ab, ab, bA, A^2) = I^2, \quad \text{which implies } I = (a, A).
   \]

   Now we can do the actual proof.

   Let \( M \in \max(D) \) and \( \pi : I \to I/MI \) the canonical epimorphism. Let \( B \subseteq I \) be a subset such that \( \pi |B \) is injective and \( \pi(B) \) is a \( D/M \)-basis of \( M/IM \). Then \( I = (B) + MI \), and \( I \supseteq (B') + MI \) for every subset \( B' \subseteq B \). We assert that \( |B| = 1 \). Indeed, suppose the contrary. Then \( B = \{ a, b \} \cup B' \), where \( a \neq b \) and \( \{ a, b \} \cap B' = \emptyset \), and if \( A = (B') + MI \triangleleft D \), then \( I = (a, b, A) \). By \( A \) we obtain \( I = (a, A) \) or \( I = (b, A) \), a contradiction. Hence \( |B| = 1 \) and \( I = bD + MI \) for some \( b \in D \).

   We assert that \( I_M = (bD)_M \neq \frac{1}{2} D_M \), and for this we must prove that \( \xi \in (bD)_M \) for all \( c \in I \). If \( c \in I \), then \( cI = bcD + cMI \subseteq (bD + cM) \), which implies \( c \in bD + cM \), say \( c = bu + cm \) for some \( u \in D \) and \( m \in M \). Hence \( c(1 - m) = bu \) and
   \[
   \frac{c}{1} = \frac{bu}{1 + m} \in (bD)_M.
   \]
It remains to prove that \( \frac{b}{a} \) is not a zero divisor in \( D_M \). Let \( c \in D \) and \( s \in D \setminus M \) be such that \( \frac{c}{s} = \frac{b}{a} \in D_M \). Then \( tcb = 0 \) for some \( t \in D \setminus M \), and we obtain \( (tcI)_M = \frac{tc}{s} D_M = \{0\}_M = (tcMI)_M \).

For \( N \in \text{max}(D) \setminus \{M\} \) we have \( M_N = D_N \) and therefore \( (tcMI)_N = (tcI)_N \). By Theorem 3.2.2 we obtain \( tcI = tcMI \), which implies \( tc \in tcM \), say \( tc = tcm \) for some \( m \in M \). Consequently,

\[
\frac{c}{s} = \frac{tc(1-m)}{std(1-m)} = 0 = \frac{1}{1} \in D_M.
\]

2. By Theorem 4.1.4.

**Theorem and Definition 4.2.5.** Let \( r \) be a finitary weak module system on \( K \). Then there exists a unique finitary weak module system \( r_a \) on \( K \) such that

\[
X_{r_a} = \bigcup_{B \in P_t(K)} ((XB)_r : B) \quad \text{for all finite subsets } X \subset K.
\]

If \( K \) is cancellative and \( r \) is a module system, then \( r_a \) is a module system.

\( r_a \) is called the completion of \( r \). It has the following properties:

1. \( 1 \leq r_a \), and \((*)\) holds for all subsets \( X \subset K \).
2. \( r_a \) is finitely cancellative, and if \( q \) is any finitely cancellative finitary weak module system on \( K \) such that \( r \leq q \), then \( r_a \leq q \). In particular, \((r_a)_a = r_a \), and \( r \) is finitely cancellative if and only if \( r = r_a \).
3. Let \( D \subset K \) be a submonoid. Then \( r[D]_a = r_a[D] \). In particular, if \( r \) is a weak \( D \)-module system, then so is \( r_a \).
4. If \( T \subset K^\bullet \) is a multiplicatively closed subset, then \( T^{-1}r_a = (T^{-1}r)_a \).
5. Let \( D \subset K \) be a GCD-monoid, \( L = q(D) \) and \( t = t(D) : \mathbb{P}(L) \to \mathbb{P}(L) \). Then \( t \) is finitely cancellative, and \( \text{Hom}_{(r,t)}(K,L) = \text{Hom}_{(r_a,t)}(K,L) \).

In particular, if \( K \) is divisible, then every \( r \)-valuation monoid of \( K \) is an \( r_a \)-valuation monoid.

**Proof.** Note that for every subset \( X \subset K \), the system \( \{(XB)_r : B \mid B \in P_t(K)\} \) is directed. Indeed, if \( B, B' \in P_t(K) \), then \( (X(BB')_r : B') \subset (XBB'_r : BB') \).

By Theorem 2.2.2 we must check the conditions \( \textbf{M1}_f \), \( \textbf{M2}_f \) and \( \textbf{M3}_f \). Suppose that \( X, Y \in P_t(K) \) and \( c \in K \).

\( \textbf{M1}_f \) If \( B \in P_t(K) \), then \( XB \cup \{0\} \subset ((XB)_r : B) \) implies \( X \cup \{0\} \subset (((XB)_r : B) \subset X_{r_a} \).

\( \textbf{M2}_f \) Suppose that \( X \subset Y_{r_a} \) and \( z \in X_{r_a} \). Then there is some \( F \in P_t(K) \) such that \( z \in ((XF)_r : F) \). As \( \{(YF)_r : B \mid B \in P_t(K)\} \) is directed, there exists some \( B \in P_t(K) \) such that \( X \subset ((YF)_r : B) \). Then \( zFB \subset (XF)_r \subset (XF)_r \subset (YF)_r = (YFB)_r \) and thus \( z \in ((YFB)_r : FB) \subset Y_{r_a} \), since \( FB \in P_t(K) \).

\( \textbf{M3}_f \) We have

\[
cX_{r_a} = \bigcup_{B \in P_t(K)} c((XB)_r : B) \subset \bigcup_{B \in P_t(K)} (c(XB)_r : B) \subset \bigcup_{B \in P_t(K)} ((cXB)_r : B) = (cX)_{r_a}.
\]

Here the first inclusion becomes an equality if \( K \) is cancellative, and the second one becomes an equality if \( r \) is a module system. Consequently, \( r_a \) is a module system if \( K \) is cancellative and \( r \) is a module system.

1. If \( X \in P_t(K) \) and \( B \in P_t(K) \), then \( X \subset (XB)_r \), hence \( X_r \subset ((XB)_r : B) \subset X_{r_a} \) and therefore \( r \leq r_a \). For every subset \( X \subset K \), we have

\[
X_{r_a} = \bigcup_{B \in P_t(K)} \left( \bigcup_{E \in P_t(X)} EB \right) = \bigcup_{B \in P_t(K)} \bigcup_{E \in P_t(X)} ((EB)_r : B) = \bigcup_{E \in P_t(X)} \bigcup_{E \in P_t(X)} E_{r_a}.
\]

If \( r \) is a module system, then \( \textbf{M3}_f \) holds for \( r_a \), and thus \( r_a \) is also a module system.
2. By Theorem 4.2.2.4 we must prove that \((EF)_r : E \subset F_r\) for all \(E \in \mathbb{P}_t(K)\) and \(F \in \mathbb{P}_t(K)\). Thus let \(E \in \mathbb{P}_t(K)\), \(F \in \mathbb{P}_t(K)\) and \(z \in ((EF)_r : E)\). Since \(zE \subset ((EF)_r : E)\), there exists some \(B \in \mathbb{P}_t(K)\) such that \(zE \subset ((EF)_r : E)\). Hence it follows that \(zEB \subset (EF)_r\) and \(z \in ((EF)_r : EB) \subset F_r\). Since \(EB \in \mathbb{P}_t(K)\).

Let now \(q\) be any finitely cancellative finitary weak module system on \(K\) such that \(r \leq q\). If \(X \in \mathbb{P}_t(K)\) and \(B \in \mathbb{P}_t(K)\), Theorem 4.2.2 implies \(((X)_r : B) \subset ((X)_q : B) \subset X_q\), and thus \(r_a \leq q\) by Theorem 2.3.2.1.

3. For \(X \subset K\), we obtain
\[
X_{,r_a[D]} = (XD)_r = \bigcup_{B \in \mathbb{P}_t(K)} ((XDB)_r : B) = \bigcup_{B \in \mathbb{P}_t(K)} ((XB)_r[D] : B) = X_{r[D]_a}.
\]

4. By Theorem 2.4.1 we must prove that \(j_T(E)(T^{-1}r_a) = T^{-1}E_{r_a}\) for all \(E \in \mathbb{P}_t(K)\). Thus assume that \(E = \{a_1, \ldots, a_n\}\), where \(n \in \mathbb{N}_0\) and \(a_1, \ldots, a_n \in K\). Then
\[
j_T(E)(T^{-1}r_a) = \bigcup_{\overline{B} \in \mathbb{P}_t(K)} ((j_T(E)\overline{B})_{T^{-1}r} : \overline{B}).
\]
Suppose that \(\overline{B} = \left\{\frac{b_1}{t_1}, \ldots, \frac{b_n}{t_m}\right\} \in \mathbb{P}_t(K), \overline{B}\).

5. By Theorem 1.5.3, every \(t\)-finitely generated \(t\)-ideal of \(D\) is principal. Hence it follows that \(M_{t,t}(D)^* = \left\{a^{-1}J \mid J \in I_{t,t}(D)^* \right\} \subset D^*\) is cancellative, and thus \(t\) is finitely cancellative.

Since \(r \leq r_a\), every \((r_a, t)\)-homomorphism is an \((r, t)\)-homomorphism. If \(\varphi : K \to L\) is an \((r, t)\)-homomorphism, then by Proposition 2.3.6.2 we must prove that \(\varphi(X_{r_a}) \subset \varphi(X)_t\) for all \(X \in \mathbb{P}_t(K)\). If \(X \in \mathbb{P}_t(K)\), \(z \in X_{r_a}\) and \(B \in \mathbb{P}_t(K)\) are such that \(zB \subset (XB)_r\), then
\[
\varphi(z) \varphi(B) \subset \varphi((XB)_r) \subset \varphi(X)_t = [\varphi(X) \varphi(B)]_t
\]
and therefore \(\varphi(z) \in ([\varphi(X) \varphi(B)]_t : \varphi(X)_t)\) by Theorem 4.2.2.

Let \(K\) be divisible, \(V \subset K\) is a valuation monoid and \(t = t(V)\). It follows by Theorem 3.4.9 that \(V\) is an \(r\)- (resp. \(r_a\))-valuation monoid if and only if \(\text{id}_K\) is an \((r, t)\)- (resp. \((r_a, t)\))-homomorphism. Hence every \(r\)-valuation monoid is an \(r_a\)-valuation monoid. \(\Box\)

**Theorem 4.2.6.** Let \(D \subset K\) be a submonoid and \(s = s(D) : \mathbb{P}(K) \to \mathbb{P}(K)\). If \(X \subset K\), \(X \cap K^* \neq \emptyset\) and \(z \in K\), then \(z \in X_{s_a}\) if and only if there exist some \(k \in \mathbb{N}_0\) and \(l \in \mathbb{N}\) such that \(z^{k+l} \in z^k X^l D\).

**Proof.** Note that \(z \in X_{s_a}\) holds if and only if \(zB \subset (XB)_r = XBD\) for some \(B \in \mathbb{P}_t(K)\).

Suppose that \(k \in \mathbb{N}_0\) and \(l \in \mathbb{N}\) are such that \(z^{k+l} \in z^k X^l D\), and let \(X_0 \subset X\) be a finite subset such that \(X_0 \cap K^* \neq \emptyset\) and \(z^{k+l} \in z^k X^l D\). Then
\[B = \bigcup_{\nu=0}^{k+l-1} X_0^{\nu} z^{k+l-\nu-1} \in \mathbb{P}_t(K), \quad X_0^{k+l-1} \subset B\], and therefore \(B \in \mathbb{P}_t(K),\).
\[ zB = \bigcup_{\nu=1}^{k+l-1} X_0^\nu z^{k+l-\nu} \cup \{ z^{k+l} \} \subset X_0 \left( \bigcup_{\nu=0}^{k+l-2} X_0^\nu z^{k+l-\nu-1} \cup z^k X_0^{l-1}D \right) \subset X_0BD \subset XBD, \]

and therefore it follows that \( z \in X_s \).

Assume now that \( z \in X_s \), and let \( B = \{ b_1, \ldots, b_n \} \in \mathcal{P}_t(K) \) be such that \( n \geq 1, \ b_1 \in K^* \) and \( zB \subset XBD \). Then there exist \( x_1, \ldots, x_n \in X \) and a map \( \sigma: [1, n] \to [1, n] \) such that \( zb_i \in b_{\sigma(i)} x_i D \) for all \( i \in [1, n] \). Let \( k \in \mathbb{N}_0 \) and \( l \in \mathbb{N} \) be such that \( \sigma^{k+l}(1) = \sigma^k(1) \). Then

\[ z^{k+l} b_1 \in b_{\sigma^{k+1}(1)} \prod_{\mu=0}^{k+l-1} x_{\sigma^{\mu}(1)} D = b_{\sigma^1(1)} \prod_{\mu=0}^{k-1} x_{\sigma^{\mu}(1)} \prod_{\mu=k}^{k+l-1} x_{\sigma^{\mu}(1)} D \subset z^k b_1 X^l D \]

and therefore \( z^{k+l} \in z^k X^l D \).

**Theorem 4.2.7.** Let \( R \) be a ring, \( D \subset R \) a subring, \( d = d(D): \mathbb{P}(R) \to \mathbb{P}(R), \ X \subset R, \ X \cap R^* \neq \emptyset \) and \( z \in R \). Then \( z \in X_{d^*} \) if and only if \( z \) satisfies an equation \( z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0 \), where \( n \in \mathbb{N} \) and \( a_i \in (X^i)_d \) for all \( i \in [1, n] \).

**Proof.** Note that \( z \in X_{d^*} \) holds if and only if \( zB \subset (XB)_d \) for some \( B \in \mathcal{P}_t^*(R) \).

Suppose that \( z \in R \) satisfies an equation \( z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0 \), where \( n \in \mathbb{N} \) and \( a_i \in (X^i)_d \) for all \( i \in [1, n] \). Let \( X_0 \subset X \) be a finite subset such that \( X_0 \cap R^* \neq \emptyset \) and \( a_i \in (X^i)_d \) for all \( i \in [1, n] \). If

\[ B = \bigcup_{\nu=0}^{n-1} X_0^{\nu} z^{n-\nu-1} \in \mathcal{P}_t(R), \text{ then } X_0^{n-1} \subset B, \text{ hence } B \in \mathcal{P}_t^*(R), \text{ and } \]

\[ zB = \{ z^n \} \cup \bigcup_{\nu=0}^{n-2} X_0^{\nu+1} z^{n-\nu-1} \subset \{ z^n \} \cup X_0 B. \]

Since

\[ z^n = - \sum_{\nu=0}^{n-1} a_{\nu+1} z^{n-\nu-1} \in \left( \bigcup_{\nu=0}^{n-1} X_0^{\nu+1} z^{n-\nu-1} \right)_d \subset (X_0 B)_d \subset (XB)_d, \]

it follows that \( zB \subset (XB)_d \) and thus \( z \in X_{d^*} \).

Assume now that \( z \in X_{d^*} \), and let \( B = \{ b_1, \ldots, b_n \} \in \mathcal{P}_t(R) \) be such that \( n \geq 1, \ b_1 \in R^* \) and \( zB \subset (XB)_d \). Then there exist elements \( x_{i,j} \in X_d \) such that

\[ z b_i = \sum_{j=1}^{n} x_{i,j} b_j \quad \text{and therefore} \quad \sum_{j=1}^{n} (\delta_{i,j} z - x_{i,j}) b_j = 0 \quad \text{for all } i \in [1, n]. \]

Hence it follows that \( \det(\delta_{i,j} z - x_{i,j})_{i,j \in [1,n]} b_1 = 0 \) and consequently \( \det(\delta_{i,j} z - x_{i,j})_{i,j \in [1,n]} = 0 \), which gives the desired equation for \( z \).

**4.3. Integrality**

*Throughout this section, let \( K \) be a monoid, and \( \mathcal{P}_t^*(K) = \{ X \in \mathcal{P}_t(K) \mid X \cap K^* \neq \emptyset \} \).*

**Remarks and Definition 4.3.1.** Let \( r \) be a finitary weak module system on \( K \).

1. Let \( X \subset K \). An element \( x \in K \) is called \( r \)-integral over \( X \) if

\[ x \in X_{rs} = \bigcup_{B \in \mathcal{P}_t^*(K)} ((XB)_r : B) \]

[equivalently: There exists some \( B \in \mathcal{P}_t^*(K) \) such that \( zB \subset (XB)_r \).]
2. Let $D \subset K$ be a submonoid and $r$ a weak $D$-module system on $K$. Then
\[ D_{ra} = \bigcup_{J \in \mathcal{M}_{r,f}(K) \backslash J \cap K^* \neq \emptyset} (J : J) \]
[an element $x \in K$ is $r$-integral over $D$ if and only if there is some $J \in \mathcal{M}_{r,f}(K)$ such that $J \cap K^* \neq \emptyset$ and $x \in (J : J)$].

**Proof.** By definition, $x \in D_{ra}$ if and only if $xB \subset (DB)_r = B_r$ and thus $xB_r \subset B_r$ for some $B \in \mathcal{P}^*_r(K)$, and this holds if and only if $xJ \subset J$ for some $J \in \mathcal{M}_{r,f}(K)$ such that $J \cap K^* \neq \emptyset$. □

3. Let $D \subset B \subset K$ be submonoids.
- $\text{cl}^B_r(D) = D_{ra} \cap B$ is called the $r$-(integral) closure of $D$ in $B$.
- $B$ is called $r$-integral over $D$ if $\text{cl}^B_r(D) = B$.
- $D$ is called $r$-(integrrally) closed in $B$ if $\text{cl}^B_r(D) = D$.

By definition, $B$ is $r$-integral over $D$ if and only if $B \subset D_{ra}$, and $D$ is $r$-integrally closed in $B$ if and only if $D_{ra} \cap B = D$.

4. If $K$ is a ring, $D \subset B \subset K$ are subrings and $r = d = d(K)$, then (by Theorem 4.2.7) the above definitions coincide with the usual ones in ring theory as follows.
- $z \in K$ is called integral over $D$ if $z$ is $d$-integral over $D$ [equivalently, $z \in D_d$].
- $\text{cl}^B_r(D) = D_{da} \cap B$ is called the integral closure of $D$ in $B$.
- $B$ is called integral over $D$ if $\text{cl}^B_r(D) = B$.
- $D$ is called integrally closed in $B$ if $\text{cl}^B_r(D) = D$.

By definition, $B$ is integral over $D$ if and only if $B \subset D_{da}$, and $D$ is integrally closed in $B$ if and only if $D_{da} \cap B = D$.

5. Let $D$ be cancellative, $K = q(D)$ and $r : \mathcal{P}(K) \rightarrow \mathcal{P}(K)$ a finitary ideal system of $D$. Then $\text{cl}_r(D) = \text{cl}^K_r(D) = D_{ra}$ is called the $r$-(integral) closure of $D$, and $D$ is called $r$-(integrally) closed if $\text{cl}_r(D) = D$. By 2. we have
\[ \text{cl}_r(D) = \bigcup_{J \in \mathcal{I}_{r,f}(D)\bullet} (J : J), \]
and consequently $D$ is $r$-closed if and only if $(J : J) = D$ for all $J \in \mathcal{I}_{r,f}(D)\bullet$.

[Indeed, $\{J \in \mathcal{M}_{r,f}(D) \mid J \cap K^* \neq \emptyset\} = \mathcal{F}_{r,f}(D)\bullet = \{c^{-1}J \mid c \in D\bullet, \ J \in \mathcal{I}_{r,f}(D)\bullet\}$, and if $c \in D\bullet$ and $J \in \mathcal{I}_{r,f}(D)\bullet$, then $(c^{-1}J : c^{-1}J) = (J : J)$.]

In particular:
(a) If $s = s(D) : \mathcal{P}(K) \rightarrow \mathcal{P}(K)$, then $\text{cl}_s(D) = \{z \in K \mid z^n \in D$ for some $n \in \mathbb{N}\}$ by Theorem 4.2.6. $\text{cl}_s(D)$ is called the root closure of $D$, and if $D = \text{cl}_s(D)$, then $D$ is called root-closed.
(b) If $D$ is a domain, and $d = d(D) : \mathcal{P}(K) \rightarrow \mathcal{P}(K)$, then $D$ is called integrally closed if it is $d$-integrally closed.

**Theorem 4.3.2.** Let $D$ be a cancellative monoid, $K = q(D)$, and let $r, q : \mathcal{P}(K) \rightarrow \mathcal{P}(K)$ be finitary ideal systems of $D$ such that $r \leq q$.

1. If $r$ is finitely cancellative, then $D$ is $r$-closed.
2. $\text{cl}_r(D) \subset \text{cl}_q(D)$, and if $D$ is $q$-closed, then $D$ is $r$-closed and, in particular, $D$ is root-closed.
4. INVERTIBILITY, CANCELLATION AND INTEGRALITY

**Remark 4.3.1.4.**

Let \( \sigma \) be a finitary ideal system of \( D \). If \( \sigma \) is invertible, then \( \sigma \) is cancellative and \( \sigma \)-integral over \( D \) if and only if, for all \( p \in rD \)-max(\( D \)), \( \sigma p \) is \( r \)-integral over \( D \). Hence \( D \) is \( r \)-closed, and since \( s(D) \leq r \), it is also root-closed by Remark 4.3.1.5.

**Theorem 4.3.3.** Let \( D \) be an integrally closed domain, \( K = \mathfrak{q}(D) \) and \( d = d(D) : \mathbb{P}(K) \to \mathbb{P}(K) \). Then \( d \sigma \) is a finitary ideal system of \( D \), \( d \sigma \)-max(\( D \)) = \( d \)-max(\( D \)), and if \( X \subset D \), then \( X d \sigma = D \) if and only if \( X d = D \).

**Proof.** \( d \sigma \) is a finitary \( D \)-module system on \( K \), and as \( D d \sigma = D \), it is even an ideal system of \( D \).

If \( X \subset D \), then \( X d \subset X d \sigma \subset D \), and therefore \( X d \) implies \( X d \sigma = D \) and \( X d = D \). Conversely, if \( X d = D \), then \( 1 \in X d \), and thus there is an equation \( 1 + a_1 + \ldots + a_n = 0 \), where \( n \in \mathbb{N} \) and \( a_i \in (X^i) d \) for all \( i \in [1, n] \). Since \( X d \subset D \) and (\( X^i \))d \subset D \) for all \( i \in [1, n] \), it follows that \( 1 \in X d \) and therefore \( X d = D \).

If \( M \subset d \)-max(\( D \)), then \( M \subset M d \sigma \subset D \), and there is some \( M^* \subset d \sigma \)-max(\( D \)) such that \( M d \sigma \subset M^* \). But \( M^* \subset \mathcal{I}(d) \), and therefore \( M = M^* \subset d \sigma \)-max(\( D \)). Conversely, if \( M \subset d \sigma \)-max(\( D \)), then \( M \subset \mathcal{I}(d) \), and there exists some \( \overline{M} \subset d \)-max(\( D \)) such that \( M \subset \overline{M} \). Since \( \overline{M} d \sigma \subset D \), we obtain \( M = \overline{M} \subset d \)-max(\( D \)).

**Theorem 4.3.4.** Let \( B \subset C \subset K \) be submonoids and \( r \) a finitary weak module system on \( K \).

1. Let \( B \) be an \( r \)-monoid and \( B' = \mathfrak{c}_r(B) \subset B \). Then \( B' \) is an \( r \)-monoid which is \( r \)-closed in \( B \).

2. Let \( B \) be \( r \)-integral over \( D \). If \( z \in K \) is \( r \)-integral over \( B \), then \( z \) is \( r \)-integral over \( D \).

3. If \( T \subset D \) is a multiplicatively closed subset, then \( \mathfrak{c}_r^{T^{-1}}(T^{-1}D) = T^{-1}\mathfrak{c}_r^K(D) \).

4. For \( P \in rD \)-max(\( D \)) let \( j_P : K \to K_P \) be the natural embedding. Then

\[
\mathfrak{c}_r^K(D) = \bigcap_{P \in rD \text{-max}(D)} j_P^{-1}(\mathfrak{c}_r^K(D_P)).
\]

In particular:

(a) An element \( z \in K \) is \( r \)-integral over \( D \) if and only if, for all \( P \in rD \)-max(\( D \)), the element \( z \) is \( rP \)-integral over \( D_P \).

(b) If \( D \subset K^x \), then \( D_P \subset K_P = K \) for all \( P \in rD \)-max(\( D \)), and

\[
\mathfrak{c}_r^K(D) = \bigcap_{P \in rD \text{-max}(D)} \mathfrak{c}_r^K(D_P).
\]

(c) If \( D \) is cancellative and \( K = \mathfrak{q}(D) \), then \( D \) is \( r \)-closed if and only if, for all \( P \in rD \)-max(\( D \)), \( D_P \) is \( rP \)-closed.

**Proof.** 1. Since \( r \leq r_a \), it follows that \( D r_a \) is an \( r \)-monoid. Hence \( B' = \mathfrak{c}_r(B) = D r_a \cap B \) is an \( r \)-monoid, and \( \mathfrak{c}_r^B(B') = B' r_a \cap B = (D r_a \cap B) r_a \cap B = D r_a \cap B \).

2. If \( B \) is \( r \)-integral over \( D \), then \( B \subset D r_a \), and therefore \( B r_a = D r_a \).

3. If \( T \subset D \) is multiplicatively closed, then \( (T^{-1}D)\text{-max}(T^{-1}D) = (T^{-1}D)\text{-max}(D) = T^{-1}D r_a \) by the Theorems 4.2.5.4 and 2.4.1.

4. Since \( D r_a \) is a \( D \)-module, Theorem 3.2.2 implies

\[
\mathfrak{c}_r^K(D) = D r_a = \bigcap_{P \in rD \text{-max}(D)} j_P^{-1}((D r_a)_P)).
\]

If \( P \in rD \)-max(\( D \)), then \( (D r_a)_P = (D_P)_r a = (D_P)_{r a} = \mathfrak{c}_r^K(D_P) \).
If \( D^* \subset K^\times \), then \( D_P \subset K_P = K \), \( j_P = id_K \), and \( (D_P)_{(r_a)_P} = \text{cl}_K^P(D_P) \) by Theorem 2.5.4.

We reformulate Theorem 4.3.4 for the classical case of integral ring extensions.

**Theorem 4.3.5.** Let \( D \subset B \subset K \) be rings.

1. \( B' = \text{cl}^B(D) \) is a subring of \( B \) which is integrally closed in \( B \).
2. If \( B \) is integral over \( D \) and \( z \in K \) is integral over \( B \), then \( z \) is integral over \( D \).
3. If \( T \subset D^* \) is a multiplicatively closed subset, then \( \text{cl}^{T^{-1}K}(T^{-1}D) = T^{-1}\text{cl}_K^D(D) \).
4. For \( P \in \text{max}(D) \) let \( j_P: K \to K_P \) be the natural embedding. Then \( \text{cl}_K^D(D) = \bigcap_{P \in \text{max}(D)} j_P^{-1}(\text{cl}_P^D(D_P)) \).

In particular:

(a) An element \( z \in K \) is integral over \( D \) if and only if, for all \( P \in \text{max}(D) \), the element \( z_P \in K_P \) is integral over \( D_P \).

(b) If \( D^* \subset K^\times \), then \( D_P \subset K_P = K \) for all \( P \in \text{max}(D) \), and \( \text{cl}_K^D(D) = \bigcap_{P \in \text{max}(D)} \text{cl}_K^D(D_P) \).

(c) If \( D \) is a domain and \( K = q(D) \), then \( D \) is integrally closed if and only if \( D_P \) is integrally closed for all \( P \in \text{max}(D) \).

**Proof.** By Theorem 4.3.4, observing that \( T^{-1}d = d(T^{-1}D) \) for every multiplicatively closed subset \( T \subset D^* \), and that \( d_P = d | \mathcal{P}(D) \). \( \square \)

### 4.4. Lorenzen monoids

**Remarks and Definition 4.4.1.** Let \( D \) be a cancellative monoid, \( K = q(D) \), \( T \) a finitary module system on \( K \) and \( D \subset \{1\}_{r_s} \) (then \( D_{r_s} = \{1\}_{r_s} \)).

By Theorem 4.2.5.2, the monoid \( \mathcal{M}_{r_s,t}(K) \) is cancellative, and \( \mathcal{M}_{r_s,t}(K)^* = \{C \in \mathcal{M}_{r_s,t}(D) \mid C^* \neq \emptyset \} \).

We define \( \Lambda_r(K) = q(\mathcal{M}_{r_s,t}(K)) \), and we call \( \Lambda_r(K)^\times = \Lambda_r(K)^* \) the Lorenzen \( r \)-group of \( K \).

For an element \( X \in \Lambda_r(K)^* \), we denote by \( X^{[-1]} \) its inverse in \( \Lambda_r(K) \). Then we obtain

\[
\Lambda_r(K) = \{C^{[-1]}A \mid A, C \in \mathcal{M}_{r_s,t}(K), C^* \neq \emptyset \}.
\]

If \( A, A' \in \mathcal{M}_{r_s,t}(K) \) and \( C, C' \in \mathcal{M}_{r_s,t}(K)^* \), then \( C^{[-1]}A = C'^{-1}A' \) if and only if \( (AC')_{r_s} = (A'C)_{r_s} \), and multiplication in \( \Lambda_r(K) \) is given by the formula \( (C^{[-1]}A) \cdot (C'^{-1}A') = (CC')_{r_s}^{-1}(AA')_{r_s} \). In particular, \( D_{r_s} = \{1\}_{r_s} \) is the unit element and \( \{0\} \) is the zero element of \( \Lambda_r(K) \). The submonoid

\[
\Lambda_r^+(K) = \{C^{[-1]}A \mid A, C \in \mathcal{M}_{r_s,t}(K), C^* \neq \emptyset, A \subset C \} \subset \Lambda_r(K)
\]

is called the Lorenzen \( r \)-monoid of \( K \).

The map \( \tau_r: K \to \Lambda_r(K) \) is defined by \( \tau_r(a) = \{a\}_{r_s} = aD_{r_s} \in \mathcal{M}_{r_s,t}(K) \subset \Lambda_r(K) \) for all \( a \in K \), is a monoid homomorphism, called the Lorenzen \( r \)-homomorphism.

By definition, \( \tau_r(D) \subset \tau_r(D_{r_s}) \subset \Lambda_r^+(K) \), and \( \tau_r | K^\times: K^\times \to \Lambda_r(K)^\times \) is a group homomorphism satisfying \( \text{Ker}(\tau_r | K^\times) = D_{r_s}^\times \).
Theorem 4.4.2. Let $D$ be a cancellative monoid, $K = \mathbb{q}(D)$, $r$ a finitary module system on $K$, $D \subset \{1\}_r$, and $t = t(\Lambda^+_r(K))$: $\mathbb{P}(\Lambda_r(K)) \to \mathbb{P}(\Lambda_r(K)).$

1. $\Lambda_r(K) = q(\Lambda^+_r(K)).$
2. If $A, C \in \mathcal{M}_{r,t}(K)$ and $C^* \neq \emptyset$, then $C^{[-1]}A \in \Lambda^+_r(K)$ if and only if $A \subset C$. In particular, $A \in \Lambda^+_r(K)$ holds if and only if $A \subset \mathcal{D}_r$.
3. $\Lambda^+_r(K)$ is a reduced GCD-monoid. If $X, Y \in \Lambda^+_r(K)$, then there exist $A, B, C \in \mathcal{M}_{r,t}(K)$ such that $C^* \neq \emptyset$, $A \cup B \subset C$, $X = C^{[-1]}A$ and $Y = C^{[-1]}B$. In this case, we have $X \mid Y$ if and only if $B \subset A$, and $\text{gcd}(X, Y) = C^{[-1]}(A \cup B)_{r_r}$.
4. If $E \in \mathbb{P}_t(D_r)$, then $E_{r_r} = \text{gcd}(\tau_r(E)) \in \Lambda^+_r(K)$. In particular, for every $X \in \Lambda_r(K)$ there exist $E, E' \in \mathbb{P}_t(D)$ such that $E'^* \neq \emptyset$, $X = E'_{r_r}$, $E_{r_r} = \text{gcd}(\tau_r(E'), r_{r_r}) = \text{gcd}(\tau_r(E'), E_{r_r})$, and then we have $X \in \Lambda^+_r(K)$ if and only if $E \subset E'_{r_r}$.
5. $r_r = \tau_r^{*}{t}$. In particular, $\tau_r$ is an $(r, t)$-homomorphism and thus also an $(r, r)$-homomorphism, $X_{r_r} = \tau_r^{-1}[\tau_r(X)]_{r_r}$ for all $X \subset K$, and $D_{r_r} = \tau_r^{-1}(\Lambda^+_r(K))$.

Proof. We will thoroughly use the fact that $r_r$ is finitely cancellative and apply Theorem 4.2.2.

1. If $X = C^{[-1]}A \in \Lambda_r(K)$, where $A, C \in \mathcal{M}_{r,t}(K)$ and $C^* \neq \emptyset$, then $(C \cup A)^{[-1]}_{r_r} C \in \Lambda^+_r(K)$, $(C \cup A)^{[-1]}_{r_r} A \in \Lambda^+_r(K)$, and $X = [(C \cup A)^{[-1]}_{r_r} C^{[-1]} [C \cup A)^{[-1]}_{r_r} A]$. 
2. Let $A, C \in \mathcal{M}_{r,t}(K)$ and $C^* \neq \emptyset$. If $A \subset C$, then $C^{[-1]}A \in \Lambda^+_r(K)$ by definition. Thus suppose that $C^{[-1]}A \in \Lambda^+_r(K)$, say $C^{[-1]}A = C_1^{[-1]}A_1$ for some $A_1, C_1 \in \mathcal{M}_{r,t}(K)$ such that $C_1^* \neq \emptyset$ and $A_1 \subset C_1$. Then $(C_1 A)_{r_r} = (CA_1)_{r_r} \subset (CC_1)_{r_r}$, and thus $A \subset C$.
3. We prove first that $\Lambda^+_r(K)$ is reduced. Let $X \in \Lambda^+_r(K)^x$, say $X = C^{[-1]}A$ and $X^{[-1]} = C_1^{[-1]}A_1$, where $A, A_1, C, C_1 \in \mathcal{M}_{r,t}(K)$, $C^* \neq \emptyset$, $C_1^* \neq \emptyset$, $A \subset C$ and $A_1 \subset C_1$. Then $(CC_1)^{[-1]}(AA_1)_{r_r} = D_{r_r}$, hence $A^*_1 \neq \emptyset$ and $(AA_1)_{r_r} = (CC_1)_{r_r} \cup (CA_1)_{r_r}$. Now it follows again that $A \subset C$, hence $\Lambda^+_r(K)$ and $X = D_{r_r}$.

Now let $X, Y \in \Lambda^+_r(K)$. As $\Lambda^+_r(K) \subset q(M_{r,t}(K),$ there exist $A, B, C \in \mathcal{M}_{r,t}(K)$ such that $C^* \neq \emptyset$, $X = C^{[-1]}A$ and $Y = C^{[-1]}B$, and by 2. we obtain $A \subset B \subset C$. 

Assume that $X \mid Y$, say $Y = X \cdot Z$, where $Z = W^{[-1]}U \in \Lambda^+_r(K)$ for some $U, W \in \mathcal{M}_{r,t}(K)$ such that $W^* \neq \emptyset$ and $U \subset W$. Therefore we obtain $C^{[-1]}B = C^{[-1]}A \cdot W^{[-1]}U = (CW)^{[-1]}(AU)_{r_r}$, which implies $(BCW)_{r_r} = (CAU)_{r_r}$, hence $(BW)_{r_r} = (AU)_{r_r} \subset (AW)_{r_r}$, and $B \subset A$ by cancelation.

Assume now that $B \subset A$. If $B^* = \emptyset$, then $B = (BA)_{r_r} \in \Lambda^+_r(K)$, $Y = C^{[-1]}B = (C^{[-1]}A) \cdot B = X \cdot B$ and therefore $X \mid Y$. If $B^* \neq \emptyset$, then $A^* \neq \emptyset$, hence $U = A^{[-1]}B \in \Lambda^+_r(K)$ and $Y = X \cdot U$, which again implies $X \mid Y$.

To prove the assertion concerning the gcd, set $Z = C^{[-1]}(A \cup B)_{r_r}$. Then $Z \mid X$ and $Z \mid Y$. We assume that $Z_1 \in \Lambda^+_r(K)$ is another element such that $Z_1 X$ and $Z_1 Y$. We must prove that $Z_1 \mid Z$. By 1., there exist $A_1, B_1, C_1, U, W \in \mathcal{M}_{r,t}(K)$ such that $C_1^* \neq \emptyset$, $A_1 \cup B_1 \cup U \cup W \subset C_1$, $X = C_1^{[-1]}A_1$, $Y = C_1^{[-1]}B_1$, $Z = C_1^{[-1]}U$ and $Z_1 = C_1^{[-1]}W$. Then it follows that $A_1 \cup B_1 \subset W$, $(CA_1)_{r_r} = (C_1 A)_{r_r}$, $(CB_1)_{r_r} = (C_1 B)_{r_r}$, and $(CU)_{r_r} = (C_1 (A \cup B))_{r_r}$. Moreover, we obtain

$$(C(A_1 \cup B_1))_{r_r} = ((CA_1)_{r_r} \cup (CB_1)_{r_r})_{r_r} = (CA_1)_{r_r} \cup (CB_1)_{r_r} = (C_1 A)_{r_r} \cup (C_1 B)_{r_r} = (C_1 (A \cup B))_{r_r},$$

and therefore $U = (A_1 \cup B_1)_{r_r} \subset W$, which implies $Z_1 \mid Z$.

4. If $E \in \mathbb{P}_t(D_{r_r})$, then $E_{r_r} = \Lambda^+_r(K)$, $\tau_r(E) \subset \Lambda^+_r(K)$, and 2. implies

$$E_{r_r} = \bigcup_{e \in E} \{e\}_{r_r} = \bigcup_{e \in E} \tau_r(e)_{r_r} = \text{gcd}\{\tau_r(e) \mid e \in E\} = \text{gcd}(\tau_r(E)).$$

If $X \in \Lambda_r(K)$, then $X = C^{[-1]}A$, where $A, C \in \mathcal{M}_{r,t}(K), A \subset C$ and $C^* \neq \emptyset$. Then there exist $E, E' \in \mathbb{P}_t(D)$ and $c \in D^*$ such that $C = (c^{-1}E')_{r_r}$ and $A = (c^{-1}E)_{r_r}$, and it follows that $E'^* \neq \emptyset$ and $X = (c^{-1}E')_{r_r} = \text{gcd}(\tau_r(E'))_{r_r} \subset \Lambda^+_r(K)$.
5. Since $t$ is finitary, it suffices to prove that $Z_{r_s} = Z_{τ_s t} = τ^{-1}(τ_s(Z)_{τ_t})$ for all $Z ∈ \mathbb{P}(K)$. Let $Z ∈ \mathbb{P}(K)$ and $a ∈ D^*$ such that $E = aZ ⊆ D$. Then $E_{r_s} = \gcd(τ_s(E))$ by 4., and therefore it follows that $τ_s(E)_{τ_t} = E_{r_s} τ^+(K)$. For $c ∈ K$, we obtain (observing that $r_s$ is a module system)
\[
c ∈ Z_{r_s} ⇔ ac ∈ aZ_{r_s} = E_{r_s} ⇔ τ_s(ac) = \{ac\}_{r_s} ⊆ E_{r_s} ⇔ E_{r_s}^{-1} τ_s(ac) ∈ \Lambda^+(K)
\]
\[
⇔ τ_s(a) τ_s(c) = τ_s(ac) ∈ E_{r_s} \Lambda^+(K) = τ_s(E)_{τ_t} = τ_s(aZ)_{τ_t} = τ_s(a) τ_s(Z)_{τ_t}
\]
\[
⇔ τ_s(c) ∈ τ_s(Z)_{τ_t} ⇔ c ∈ τ^{-1} s(τ_s(Z)_{τ_t}).
\]
The remaining assertions are obvious.

Theorem 4.4.3 (Universal property of the Lorenzen monoid). Let $D$ be a cancellative monoid, $K = q(D)$, r a finitary module system on $K$, $D ⊆ \{1\}_{r_s}$ and $t = t(\Lambda^+(K)): \mathbb{P}(\Lambda^+(K)) → \mathbb{P}(\Lambda_s(K))$.

1. Let $G$ be a reduced GCD-monoid, $L = q(G)$ and $t' = t(G): \mathbb{P}(L) → \mathbb{P}(L)$. Then there is a bijective map
\[
\text{Hom}_{(t,t')}(\Lambda_s(K), L) → \text{Hom}_{(t,t')}(K, L), \text{ given by } Φ ↦ Φ τ_s.
\]

2. Let $V$ be the set of all $r$-valuation monoids of $K$ and $W$ the set of all $t$-valuation monoids of $\Lambda_r(K)$.

(a) Suppose that $W ∈ W$, and let $w: \Lambda_r(K)^X → \Gamma$ be a valuation morphism of $W$. Then $V = τ^{-1}(W) ∈ V$, and $w τ_s | K^X: K^X → Γ$ is a valuation morphism of $V$. If $E ∈ \mathbb{P}(K)$, then $w(E_{r_s}) = \min\{w(σ) | E_s(σ)\}$.

(b) The assignment $W → τ^{-1}(W)$ defines a bijective map $τ_s: W → V$.

Proof. 1. If $Φ: \Lambda_r(K) → L$ is a $(t,t')$-homomorphism, then $Φ τ_s: K → L$ is an $(r,t')$-homomorphism, since $τ_s$ is an $(r,t)$-homomorphism. We prove that for every $(r,t')$-homomorphism $ϕ: K → L$ there is a unique $(t,t')$-homomorphism $Φ: \Lambda_r(K) → L$ such that $Φ τ_s = ϕ$.

Thus let $ϕ ∈ \text{Hom}_{(t,t')}(K, L)$ (see Theorem 4.2.5.5). By Theorem 2.6.5, the map $\text{Hom}_{(t,t')}(\Lambda_r(K), L) → \text{Hom}_{\mathbb{QCD}}(\Lambda_r(K), G)$, $Φ ↦ Φ | \Lambda_r(K)$, is bijective, and if $Φ ∈ \text{Hom}_{(t,t')}(\Lambda_r(K), L)$, then $ϕ = Φ τ_s$ if and only if $ϕ | D = Φ(\Lambda_r(K)) τ_s | D$. Hence it suffices to prove that there exists a unique $ψ ∈ \text{Hom}_{\mathbb{QCD}}(\Lambda_r(K), G)$ such that $ψ τ_s(a) = ϕ(a)$ for all $a ∈ D^*$.

Uniqueness: Let $ψ ∈ \text{Hom}_{\mathbb{QCD}}(\Lambda_r(K), G)$ be such that $ψ τ_s(a) = ϕ(a)$ for all $a ∈ D^*$, and assume that $X ∈ \Lambda^+_r(K)$, say $X = \gcd(τ_s(E'))^{-1} \gcd(τ_s(E))$, where $E, E' ∈ \mathbb{P}_t(D)$, $E^* ≡ \emptyset$ and $E_{r_s} ⊆ E_{r_s}$. Then it follows that $ψ(X) = g(x(ψ(τ_s(E')))^{-1} \gcd(ψ(τ_s(E))) = \gcd(ϕ(E'))^{-1} \gcd(ϕ(E))$, and thus $ψ$ is uniquely determined by $ϕ$.

Existence: Define $ψ$ provisionally by
\[
ψ(X) = \gcd(ϕ(E'))^{-1} \gcd(ϕ(E)) ∈ L \text{ if } X = \gcd(τ_s(E'))^{-1} \gcd(τ_s(E)) \in E_{r_s} τ_s(\Lambda_r(K)),
\]
where $E, E' ∈ \mathbb{P}_t(D)$, $E^* ≡ \emptyset$, and $E ∈ E_{r_s}$. We must prove: 1) $ψ(X) ∈ G$; 2) the definition is independent of the choice of $E$ and $E'$; 3) $ψ$ is a GCD-homomorphism.

If this is done and $a ∈ D$, then (putting $E' = \{1\}$ and $E = \{a\}$) we obtain $ψ τ_s(a) = ϕ(\{a\}_{r_s}) = ϕ(a)$.

1) Since $ϕ$ is an $(r_s,t')$-homomorphism, we obtain $ϕ(E) ⊆ ϕ(E_{r_s}) ⊆ ϕ(E')$, and therefore $\gcd(ϕ(E)) G = ϕ(E') G ⊆ ϕ(E') G = \gcd(ϕ(E') G).$ Hence $ψ(X) = g(x(ϕ(E'))^{-1} \gcd(ϕ(E)) ∈ G.$

2) Suppose that $X = E_{r_s}^{r_s} E_r = F_{r_s}^{r_s} F_r$, where $E, E', F, F' ∈ \mathbb{P}_t(D)$, $E^* ≡ \emptyset$, $F^* ≡ \emptyset$, $E ∈ E_{r_s}$ and $F ∈ F_{r_s}$. Then $(EF')_{r_s} = (EF)^{r_s}$, and since $ϕ$ is an $(r_s,t')$-homomorphism, we obtain $ϕ(EF') ⊆ ϕ((EF)_{r_s}) = ϕ((EF)^{r_s}) ∈ ϕ(EF')$. and $ϕ(EF') ∈ ϕ(EF)^{r_s}$. 

\[
ϕ(EF'') ⊆ ϕ((EF)'')_{r_s} = ϕ((EF')^{r_s}) ∈ ϕ(EF') \text{ and } ϕ(EF') ∈ ϕ(EF)^{r_s}.
\]
Similarly, \( \varphi(E^F)_\nu \subset \varphi(EF^F)_\nu \), and thus equality holds. Therefore it follows that
\[
gcd(\varphi(E)) gcd(\varphi(F'))G = gcd(\varphi(EF'))G = \varphi(EF')_\nu
\]
\[
= \varphi(E')_\nu = gcd(\varphi(EF'))G = gcd(\varphi(E')) gcd(\varphi(F))G,
\]
hence \( gcd(\varphi(E)) gcd(\varphi(F')) = gcd(\varphi(E')) gcd(\varphi(F)) \) (since \( G \) is reduced), which finally implies that \( gcd(E')^{-1} E = gcd(F')^{-1} gcd(F) \).

3) Let \( X_1, X_2 \in \Lambda^+(K) \) and \( E, E_1, E_2 \in \mathbb{P}_1(D) \) be such that \( E^* \neq \emptyset \), \( E_1 \cup E_2 \subset E_{r_0} \) and \( X_i = E_{r_0}^{-1}(E_i)_{r_0} \) for \( i \in \{1, 2\} \). Then \( gcd(X_1, X_2) = E_{r_0}^{-1}(E_1 \cup E_2)_{r_0} \),
\[
\psi(X_1 \cdot X_2) = \psi((E^2)_{r_0}^{-1}(E_1 E_2)_{r_0}) = gcd(\varphi(E^2))^{-1} gcd(\varphi(E_1 E_2))
\]
\[
= [gcd(\varphi(E))^{-1} gcd(\varphi(E_1))] [gcd(\varphi(E))^{-1} gcd(\varphi(E_2))] = \psi(X_1) \psi(X_2)
\]
and
\[
\psi(gcd(X)) = gcd(\varphi(E))^{-1} gcd(\varphi(E_1 \cup E_2)) = gcd(\varphi(E))^{-1} gcd[gcd(\varphi(E_1)), gcd(\varphi(E_2))]
\]
\[
= gcd(gcd(\varphi(E))^{-1} gcd(\varphi(E_1)), gcd(\varphi(E))^{-1} gcd(\varphi(E_2))) = \psi(X_1) \psi(X_2).
\]

2. (a) If \( W \in \mathcal{W} \), then \( \tau_{r_0}^{-1}(W) \) is an \( r_0 \)-valuation monoid (and hence also an \( r \)-valuation monoid) by Theorem 3.4.10, and therefore \( \tau_{r_0}^{-1}(W) \in \mathcal{V} \).

If \( E \in \mathbb{P}_1(D) \), then \( E_{r_0} = gcd(\tau_{r_0}(E)) \). Hence it follows that \( E_{r_0} \Lambda^+(K) = \tau_{r_0}(E)_i \), \( E_{r_0} W = \tau_{r_0}(E) W \) and \( w(E_{r_0}) = \min\{w(\tau_{r_0}(E)) \in w(\tau_{r_0}(K^*)) \} \) by Theorem 3.4.2.2, and \( w(M_{r_0}, (K)^*) = w(\tau_{r_0}(K^*)) \subset \Gamma \) is a subgroup. Since \( \Lambda_r(K)^* = q(M_{r_0}, (K^*)) \), we obtain \( \Gamma = q(w(\tau_{r_0}(K^*))) = w(\tau_{r_0}(K^*)) \). By definition, \( V = \tau_{r_0}^{-1}(W) = (w(\tau_{r_0})^{-1}(\Gamma_+)) \), and since \( w(\tau_{r_0}) : K^* : K^* \rightarrow \Gamma \) is surjective, it is a valuation morphism of \( V \).

(b) By (a) we must prove that \( \tau_{r} \) is bijective.

\( \tau_{r} \) is injective: For \( i \in \{1, 2\} \), let \( W_i \in \mathcal{W} \) be such that \( \tau_{r_0}^{-1}(W_i) = V \in \mathcal{V} \), and let \( w_i : \Lambda_r(K)^* \rightarrow \Gamma_i \) be a valuation morphism of \( W_i \). Then \( w_i \circ \tau_{r} | K^* : K^* \rightarrow \Gamma_i \) is a valuation morphism of \( V \), and by Theorem 3.4.2.2 there exists an order isomorphism \( \varphi : \Gamma_1 \rightarrow \Gamma_2 \) such that \( \varphi w_1 \circ \tau_{r} | K^* = w_2 \circ \tau_{r} | K^* \).

If \( X \in \Lambda_r(K)^* \), then \( X = E_{r_0}^{-1}(E_{r_0}) \) for some \( E, E' \in \mathbb{P}_1(D) \). Hence we obtain
\[
w_2(X) = w_2(E_{r_0}) - w_2(E_{r_0}') = \min\{w_2(\tau_{r_0}(E^*)) - \min\{w_2(\tau_{r_0}(E'^*))
\]
\[
= \min\{\varphi w_1 \circ \tau_{r}(E^*) - \min\{\varphi w_1 \circ \tau_{r}(E'^*)\} = \varphi(\min\{w_1 \circ \tau_{r}(E^*) - \min\{w_1 \circ \tau_{r}(E'^*)\})
\]
\[
= \varphi(\psi_1(E_{r_0}) - E_{r_0}')) = \varphi(\psi_1(X)).
\]
Therefore \( w_2(X) \geq 0 \) holds if and only if \( w_1(X) \geq 0 \), and consequently \( W_1 = W_2 \).

\( \tau_{r} \) is surjective: Let \( \mathcal{V} \), and let \( \varepsilon : K \rightarrow K/VX \) be the natural epimorphism. By the Theorems 2.3.7 and 3.4.10, \( V/V^* \) is an \( \varepsilon(r) \)-monoid of \( K/VX^* \), and if \( t^* = t(V/VX^*) \), then \( \varepsilon(r) = t^* \), since \( \varepsilon(r) \) is finitary, and \( \varepsilon \) is an \((r, t^*)\)-homomorphism.

By 1. the map \( \text{Hom}_{(t, t^*)}(\Lambda_r(K), K/VX^*) \rightarrow \text{Hom}_{(r, r^*)}(\Lambda_r(K), K/VX^*) \), given by \( \Phi \mapsto \Phi \circ \tau_{r} \), is bijective. Hence there exists a unique \((t, t^*)\)-homomorphism \( \Phi : \Lambda_r(K) \rightarrow K/VX^* \) such that \( \Phi \circ \tau_{r} \varepsilon = \varepsilon \), and we set \( W = \Phi(1)(K/VX^*) \subset \Lambda_r(K) \). Then \( \tau_{r_0}^{-1}(W) = (\Phi \circ \tau_{r})^{-1}(V/VX^*) = \varepsilon^{-1}(V/VX^*) \varepsilon(V/VX^*) = V \), and since \( \Phi \) is a \((t, t^*)\)-homomorphism, Theorem 3.4.10 implies \( W \in \mathcal{W} \). \( \square \)

**Theorem 4.4.4.** Let \( D \) be a cancellative monoid, \( K = q(D) \), \( r \) a finitary module system on \( K \), \( D \subset \{1\}_{r_0} \) and \( \mathcal{V}_r \) the set of all \( r \)-valuation monoids of \( K \). Then \( \mathcal{V}_r = \mathcal{V}_{r_0} \), and for all \( X \subset K \) we have
\[
X_{r_0} = \bigcap_{V \in \mathcal{V}_r(D)} XV.
\]
In particular, \( \text{cl}_r(D) = D_{r_0} \) is the intersection of all \( r \)-valuation monoids of \( K \).
Proof. By Theorem 4.2.5.5 we have $V_r = V_{r_a}$. Let $\tau_r : K \to \Lambda_r(K)$ be the Lorenzen $r$-homomorphism, $t = t(\Lambda_r^+(K))$ and $W$ the set of all $t$-valuation monoids of $\Lambda_r(K)$. Then $\tau_r^* t = r_a$ and $V_r = \{ \tau_r^{-1}(W) \mid W \in W \}$. If $X \subset K$, then $\tau_r^{-1}(\tau_r(X)) = XD_r^{x_r}$, and therefore, using Theorem 3.4.9.3,

$$X_r = \tau_r^{-1}(\tau_r(X)) = \tau_r^{-1} \left( \bigcap_{W \in W} \tau_r(X)W \right) = \bigcap_{W \in W} \tau_r^{-1}(\tau_r(X)) \tau_r^{-1}(W) = \bigcap_{V \in V_r(D)} XD_r^{x_r}V = \bigcap_{V \in V_r(D)} XV. \qed$$

Corollary 4.4.5. Let $D$ be a domain, $K = q(D)$ and $d = d(D) : \mathbb{P}(K) \to \mathbb{P}(K)$. Let $r$ be a finitary module system on $K$ such that $d \leq r$.

1. Let $V \subset K$ be a subset.
   (a) $V$ is an $r$-valuation monoid of $K$ if and only if $V$ is a valuation domain satisfying $V_r = V$. If this is the case, then $D \subset D_r \subset V$.
   (b) $V$ is a $d$-valuation monoid of $K$ if and only if $V$ is a valuation domain satisfying $D \subset V$.

2. The $r$-closure $\text{cl}_r(D)$ of $D$ is the intersection of all valuation domains $V$ of $K$ satisfying $V_r = V$.

In particular, the integral closure $\text{cl}_d(D)$ of $D$ is the intersection of all valuation domains $V$ of $K$ containing $D$.

Proof. Obvious by the Theorems 4.4.3 and 4.4.4. \qed
CHAPTER 5

Complete integral closures

Throughout this Chapter, let $D$ be a cancellative monoid, $K = q(D) \neq D$, $v = v(D)$ and $t = t(D)$.

5.1. Strong ideals

Theorem and Definition 5.1.1.

1. For an ideal $I \subset D$, the following assertions are equivalent:
   
   (a) $I^{-1} \subset (I:I)$.
   
   (b) $I^{-1} = (I:I)$.
   
   (c) $I^{-1}$ is an overmonoid of $D$.
   
   (d) There exists an overmonoid $T \supset D$ such that $I = T^{-1} = (D:T)$.
   
   (e) $I_v = (I^{-1})_v$.

A non-zero ideal $I \subset D$ satisfying these conditions is called strong (in $D$).

2. Let $D$ be a Mori domain and $\{0\} \neq P \in v$-$\text{spec}(D)$.
   
   (a) $P$ is not strong if and only if $D_P$ is a dv-monoid (and then $P \in \mathbb{X}(D)$).
   
   (b) If $P \in v$-$\text{max}(D)$, then $P$ is not strong if and only if $P$ is $v$-invertible.
   
   (c) If $T \subset D^*$ is a multiplicatively closed subset, then $P$ is strong if and only if $T^{-1}P$ is strong in $T^{-1}D$.

Proof. 1. (a) $\Rightarrow$ (b) $(I:I) \subset (D:I) = I^{-1}$.

(b) $\Rightarrow$ (c) $(I:I) \supset D$ is an overmonoid.

(c) $\Rightarrow$ (d) Obvious.

(d) $\Rightarrow$ (e) Let $T \supset D$ be an overmonoid such that $I = T^{-1}$. Then $I^{-1} = T_v \supset T$ is a monoid, and by Theorem 2.6.2.2 we obtain $(II^{-1})^{-1} = (I^{-1};I^{-1}) = (T_v:T_v) = T_v = I^{-1}$. Hence $(II^{-1})_v = I_v$.

(e) $\Rightarrow$ (a) $(I:I) = (II^{-1})^{-1} = (I^{-1})^{-1}_v = I_v^{-1} = I^{-1}$ (by Theorem 2.6.2.2, applied with $X = I^{-1}$).

2. (a) If $P$ is not strong and $a \in P^{-1} \setminus (P:P)$, then $aP \subset D$ and $aP \not\subset P$, which implies that $aP_P = D_P$. Since $D_P$ is a Mori monoid, it satisfies the ascending chain condition on principal ideals. Hence it is atomic by Theorem 1.5.5, and by Theorem 3.4.8, it is a dv-monoid.

   If $P$ is strong, then $(D:P) = (P:P)$ implies $(D_P:P_P) = (P_P:P_P)$, and therefore $D_P$ is not a dv-monoid.

   (b) Assume that $P \in v$-$\text{max}(D)$. If $P$ is strong, then $(PP^{-1})_v = P$ by 1., and therefore $P$ is not $v$-invertible. If $P$ is not strong, then $D_P$ is a dv-monoid and $P_P$ is a principal ideal of $D_P$. If $M \in v$-$\text{max}(D) \setminus \{P\}$, then $P_M = D_M$. Hence $P$ is $t$-invertible (and thus $v$-invertible) by Theorem 4.1.4.

   (c) By Theorem 1.3.8 we have $D_P = (T^{-1}D)_{T^{-1}P}$, and thus the assertion follows by (a). $\square$

Theorem 5.1.2. Let $I \subset D$ be a strong ideal, $C = (D:I) = (I:I)$ and $Q \subset C$ a prime ideal such that $I = c\sqrt[\mathbb{K}]{I} \subset Q$. Then $(Q:Q) = C$. 

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Proof. It suffices to prove that \((Q:Q)\subset (I:I)\). Indeed, then \(C \subset (Q:Q) \subset (I:I) = C\), hence \((Q:Q) = C\), and if \(Q\) is strong, then \((C:Q) = C\) and \(Q_{v(C)} = C \neq Q\).

Thus assume that \(x \in (Q:Q)\) and \(y \in I\). We must prove that \(xy \in I\), and since
\[
I = \sqrt{I} = \bigcap_{P \in \mathcal{P}_C(I)} P,
\]
it suffices to prove that \(xy \in P\) for all \(P \in \mathcal{P}_C(I)\). If \(Q \in \mathcal{P}_C(I)\), then \(xy \in (Q:Q)I \subset (Q:Q)Q \subset Q\). If \(P \in \mathcal{P}_C(I)\) and \(P \neq Q\), then \(Q \not\subset P\) and \(xyQ \subset I(Q:Q)Q \subset IQ \subset I \subset P\), which implies \(xy \in P\). □

Theorem 5.1.3. Let \(I \subset D\) be a strong ideal, \(C = (D:I) = (I:I)\) and \(v^* = v(C)\).

1. If \(D\) is a Mori monoid, then \(C\) is also a Mori monoid, and \(\mathcal{F}_v(C) \subset \mathcal{F}_{v^*}(D)\).

2. The assignment \(P \mapsto (P:I)\) defines a bijective map
\[
\Phi: \{P \subset D \mid P\text{ is a prime ideal, } I \not\subset P\} \to \{Q \subset C \mid Q\text{ is a prime ideal, } I \not\subset Q\},
\]
whose inverse is given by \(Q \mapsto Q \cap D\).

3. Let \(P \subset D\) be a prime ideal such that \(I \not\subset P\) and \(Q = (P:I)\).

\(a\) \quad \(D_P = C_Q\).

\(b\) \quad If \(J \subset D\) and \(J^* \subset C\) are ideals such that \(J^* \cap D = J \subset P\), then \(J^* \subset Q\).

\(c\) \quad If \(P \in v^*\text{-spec}(D)\), then \(Q \in v^*\text{-spec}(C)\).

\(d\) \quad If \(D\) is a Mori monoid and \(P \in v\text{-max}(D)\), then \(Q \in v^*\text{-max}(C)\).

Proof. 1. Since \((D:I) \in \mathcal{F}_v(D) \subset \mathcal{F}_l(D)\), Theorem 2.6.6.3 implies that \(C\) is a Mori monoid, and \(\mathcal{F}_{v^*}(C) = \mathcal{F}_{l(C)}(C) \subset \mathcal{F}_l(D) = \mathcal{F}_v(D)\).

2. Let \(P \subset D\) be a prime ideal, \(I \not\subset P\), and \(Q = (P:I)\).

Clearly, \(Q \subset (D:I) = C\), and \(CQI \subset QI \subset P\) implies \(CQ \subset (P:I) = Q\). Hence \(Q \subset C\) is an ideal, and we prove that it is a prime ideal of \(C\). Suppose that \(x, y \in C\), \(xy \in Q\) and \(x \not\in Q\). Then
\[
xyP \subset (P:I)P = PI \subset P,
\]
and since \(xP \not\subset P\), we obtain \(yP \subset P\) and \(y \in (P:I) = Q\).

Next we prove that \(Q \cap D = P\). Clearly, \(PI \subset P\) implies \(P \subset (P:I) \cap D = Q \cap D\). Conversely, if \(z \in Q \cap D\), then \(zI \subset P\) and \(I \not\subset P\) implies \(z \in P\).

It remains to prove that \(\Phi\) is surjective. Thus let \(R \subset C\) be a prime ideal ideal such that \(I \not\subset R\).

Then \(R \cap D \subset D\) is a prime ideal, \(I \not\subset R \cap D\), and we assert that \(R = (R \cap D : I)\). If \(x \in R\), then \(R \subset C = (D:I)\) implies \(xI \subset R \cap D\) and \(x \in (R \cap D : I)\). Conversely, if \(x \in (R \cap D : I)\), then \(xI \subset R\) and \(I \not\subset R\) implies \(x \in R\).

3. (a) Since \(D \setminus P \subset C \setminus Q\), we obtain \(D_P \subset C_Q\). Thus let \(z = s^{-1}c \in C_Q\), where \(c \in C\) and \(s \in C \setminus Q\). If \(y \in I \setminus P\), then \(cy \in CI = I \subset D\), and \(sI \not\subset P\) implies \(sy \in CI \setminus P \subset D \setminus P\). Hence it follows that \(z = (sy)^{-1}cy \in D_P\).

(b) Let \(J \subset D\) and \(J^* \subset C\) be ideals such that \(J \subset P\) and \(J^* \cap D = J\). Then it follows that \(J^*I \subset J^* \cap CI \subset J^* \cap D = J \subset P\), and therefore \(J^* \subset (P:I) = Q\).

(c) Suppose that \(P \in v\text{-spec}(D)\). We must prove that \((P:I)_{v^*} \subset (P:I)\). We have
\[
I(P:I)_{v^*} = I(C:C:(P:I))) = I(I^{-1}(I:(I:I):(P:I)))) \subset (II^{-1}(I:I)P) \subset (D:(I:I)(P:I))\,
\]
and we shall prove that \(P^{-1} \subset (I:I)\). If this is done, then \((P:I)_{v^*} \subset (I:I(P:I))^{-1} \subset Pv = P\), and therefore \((P:I)_{v^*} \subset (P:I)\).

(d) Suppose that \(D\) is a Mori monoid and \(P \in v\text{-max}(D)\). Then \(Q \in v^*\text{-spec}(C)\), and since \(C\) is a Mori monoid, there exists some \(M \in v^*\text{-max}(C)\) such that \(M \supset Q\). Then \(M \cap D \in \mathcal{F}_v(D)\) is a prime ideal of \(D\), hence \(M \cap D \in v\text{-spec}(D)\), and \(P \subset M \cap D\). Hence \(P = M \cap D\), \(I \not\subset M\), and by 1. it follows that \(Q = M \in v^*\text{-max}(D)\). □
5.2. Complete integral closures and Krull monoids

Definition 5.2.1.
1. An element \( a \in K \) is called \textit{almost integral} over \( D \) if there exists some \( c \in D^\bullet \) such that \( ca^n \in D \) for all \( n \in \mathbb{N} \).
2. The set \( \tilde{D} = \{ a \in K \mid \text{a is almost integral over } D \} \) is called the \textnormal{complete integral closure} of \( D \), and \( F_D = (D:\tilde{D}) \) is called the \textnormal{conductor} of \( D \).
3. \( D \) is called \textit{completely integrally closed} if \( D = \tilde{D} \).
4. \( D \) is called a \textit{Krull monoid} if \( D \) is a completely integrally closed Mori monoid.

Theorem 5.2.2. Let \( r \) be an ideal system on \( D \).
1. \( \tilde{D} \) is a submonoid of \( K \),
   \[
   \tilde{D} = \bigcup_{I \in c_I(D)} I^{-1} = \bigcup_{J \in \mathcal{F}_r(D)^\bullet} (J:J) = \bigcup_{J \in \mathcal{F}_r(D)^\bullet} (J:J) = \bigcup_{J \in \mathcal{I}_r(D)^\bullet} (J:J)
   \]
   and if \( r \) is finitary, then \( \bigcap_{r} = \tilde{D} \).
   In particular, if \( D \) is a domain, then \( \tilde{D} \) is also a domain.
2. \( D/ \tilde{D}^\times = \tilde{D}/D^\times \). In particular, \( D \) is completely integrally closed if and only if \( D/D^\times \) is complete integrally closed.
3. \( \text{cl}_r(D) \subseteq \tilde{D} \), and if \( D \) is \( r \)-noetherian, then \( \tilde{D} = \text{cl}_r(D) \). In particular, if \( D \) is completely integrally closed, then \( D \) is \( r \)-closed, and the converse holds if \( D \) is \( r \)-noetherian.
4. \( F_D \) is the intersection of all strong \( v \)-ideals of \( D \).
5. \( F_D \neq \emptyset \) if and only if \( D \) contains a smallest strong \( v \)-ideal \( F \). If \( F \) is the smallest strong \( v \)-ideal of \( D \), then \( F_D = F \), \( \tilde{D} = F^{-1} \in \mathcal{I}_r(D) \), and \( \tilde{D} \) is completely integrally closed.

Proof. 1. We show that
   \[
   \tilde{D} \subseteq \bigcup_{I \in c_I(D)} I^{-1} \subset \bigcup_{J \in \mathcal{F}_r(D)^\bullet} (J:J) \subset \bigcup_{J \in \mathcal{F}_r(D)^\bullet} (J:J) = \bigcup_{J \in \mathcal{I}_r(D)^\bullet} (J:J) \subseteq \tilde{D}.
   \]
   If \( x \in \tilde{D} \), then there is some \( c \in D^\bullet \) such that \( X = \{ cx^n \mid n \in \mathbb{N} \} \subseteq D \). By Theorem 5.1.1.1(b), \( I \in \mathcal{I}_r(D)^\bullet \) is strong, and since \( xX \subseteq X \), it follows that \( xX \subseteq X \), and thus \( x \in I^{-1} \).
   The two following inclusions are obvious. If \( J \in \mathcal{F}_r(D)^\bullet \) and \( c \in D^\bullet \) is such that \( cJ \subseteq D \), then \( (J:J) = (cJ:cJ) \). If \( J \in \mathcal{I}_r(D)^\bullet \), \( c \in J^\bullet \) and \( x \in (J:J) \), then \( x^n \in (J:J) \) and therefore \( cx^n \in J \subseteq D \) for all \( n \in \mathbb{N} \), which implies \( x \in \tilde{D} \).
   If \( J, J' \in \mathcal{I}_r(D)^\bullet \), then \( (JJ')_r : ((JJ')_r) > (J:J) \). Therefore \( \{(J:J) \mid J \in \mathcal{I}_r(D)^\bullet \} \) is a directed set of \( r \)-monoids. Hence \( \tilde{D} \) is a monoid, and if \( r \) is finitary, then \( \bigcap_{r} = \tilde{D} \).
   If \( D \) is a domain, then \( \tilde{D}_{d(D)} = \tilde{D} \). Hence \( \tilde{D} \) is a \( D \)-module and therefore itself a domain.
   2. By definition, \( q(D/D^\times) = K/D^\times \), and if \( x \in K \), then \( x \in \tilde{D} \) if and only if \( xD^\times \in D/\tilde{D}^\times \). Hence \( \tilde{D}/\tilde{D}^\times = \tilde{D}/D^\times \), and \( D = \tilde{D} \) if and only if \( D/D^\times = \tilde{D}/\tilde{D}^\times \).
   3. By Theorem 4.3.3 we have
   \[
   \text{cl}_r(D) = \bigcup_{J \in \mathcal{I}_r(D)} (J:J).
   \]
   Hence \( \text{cl}_r(D) \subseteq \tilde{D} \). If \( D \) is \( r \)-noetherian, then \( \mathcal{I}_r(D) = \mathcal{I}_{r,f}(D) \), and therefore equality holds.
4. By 1., we obtain
\[ F_D = \hat{\mathcal{D}}^{-1} = \left( \bigcup_{I \in \mathcal{I}_v(D)} I^{-1} \right)^{-1} = \bigcap_{I \in \mathcal{I}_v(D)} I. \]

5. If \( F_D^* \neq \emptyset \), then \( F_D \) is a strong \( v \)-ideal by Theorem 5.1.1, and by 4. it is the smallest strong \( v \)-ideal of \( D \). Conversely, if \( F \) is the smallest strong \( v \)-ideal of \( D \), then \( F = F_D \) by 4. Hence \( \hat{\mathcal{D}}_v = F^{-1} \subset \hat{\mathcal{D}} \), and therefore \( F^{-1} = \hat{\mathcal{D}} \in \mathcal{F}_v(D) \). In particular, if \( \mathcal{F}(D) \) resp. \( \mathcal{F}(\hat{\mathcal{D}}) \) denotes the set of all fractional semigroup ideals of \( D \) resp. \( \hat{\mathcal{D}} \), then \( \mathcal{F}(\hat{\mathcal{D}}) \subset \mathcal{F}(D) \), hence
\[ \hat{\mathcal{D}} = \bigcup_{J \in \mathcal{F}(\hat{\mathcal{D}})} (J : J) \subset \bigcup_{J \in \mathcal{F}(D)} (J : J) \subset \hat{\mathcal{D}}, \]
and therefore equality follows. \( \Box \)

**Theorem 5.2.3.** The following assertions are equivalent:

(a) \( D \) is completely integrally closed.

(b) \( \mathcal{F}_v(D)^* = \mathcal{F}_v(D)^\times \) [equivalently: every non-zero (fractional) \( v \)-ideal of \( D \) is \( v \)-invertible].

(c) \( D \) is the only strong \( v \)-ideal of \( D \).

**Proof.** (a) \( \Rightarrow \) (b) If \( J \in \mathcal{F}_v(D)^* \), then \( (J : J) \subset \hat{\mathcal{D}} = D \), hence \( (J : J) = D \), and therefore \( J \in \mathcal{F}_v(D)^\times \) by Theorem 4.1.2.

(b) \( \Rightarrow \) (c) If \( J \in \mathcal{F}_v(D)^* \) is strong and invertible, then \( J = J \cdot_v J^{-1} = D \).

(c) \( \Rightarrow \) (a) By Theorem 5.2.2.1. \( \Box \)

**Theorem 5.2.4.** Let \( D \) be a Mori monoid.

1. If \( F_D^* \neq \emptyset \), then \( \hat{\mathcal{D}} \) is a Krull monoid.

2. Let \( T \subset D^\times \) be a multiplicatively closed subset. Then \( \hat{T^{-1}D} = T^{-1}\hat{D} \). In particular, if \( D \) is a Krull monoid, then \( T^{-1}D \) is a Krull monoid.

3. \( \hat{D}^\times \cap D = D^\times \).

**Proof.** 1. If \( F_D^* \neq \emptyset \), then \( \hat{\mathcal{D}} \) is completely integrally closed by Theorem 5.2.5, and \( \hat{\mathcal{D}} \) is a Mori monoid by Theorem 5.1.1.2.

2. Observe that \( T^{-1}t = t(T^{-1}D) \), and \( \hat{T^{-1}D} = \text{cl}_{T^{-1}}(T^{-1}D) = T^{-1}\text{cl}_t(D) = T^{-1}\hat{D} \) by the Theorems 2.6.6.2 and 4.3.4.3.

3. Obviously, \( D^\times \subset \hat{D}^\times \cap D \). If \( a \in \hat{D}^\times \cap D \), then there is some \( c \in D^\times \) such that \( ca^{-n} \in D \) for all \( n \in \mathbb{N} \). Hence it follows that \( c \in a^nD \) for all \( n \in \mathbb{N} \), and therefore the set \( \{ a^nD \mid n \in \mathbb{N} \} \subset \mathcal{I}_v(D) \) has a smallest element. Consequently, there is some \( n \in \mathbb{N} \) such that \( a^nD = a^{n+1}D \), which implies \( D = aD \) and \( a \in D^\times \). \( \Box \)

**Theorem 5.2.5.**

1. The following assertions are equivalent:

(a) \( D \) is a Krull monoid.

(b) \( \mathcal{F}_t(D)^* = \mathcal{F}_t(D)^\times \) [equivalently: every non-zero (fractional) \( t \)-ideal of \( D \) is \( t \)-invertible].

(c) \( D \) is a Mori monoid, and for every \( M \in \text{t-max}(D) \), \( D_M \) is a dv-monoid.

In particular, if \( D \) is a Krull monoid, then \( \text{t-max}(D) = \mathcal{X}(D) \), and therefore \( D_P \) is a discrete valuation monoid for every non-zero prime \( t \)-ideal.
2. $D$ is factorial if and only if $D$ is a Krull monoid and $\mathcal{C}_v(D) = 0$.

3. $D$ is a dv-monoid if and only if $D$ is a t-local Krull monoid.

**Proof.** 1. (a) $\Rightarrow$ (b) $v = t$, and by Theorem 5.2.3 we have $\mathcal{F}_v(D) = \mathcal{F}_t(D)$.

(b) $\Rightarrow$ (c) Since every non-zero $t$-ideal is $t$-invertible and hence $t$-finitely generated, it follows that $D$ is a Mori monoid. If $M \in t$-max($D$), then $D_M$ is $t$-noetherian, hence atomic, and $M_M$ is a principal ideal. By Theorem 3.4.8, $D_P$ is a dv-monoid.

(c) $\Rightarrow$ (a) If $J \in \mathcal{F}_t(D)^*$, then $J_M$ is principal for all $M \in t$-max($D$). Hence $J$ is $t$-invertible, and as $t = v$, $D$ is completely integrally closed by Theorem 5.2.3.

In particular, if $D$ is a Krull monoid and $P \in t$-max($D$), then $P$ is $t$-invertible and thus $P \in \mathcal{X}(D)$ by Theorem 5.1.1.4.

2. By Theorem 2.6.3.2, $D$ is factorial if and only if every non-zero $t$-ideal is principal. However, this holds if and only if every $J \in \mathcal{I}_t(D)$ is $t$-invertible and principal. By 1., the assertion follows.

3. Obvious by 1.(c). \qed

**Theorem 5.2.6.** Let $D$ be a Krull monoid. Then $\Lambda_t(K) = \mathcal{F}_{t,t}(D)$, and $\Lambda_t^+(K) = \mathcal{I}_{t,t}(D)$ is free with basis $t$-max($D$).

**Proof.** Since $\mathcal{M}_{t,t}(K) = \mathcal{F}_{t,t}(D)$ and $\mathcal{F}_{t,t}(D)^*$ is a group, it follows that $t$ is finitely cancellative, hence $t = t_a$. $\Lambda_t(K) = q(\mathcal{M}_{t,t}(K)) = \mathcal{F}_{t,t}(D)$, and $\Lambda_t^+(K) = \{C \in \mathcal{F}_{t,t}(D) \mid C \subseteq D_t = D\} = \mathcal{I}_{t,t}(D)$ is a reduced GCD-monoid by Theorem 4.4.2. Moreover, for all $I$, $J \in \mathcal{I}_{t,t}(D)$ we have $I \mid J$ in $\mathcal{I}_{t,t}(D)$ if and only if $J \subseteq I$. Hence $\Lambda_t^+(D)$ satisfies the ACC for principal ideals, and as it is a reduced GCD-monoid, it is factorial and therefore free with the set of prime elements as a basis. An element $P \in \mathcal{I}_{t,t}(D) \setminus \{D\}$ is a prime element if and only if it is maximal with respect to inclusion, that is, if and only if it is a $t$-maximal $t$-ideal. \qed

**Definition 5.2.7.** A domain $D$ is called a

- **Krull domain** if it is a Krull monoid;
- **Dedekind domain** if it is a Krull domain, and $d(D) = t$ [equivalently, every ideal is divisorial].

**Theorem 5.2.8.** For a domain $D$, the following assertions are equivalent:

(a) $D$ is a Dedekind domain.

(b) $D$ is a Krull domain and $\dim(D) = 1$ [equivalently, every non-zero prime ideal of $D$ is maximal].

(c) Every non-zero ideal of $D$ is invertible.

(d) $D$ is noetherian, and for every non-zero prime ideal $P$, $D_P$ is a discrete valuation domain.

(e) $D$ is noetherian, integrally closed, and $\dim(D) = 1$ [equivalently, every non-zero prime ideal of $D$ is maximal].

**Proof.** Set $d = d(D)$.

(a) $\Rightarrow$ (b) If $P \in \text{spec}(D) = t$-spec($D$) and $P^* \neq \emptyset$, then $P$ is not strong by Theorem 5.2.3, and thus $P \in \mathcal{X}(D)$ by Theorem 5.1.1.4.

(b) $\Rightarrow$ (c) Let $J \in \mathcal{I}(D)^*$ be a non-zero ideal. Then $J_t \in \mathcal{I}_{t,t}(D)$, and by Theorem 4.1.4 we must prove that $J_{t'}$ is principal for all $P \in \text{max}(D)$. If $P \in t$-max($D$), then $D_P$ is a discrete valuation domain and therefore $J_P$ is principal. However, $\text{max}(D) = \mathcal{X}(D)$ by assumption, and by the Theorems 3.1.6.4 and 5.2.5 it follows that $\text{max}(D) = t$-max($D$).

(c) $\Rightarrow$ (a) Every non-zero ideal of $D$ is invertible, hence a $t$-ideal by Theorem 4.1.2. Therefore $t = d$, and $D$ is a Krull domain by Theorem 5.2.5.
(a) ⇒ (d) Obvious by Theorem 5.2.5.

(d) ⇒ (e) If \( P \in \text{spec}(D) \) and \( P^* \neq \emptyset \), then \( D_P \) is a discrete valuation domain, hence primary, and therefore \( P \in \mathfrak{X}(D) \) by Theorem 3.4.6.3. Hence \( \dim(D) = 1 \). Moreover, for all non-zero \( P \in \text{spec}(D) \), \( D_P \) is a Krull domain and thus (completely) integrally closed. Hence \( D \) is integrally closed by Theorem 4.3.4.4.

(e) ⇒ (a) It suffices to prove that \( \mathcal{I}(D) \subset \mathcal{I}_v(D) \). Since \( \dim(D) = 1 \), we have \( \max(D) = t\max(D) \), and we assert that, for every \( g \in \mathcal{I}(D) \), and let \( a \in \mathcal{I}_v(D) \), we obtain \( J_t = \bigcap_{P \in t\max(D)} (J_t)_P = \bigcap_{P \in \max(D)} J_P = J \in \mathcal{I}_v(D) \).

\( \square \)

The following example shows that the complete integral closure need not be completely integrally closed.

**Example 5.2.9.** Let \( K \) be a field,

\[ R = K[\{X^{2n+1}Y^{(n+1)} | n \in \mathbb{N}_0\}] \quad \text{and} \quad S = K[\{XY^n | n \in \mathbb{N}_0\}] .\]

Then \( R \subset S \subset K[X,Y] = q(R) \), \( S = \hat{R} \) and \( K[X,Y] = \hat{S} \).

**Proof.** By definition, \( R \subset S \subset K[X,Y] \), and for all \( n \in \mathbb{N}_0 \), \( (XY^n)^{2n+1} \in R \). Hence \( S \) is integral over \( R \), and therefore \( S \subset \hat{R} \). Since \( \{X, X^3Y^3, X^5Y^5\} \subset R \), we obtain \( Y = X^3(X^3Y^3)^{-3}(X^5Y^5) \in q(R) \) and therefore \( q(R) = K[X,Y] \). Since \( XY^n \in S \) for all \( n \in \mathbb{N}_0 \), it follows that \( K[X,Y] \subset \hat{S} \). On the other hand, \( K[X,Y] \) is factorial, hence a Krull domain and therefore completely integrally closed. Thus we obtain \( \hat{S} \subset K[X,Y] = K[X,Y] \), and it remains to prove that \( \hat{R} \subset S \). We show the following two assertions:

**A.** \( K[X,Y] = S + K[Y] \).

**B.** \( K[Y] \cap \hat{R} = K \).

Suppose that **A** and **B** hold, and let \( f \in \hat{R} \subset K[X,Y] \). By **A** we have \( f = g + h \), where \( g \in S \) and \( h \in K[Y] \). Since \( S \subset \hat{R} \), it follows that \( h = f - g \in K[Y] \cap \hat{R} = K \) and therefore \( f = g + h \in S \).

**Proof of A.** It suffices to prove that \( X^iY^j \in S + K[Y] \) for all \( i, j \in \mathbb{N}_0 \). This is obvious for \( i = 0 \), and if \( i \geq 1 \), then \( X^iY^j = X^{i-1}(XY^j) \in S \), since \( X \in S \). \( \square \)**A**.

**Proof of B.** Assume to the contrary, that there is some \( f \in K[Y] \cap \hat{R} \) such that \( \deg(f) = n \geq 1 \), and let \( a \in K^\times \) be the leading coefficient of \( f \). Then there exists some \( g \in R^* \) such that \( gf^k \in R \) for all \( k \in \mathbb{N} \). Suppose that \( g = (bX^l + h_1)Y^r + g_0 \), where \( l, r \in \mathbb{N}_0 \), \( b \in K^\times \), \( h_1 \in K[X] \), \( \deg(h_1) < l \), \( g_0 \in K[X,Y] \) and \( \deg_Y(g_0) < r \). Let \( k \in \mathbb{N} \) be such that

\[ r + nk > \sum_{i=0}^{l} i(2i+1). \]

Then \( gf^k = ba^kX^lY^{r+nk} + g_k \), where \( g_k \in K[X,Y] \) and \( \deg_Y(g_k) < r + nk \). A \( K \)-basis of \( R \) is given by the set of all products of the form

\[ \prod_{\nu=0}^N X^{2s_\nu+1}Y^{s_\nu(2s_\nu+1)}, \quad \text{where} \quad N \in \mathbb{N}_0 \quad \text{and} \quad s_0, \ldots, s_N \in \mathbb{N}_0. \]
5.3. OVERMONOIDS OF MORI MONOIDS

Hence there exist some \( N \in \mathbb{N} \) and \( s_0, \ldots, s_N \in \mathbb{N}_0 \) such that
\[
X^l Y^{r+nk} = \prod_{\nu=0}^{N} X^{2s_{\nu}+1} Y^{s_{\nu}(2s_{\nu}+1)}. 
\]

For \( i \in \mathbb{N}_0 \), we define \( r_i = |\{\nu \in [0, N] \mid s_{\nu} = i\}| \), and then we obtain
\[
l = \sum_{\nu=0}^{N} (2s_{\nu} + 1) = \sum_{i \geq 0} r_i (2i + 1) \quad \text{and} \quad r + nk = \sum_{\nu=0}^{N} s_{\nu} (2s_{\nu} + 1) = \sum_{i \geq 0} r_i (2i + 1).
\]

Hence it follows that \( r_i \leq l \) for all \( i \geq 0 \), and
\[
r + nk \leq l \sum_{i \geq 0} i(2i + 1) < r + nl, \quad \text{a contradiction.} \quad \Box
\]

5.3. Overmonoids of Mori monoids

**Theorem 5.3.1.** Let \( (D_\lambda)_{\lambda \in \Lambda} \) be a family of monoids such that \( D \subset D_\lambda \subset K \) for all \( \lambda \in \Lambda \),
\[
D' = \bigcap_{\lambda \in \Lambda} D_\lambda,
\]
and assume that, for every \( a \in D^* \), the set \( \{\lambda \in \Lambda \mid a \notin D_\lambda^x\} \) is finite.

1. If \( T \subset D^* \) is a multiplicatively closed subset, then
\[
T^{-1}D' = \bigcap_{\lambda \in \Lambda} T^{-1}D_\lambda.
\]

2. If \( (D_\lambda)_{\lambda \in \Lambda} \) is a family of Mori monoids, then \( D' \) is a Mori monoid.

**Proof.** 1. Obviously, \( T^{-1}D' \subset T^{-1}D_\lambda \) for all \( \lambda \in \Lambda \). Thus suppose that
\[
x \in \bigcap_{\lambda \in \Lambda} T^{-1}D_\lambda, \quad \text{say} \quad x = a^{-1}b, \quad \text{where} \quad a \in D^* \quad \text{and} \quad b \in D.
\]

The set \( \Delta = \{\lambda \in \Lambda \mid a \notin D_\lambda^x\} \) is finite, and if \( \lambda \in \Lambda \setminus \Delta \), then \( x \in D_\lambda \). For each \( \lambda \in \Delta \), there exist \( a_\lambda \in D_\lambda \) and \( t_\lambda \in T \) such that \( x = t_\lambda^{-1}a_\lambda \), and we set
\[
t = \prod_{\lambda \in \Delta} t_\lambda.
\]

Then it follows that \( t \in T \), \( tx \in D' \) and \( x = t^{-1}(tx) \in T^{-1}D' \).

2. For every subset \( X \subset D' \), we set
\[
X' = \bigcap_{\lambda \in \Lambda} X_{v(D_\lambda)}, \quad \text{and we assert that} \quad X \subset X' \subset X_{v(D')},
\]
Obviously, \( X \subset X' \), and if \( c \in K \) is such that \( X \subset D'c \), then \( X_{v(D_\lambda)} \subset D'_{v(D_\lambda)}c \subset D_\lambda c \) for all \( \lambda \in \Lambda \), and therefore \( X' \subset D'c \). Hence it follows that
\[
X' \subset \bigcap_{c \in K} \bigcap_{X \subset D'c} D'c = X_{v(D')}.
\]
We prove that for every subset \( X \subset D' \) there exists some \( E \in \mathbb{P}_f(X) \) such that \( X \subset E_{v(D')} \). Thus let \( X \subset D' \). We may assume that \( X^* \neq \emptyset \), and we fix some \( a \in X^* \). Then the set \( \Delta = \{\lambda \in \Lambda \mid a \notin D_\lambda^x\} \) is
finite, and for every \( \lambda \in \Delta \), there is some \( E_\lambda \in \mathbb{P}_t(X) \) such that \( a \in E_\lambda \) and \( X_{v(D_\lambda)} = (E_\lambda)_{v(D_\lambda)} \). Now we consider the set

\[
E = \bigcup_{\lambda \in \Delta} E_\lambda \in \mathbb{P}_t(X).
\]

If \( \lambda \in \Delta \), then \( E_{v(D_\lambda)} \supset (E_\lambda)_{v(D_\lambda)} = X_{v(D_\lambda)} \), and if \( \lambda \notin \Lambda \setminus \Delta \), then \( E_{v(D_\lambda)} = D_\lambda = X_{v(D_\lambda)} = D_\lambda \).

Hence we obtain

\[
E_{v(D')} \supset E' = \bigcap_{\lambda \in \Lambda} E_{v(D_\lambda)} \supset \bigcap_{\lambda \in \Lambda} X_{v(D_\lambda)} = X' \supset X.
\]

\[\square\]

**Definition 5.3.2.** Let \( D \) be a Mori monoid. We define

\[
\mathcal{S}(D) = \{P \in v\text{-max}(D) \mid P \text{ strong}\} \quad \text{and} \quad \mathcal{R}(D) = \{P \in v\text{-max}(D) \mid P \text{ not strong}\},
\]

\[
\hat{D} = \bigcap_{P \in \mathcal{R}(D)} D_P \cap \bigcap_{P \in \mathcal{S}(D)} (D_P : P_P), \quad \hat{v} = v(\hat{D}) \quad \text{and} \quad \hat{t} = t(\hat{D}).
\]

If \( P \in v\text{-max}(D) \), then Theorem 5.1.1.2 implies that \( P \in \mathcal{R}(D) \) if and only if \( D_P \) is a dv-monoid, and \( P \in \mathcal{S}(D) \) if and only if \( D_P \) is not a dv-monoid. In particular, Theorem 5.2.5 implies that \( D \) is a Krull monoid if and only if \( \mathcal{S}(D) = \emptyset \).

**Theorem 5.3.3.** Let \( D \) be a Mori monoid.

1. \( \hat{D} \in \mathcal{M}_t(K) \) is a Mori monoid, and \( \hat{D} \subset \text{cl}_t(D) \subset \hat{D} \).

2. If \( R \in \mathcal{S}(D) \), then \( \hat{D}_R = (D_R : Q_R) = (Q_R : Q_R) \).

3. If \( R \in v\text{-spec}(\hat{D}) \), then \( R \cap D \in v\text{-spec}(D) \), and if \( R \) is strong, then \( R \cap D \) is strong, too.

**Proof.** 1. If \( P \in \mathcal{S}(D) \), then \( (D_P : P_P) = (D : P)_P = (P : P)_P \subset D \) is an overmonoid, and therefore \( \hat{D} \supset D \) is an overmonoid. If \( P \in v\text{-max}(D) \), then \( (D_P)_t = D_P \in \mathcal{M}_t(K) \) by Theorem 2.5.4, hence \( (D_P : P_P) \in \mathcal{M}_t(K) \), and therefore it follows that \( \hat{D} \in \mathcal{M}_t(K) \). By Theorem 2.6.6 it follows that \( D_P \) is a Mori monoid for all \( P \in \mathcal{R}(D) \), and that \( (D_P : P_P) = (P : P)_P \) is a Mori monoid for all \( P \in \mathcal{S}(D) \). If \( a \in D^* \), then the set \( \{P \in v\text{-spec}(D) \mid a \in P \} \) is finite by Theorem 3.2.7.2. If \( P \in \mathcal{R}(D) \) and \( a \notin P \), then \( a \in D^* \). If \( P \in \mathcal{S}(D) \) and \( a \notin P \), then \( a^{-1} \in (P : P)_P = (D_P : P_P), \) and therefore \( a \in (D_P : P_P)^* \).

By Theorem 5.3.2.2 it follows that \( \hat{D} \) is a Mori monoid.

If \( P \in \mathcal{S}(D) \), then \( (D_P : P_P) = (D : P)_P = (P : P)_P \subset \text{cl}_t(D_P) \), and therefore

\[
\hat{D} \subset \bigcap_{P \in \text{v-max}(D)} \text{cl}_t(D_P) = \text{cl}_t(D) \subset \hat{D}.
\]

2. Assume that \( Q \in \mathcal{S}(D) \). If \( P \in v\text{-max}(D) \) and \( P \neq Q \), then \( (D : Q) \subset D_P \) by Theorem 1.3.9.1, and therefore \( (D_Q : Q_Q) = (D_Q : Q_Q) \subset (D : Q)_Q \subset (D : P_P)_Q \). If \( P \in \mathcal{R}(D) \), then \( D_P \) is a dv-monoid, and since \( P \notin Q \), Theorem 1.3.9.2 implies that \( D_P \subsetneq (D : Q)_Q \). By Theorem 3.4.8, \( D_P \) is primary, and by Theorem 3.4.6 we obtain \( (D : Q)_Q = K \). Collecting these arguments, we obtain, using Theorem 5.3.1.1,

\[
\hat{D}_Q = \bigcap_{P \in \mathcal{R}(D)} (D_P : P_P) \cap \bigcap_{P \in \mathcal{S}(D)} (D : P_P)_Q = \bigcap_{P \in \mathcal{S}(D)} (D_P : P_P)_Q \cap (D : Q)_Q = (D : Q)_Q.
\]

Finally, \( (D : Q)_Q = (Q : Q)_Q \), since \( Q_Q \) is strong in \( D_Q \).

3. By Theorem 2.5.3.4, \( \mathcal{t}(\hat{D}) \) is an ideal system of \( \hat{D} \), and therefore \( t(\hat{D}) \leq \hat{t} \). If \( R \in v\text{-spec}(\hat{D}) \), then \( (R \cap D)_v = (R \cap D)_t \subset R \subset R \subset R \cap D \) and therefore \( R \cap D \in v\text{-spec}(D) \).

If \( R \cap D \) is not strong, then \( D_{R \cap D} \) is a dv-monoid, and since \( D_{R \cap D} \subset \hat{D}_R \subset K \), it follows that \( \hat{D}_R = D_{R \cap D} \). Hence \( R \) is not strong. \[\square\]
Theorem 5.3.4. Let $D$ be a Mori monoid and $P \in v\text{-spec}(D) \setminus S(D)$. Then there exists a unique $\bar{P} \in \bar{v}\text{-spec}(\bar{D})$ such that $\bar{P} \cap D = P$, and the following assertions hold:

- $D_P = \bar{D}_{\bar{P}}$.
- $P$ is strong if and only if $\bar{P}$ is strong.
- If $P \in R(D)$, then $\bar{P} \in R(\bar{D})$.
- If $I \in I_\bar{v}(D)$, $\bar{I} \in I_\bar{v}(\bar{D})$ and $\bar{I} \cap D = I \subset P$, then $\bar{I} \subset \bar{P}$, and $\bar{I}_P = I_P$.

Proof. We assume first that all statements of the Theorem except the equality $\bar{I}_P = I_P$ in the last assertion hold, and we show how this equality follows. Since $D_P = \bar{D}_{\bar{P}}$, we obtain $\bar{P}_P = P_P \subset \bar{P}_P$, and since $D \setminus P \subset \bar{D} \setminus \bar{P}$, it follows that $\bar{P}_P \subset \bar{P}_P$, and therefore $\bar{P}_P = P_P$. Let now $I \in I_\bar{v}(D)$, and $\bar{I} \in I_\bar{v}(\bar{D})$ be such that $\bar{I} \cap D = I \subset P$ and $\bar{I} \subset P$. Then $P \cap \bar{I} = I$, and $\bar{I}_P = \bar{I}D_{\bar{P}} = \bar{I}D_P = \bar{I}_P = P_P \cap I_P = P_P \cap \bar{I}_P = (P \cap \bar{I})_P = I_P$.

For the main part of the proof we distinguish two cases. Since $P \in v\text{-spec}(D) \setminus S(D)$, it follows that either $P \in R(D)$, or that $P$ is not $v$-maximal. In this second case, there is some $M \in v\text{-max}(D)$ such that $P \subsetneq M$, and then necessarily $M \in S(D)$.

CASE 1: $P \in R(D)$. If this case, $D_P$ is a dv-monoid, $\bar{D} \subset D_P$, and we set $\bar{P} = P_P \cap \bar{D}$. Then $\bar{P} \subset \bar{D}$ is a prime ideal, and $P \cap D = P_P \cap D = P$. Suppose now that $P' \subset \bar{D}$ is another prime ideal satisfying $P' \cap D = P$. Then $D_P \subset \bar{D}_P \subset K$, hence $D_P = \bar{D}_P$, and $P_P = P'_P$ is a principal ideal. Therefore it follows that $\bar{P} = P_P \cap \bar{D} = P'_P \cap \bar{D} = P' \subset \bar{v}\text{-spec}(\bar{D})$ by Theorem 2.6.6.2 (c). Since $\bar{D}_P = D_P$ is a dv-monoid, $\bar{P}$ is not strong, and we assert that $\bar{P} \in \bar{v}\text{-max}(\bar{D})$. Indeed, if $\bar{P} \in \bar{v}\text{-spec}(\bar{D})$ is such that $\bar{P} \subset \bar{P}$, then $P = \bar{P} \cap D \subset \bar{P} \cap D$, and since $\bar{P} \cap D \in v\text{-spec}(D)$ by Theorem 5.3.3.3, it follows that $\bar{P} \cap D = P$ and therefore $\bar{P} = \bar{P} \in \bar{v}\text{-max}(\bar{D})$ by the uniqueness of $\bar{P}$.

Assume finally that $I \in I_\bar{v}(D)$, $\bar{I} \in I_\bar{v}(\bar{D})$ and $\bar{I} \cap D = I \subset P$. We must prove that $\bar{I} \subset \bar{P}$, and we may assume that $\bar{I} \neq \emptyset$. Then Theorem 3.2.7.2 implies that $\{P' \in \bar{v}\text{-max}(\bar{D}) \mid \bar{I} \subset P'\} = \{P'_1, \ldots, P'_n\}$ for some $n \in \mathbb{N}$. For $i \in [1, n]$, we set $P_i = P'_i \cap D$, and then we obtain

$$P \supset \bar{I} \cap D = \bar{I} = \bigcap_{P' \in \bar{v}\text{-max}(\bar{D})} I_{P'} \cap D = \bar{I}_{P_1} \cap \ldots \cap \bar{I}_{P_n} \cap D \supset I_{P_1} \cap \ldots \cap I_{P_n} \cap D.$$  

Hence there exists some $i \in [1, n]$ such that $I_{P_i} \cap D \subset P$, and therefore

$$P \supset \sqrt{I_{P_i}} \cap D \supset \sqrt{I_{P_i}} \cap D = \sqrt{I_{P_i}} \cap D = \bigcap_{Q \in \mathcal{P}(I_{P_i})} Q \cap D.$$  

Hence it follows that $Q \cap D \subset P$ for some $Q \in \mathcal{P}(I_{P_i}) \subset v_{P_i}\text{-spec}(D_{P_i})$, and since $Q \cap D \in v\text{-spec}(D)$ and $P \in \mathcal{X}(D)$, we obtain $P = Q \cap D \subset (P_i)_{P_i} \cap D = P_i$. As $P \in v\text{-max}(D)$, we get $P = P_i$, and (by the uniqueness of $\bar{P}$) $\bar{P} = \bar{P} \supset \bar{I}$.  

CASE 2: There is some $M \in S(D)$ such that $P \subsetneq M$.

In this case, $\bar{D}_M = (D_M : M_M) = (M_M : M_M)$ by Theorem 5.3.3, and $P_M \in v_M\text{-spec}(D_M)$. By Theorem 5.1.3, applies to the extension $D_M \subset \bar{D}_M$, there exists a unique $P^* \in \bar{v}_M\text{-spec}(\bar{D}_M)$ such that $P^* \cap D_M = P_M$, and the following assertions hold:

- $(\bar{D}_M)_{P^*} = (D_M)_{P_M}$.
- If $P_M \in v_M\text{-max}(D_M)$, then $P^* \in \bar{v}_M\text{-max}(\bar{D}_M)$.
- If $J \subset D_M$ and $J^* \subset \bar{D}_M$ are ideals such that $J^* \cap D_M = J \subset P_M$, then $J^* \subset P^*$.  


Now we set \( \tilde{P} = P^* \cap \bar{D} \). Then \( \tilde{P} \cap D = P^* \cap D_M \cap D = P_M \cap D = P \), and by the Theorems 2.6.6.2(c) and 1.3.6.2, it follows that \( \tilde{P} \in \bar{v}\text{-spec}(\bar{D}) \) and \( P^* = \tilde{P}_M \).

To prove the uniqueness of \( \tilde{P} \), suppose that \( P' \in \bar{v}\text{-spec}(\bar{D}) \) is such that \( P' \cap D = P \). Then \( P'_M \in \bar{v}_M\text{-spec}(\tilde{D}_M) \) and \( \bar{P}'_M \cap D_M = (P' \cap D)_M = P_M \), hence \( P'_M = P^* \) (by the uniqueness of \( P^* \)), and \( P' = P'_M \cap \bar{D} = P^* \cap \bar{D} = \tilde{P} \).

It remains to prove that \( \tilde{P} \) has the asserted properties. By Theorem 1.3.8 we obtain
\[
D_P = (D_M)_{P_M} = (\tilde{D}_M)_{\tilde{P}_M} = \tilde{D}_{\tilde{P}}.
\]
Hence \( D_P \) is a dv-monoid if and only if \( \tilde{D}_{\tilde{P}} \) is a dv-monoid, and therefore \( P \) is strong if and only if \( \tilde{P} \) is strong. If \( P \in R(D) \subset \bar{v}\text{-max}(\tilde{D}) \), then \( P_M \in \bar{v}_M\text{-max}(D_M) \), hence \( P^* = \tilde{P}_M \in \bar{v}_M\text{-max}(\tilde{D}_M) \), and therefore \( \tilde{P} \in \bar{v}\text{-max}(\bar{D}) \). Since \( P \) is not strong, it follows that \( \tilde{P} \in R(\bar{D}) \). Assume finally that \( I \in \mathcal{I}_v(D) \), \( I \subset I \cap D = I \subset P \). Then \( I_M \in D_M \), \( I_M \subset \tilde{D}_M \) and \( \tilde{I}_M \cap \tilde{D}_M = (\tilde{I} \cap \tilde{D})_M = I_M \subset P_M \).

Hence it follows that \( \tilde{I}_M \subset P^* = \tilde{P}_M \), and \( I \subset \tilde{I}_M \cap \tilde{D} \subset P_M \cap \bar{D} = \tilde{P} \).

**Theorem 5.3.5.** Let \( D \) be a Mori monoid, \( I \in \mathcal{I}_v(D)^* \), and suppose that there is no \( P \in S(D) \) such that \( I \subset P \). Then there exists a unique \( \tilde{I} \in \mathcal{I}_v(\tilde{D}) \) such that \( \tilde{I} \cap D = I \), and there is no \( P^* \in S(\tilde{D}) \) such that \( P^* \supset \tilde{I} \).

**Proof.** By Theorem 3.2.7.2, \( \{ P \in v\text{-max}(D) \mid I \subset P \} = \{ P_1, \ldots, P_n \} \) for some \( n \in \mathbb{N} \). For \( i \in [1, n] \) we have \( P_i \in R(\bar{D}) \), and by Theorem 5.3.4 there exists some \( P_i \in R(\bar{D}) \) such that \( P_i \cap D = P_i \), \( D_{P_i} = \tilde{D}_{P_i} \), and \( I' \in \mathcal{I}_v(\tilde{D}) \) is such that \( I' \cap D = I \), then \( I' \subset P_i \) and \( I'_P = I_P \).

We set \( \tilde{I} = I_{P_1} \cap \ldots \cap I_{P_n} \cap \tilde{D} \). For \( i \in [1, n] \), \( D_{P_i} = \tilde{D}_{P_i} \) is a dv-monoid, hence \( I_{P_i} \cap \bar{D} \cap \tilde{D} \) is a principal ideal, and therefore \( I_{P_i} \cap \tilde{D} \in \bar{v}\text{-spec}(\tilde{D}) \). Hence it follows that \( \tilde{I} \in \mathcal{I}_v(\tilde{D}) \) and \( \tilde{I} \cap D = I \), since \( I = \bigcap_{P \in v\text{-max}(D)} P = I_{P_1} \cap \ldots \cap I_{P_n} \cap D \).

If \( P^* \in \bar{v}\text{-max}(\tilde{D}) \) is such that \( P^* \supset \tilde{I} \), then \( P^* \cap D \in v\text{-spec}(\tilde{D}) \) and \( P^* \cap D \supset \tilde{I} \). Hence there exists some \( i \in [1, n] \) such that \( P^* \cap D \subset P_i \), and as \( P_i \in \mathcal{X}(D) \), we obtain \( P^* \cap D = P_i \) and therefore \( P^* = \tilde{P}_i \in R(\bar{D}) \).

It remains to prove the uniqueness of \( \tilde{I} \). Let \( I' \in \mathcal{I}_v(\tilde{D}) \) be such that \( I' \cap D = I \). Then \( I'_P = I_{P_i} = \tilde{I}_{P_i} \) for all \( i \in [1, n] \), and it suffices to prove that \( \{ \tilde{P}_1, \ldots, \tilde{P}_n \} = \{ P' \in \bar{v}\text{-max}(\tilde{D}) \mid P' \supset I' \} \). Indeed, once this is done, we obtain
\[
I' = \bigcap_{P' \in \bar{v}\text{-max}(\tilde{D})} I'_{P'} = I'_{P_1} \cap \ldots \cap I'_{P_n} \cap \tilde{D} = I_{P_1} \cap \ldots \cap I_{P_n} \cap \tilde{D} = \tilde{I}.
\]
For \( i \in [1, n] \), we have \( \tilde{P}_i = (\tilde{P}_i)_{P_i} \cap \tilde{D} \supset I'_{P_i} \cap \bar{D} \supset I' \). Conversely, assume that \( P' \in \bar{v}\text{-max}(\tilde{D}) \) is such that \( P' \supset I' \). Then \( P' \cap D \in v\text{-spec}(\tilde{D}) \), \( P' \cap D \supset I' \cap D = I \), and therefore there exists some \( i \in [1, n] \) such that \( P' \cap D \subset P_i \). Since \( P_i \in \mathcal{X}(D) \), we obtain \( P' \cap D = P_i \) and \( P' = \tilde{P}_i \). \( \square \)

**5.4. Seminormal Mori monoids**

**Theorem und Definition 5.4.1.**

1. The following assertions are equivalent:
   a) if \( x \in K \) and \( \{ x^2, x^3 \} \subset D \), then \( x \in D \).
   b) if \( x \in K \) and \( x^n \in D \) for all sufficiently large \( n \in \mathbb{N} \), then \( x \in D \).
If $D$ satisfies these conditions, then it is called \textit{seminormal}.

If $D$ is root-closed, then $D$ is seminormal.

2. Let $D$ be seminormal and $T \subset D^\bullet$ a multiplicatively closed subset. Then $T^{-1}D$ is seminormal.

3. Let $(D_\lambda)_{\lambda \in \Lambda}$ be a family of seminormal monoids such that $D_\lambda \subset K$ for all $\lambda \in \Lambda$ and

$$D = \bigcap_{\lambda \in \Lambda} D_\lambda.$$

Then $D$ is seminormal.

4. Let $D$ be seminormal, $x, y \in D^\bullet$ and $k \in \mathbb{N}$ such that $x^k(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$. Then it follows that already $x(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$.

\textbf{Proof.} (a) $\Rightarrow$ (b) Let $x \in K$, and let $m \in \mathbb{N}_0$ be minimal such that $x^m \in D$ for all $n > m$. We must prove that $m = 0$, and we assume to the contrary that $m \geq 1$. Then $x^m \notin D$, and since $3m > 2m > m$, we obtain $\{(x^m)^2, (x^m)^3\} \subset D$, a contradiction.

(b) $\Rightarrow$ (a) If $x \in K$ is such that $\{x^2, x^3\} \subset D$, then $x^k \in D$ for all $k \geq 2$, and thus also $x \in D$.

2. Let $x \in K$ be such that $\{x^2, x^3\} \subset T^{-1}D$. Then there exist $a, b \in D$ and $t \in T$ such that $x^2 = t^{-1}a$ and $x^3 = t^{-1}b$, and therefore $(tx)^2 = ta \in D$ and $(tx)^3 = t^2a \in D$. Since $D$ is seminormal, it follows that $tx \in D$ and $x = t^{-1}(tx) \in T^{-1}D$.

3. Let $x \in K$ be such that $\{x^2, x^3\} \subset D$. For all $\lambda \in \Lambda$, this implies $\{x^2, x^3\} \subset D_\lambda$, hence $x \in D_\lambda$, and therefore we obtain $x \in D$.

4. If $n \in \mathbb{N}$, then it follows that $[x(xy^{-1})^n]^j = x^k(xy^{-1})^{nj} x^{i-k} \in D$ for all $j \geq k$, which implies $x(xy^{-1})^n \in D$. \qed

\textbf{Theorem 5.4.2.} Let $D$ be a seminormal Mori monoid.

1. If $x, y \in D^\bullet$, then $xy^{-1} \in \hat{D}$ if and only if $x(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$.

2. $\hat{D}$ is completely integrally closed.

\textbf{Proof.} 1. By definition, if $x, y \in D^\bullet$ and $x(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$, then $xy^{-1} \in \hat{D}$.

Thus assume that $x, y \in D^\bullet$, $xy^{-1} \in \hat{D}$, and let $c \in D^\bullet$ be such that $c(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, we consider the ideal

$$I_n = \bigcap_{i=0}^{n}((x^{-1}y)^iD \cap D).$$

By definition, $I_n \in I_n(D)$, $I_n \supset I_{n+1}$ and $c \in I_n$ for all $n \in \mathbb{N}$. As $D$ is a Mori monoid, there exists some $k \in \mathbb{N}$ such that $I_k = I_{k+n}$ for all $n \in \mathbb{N}$, and since $y^k = (x^{-1}y)^k x^k \in I_k$, we obtain $y^k \in I_{k+n}$ for all $n \in \mathbb{N}$. Hence for every $n \in \mathbb{N}$ there exists some $b_n \in D$ such that $y^k = (x^{-1}y)^{k+n} b_n$ and therefore $x^k(xy^{-1})^n = x^k b_n \in D$. Consequently, $x(xy^{-1})^n \in D$ for all $n \in \mathbb{N}$ holds by Theorem 5.4.1.4.

2. Suppose that $u = y^{-1}x \in \hat{D}$, where $x, y \in D^\bullet$, and let $d \in D^\bullet$ be such that $du^n = dx^n(y^n)^{-1} \in \hat{D}$ for all $n \in \mathbb{N}$. We may assume that $d \in D^\bullet$. By 1. it follows that $dx^n[dx^n y^{-n}]^m \in D$ for all $m, n \in \mathbb{N}$. For $m \in \mathbb{N}$ and $n \geq m + 1$, this implies that $[dx(y^{-1}x)^m]^n = dx^n(dx^n(y^{-n})^m d^{-m-1} \in D$, hence $dx(y^{-1}x)^m \in D$, since $D$ is seminormal and therefore $u = y^{-1}x \in \hat{D}$. Hence $\hat{D}$ is completely integrally closed. \qed

\textbf{Theorem 5.4.3.} Let $D$ be a seminormal Mori domain.

1. Let $I \subset D$ be a strong ideal and $C = (D : I) = (I : I)$. If $I$ is a radical ideal of $C$, then $C$ is seminormal.

2. $\hat{D}$ is seminormal, and if $P \in S(D)$, then $P\hat{D}_P = P_P$ is a radical ideal of $\hat{D}_P$.\
3. If \( Q \in S(\DD) \), then \( Q \cap D \notin \nu\text{-}\text{max}(D) \). In particular, the assignment \( Q \mapsto Q \cap D \) defines a bijective map

\[
\{Q \in \overline{v}\text{-}\text{spec}(\DD) \mid Q \text{ strong} \} \rightarrow \{P \in \nu\text{-}\text{spec}(D) \mid P \text{ strong}, \ P \notin S(D) \}.
\]

**Proof.** 1. By Theorem 5.1.3.1 \( C \) is a Mori monoid. If \( v^* = v(C) \), then

\[
C = \bigcap_{P \in \nu^*\text{-}\text{max}(C)} C_P,
\]

and therefore it suffices to prove that \( C_P \) is seminormal for all \( P \in \nu^*\text{-}\text{max}(C) \). Suppose that \( P \in \nu^*\text{-}\text{max}(C) \), and consider the following two cases.

**CASE 1:** \( I \not\subset P \). Theorem 5.1.3 implies that \( C_P = D_{D \cap P} \), and the latter monoid is seminormal by Theorem 5.4.1.2.

**CASE 2:** \( I \subset P \). By Theorem 5.1.2 we obtain \( (P : P) = C \), and since \( P \in \nu^*\text{-}\text{max}(C) \), it follows that \( (C : P) \supseteq C \). Hence \( P \) is not strong, and by Theorem 5.1.1.2 \( C_P \) is a dv-monoid. Hence \( C_P \) is root-closed and therefore seminormal.

2. If \( P \in R(D) \), then \( D_P \) is a dv-monoid, hence it is root-closed and therefore seminormal.

Assume now that \( P \in S(D) \). Then \( \DD_P = (P_P : P_P) \) by Theorem 5.3.3, and therefore we get

\[
P \DD_P = P_D \DD_P = P_P(P_P : P_P) = P_P.
\]

We show that \( P_P \) is a radical ideal of \( (P_P : P_P) \). Thus let \( x \in (P_P : P_P) \) be in the radical of \( P_P \). Then \( x^n \in P_P \subset D_P \) for all sufficiently large \( n \in \mathbb{N} \), and as \( D_P \) is seminormal, it follows that \( x \in D_P \). Hence \( x \in P_P \), since \( P_P \subset D_P \) is a prime ideal. By 1. it follows that \( \DD_P \) is seminormal.

Now \( D \) is seminormal, since

\[
\DD = \bigcap_{\nu \in R(D)} D_{\nu} \cap \left( \bigcap_{P \in S(D)} (P_P : P_P) = \bigcap_{P \in R(D)} D_P \cap \left( \bigcap_{P \in S(D)} \DD_P \right) \right.
\]

3. Suppose to the contrary that \( Q \in S(\DD) \) and \( P = Q \cap D \notin \nu\text{-}\text{max}(D) \). Then Theorem 5.3.3 yields \( P \in S(D) \) and \( \DD_P = (D_P : P_P) = (P_P : P_P) \). By 2., \( P_P \) is a radical ideal of \( \DD_P \), and since \( P_P \subset Q_P \), Theorem 5.1.2 implies \( (Q_P : Q_P) = \DD_P \). On the other hand, \( Q_P \) is strong, hence \( (D_P : Q_P) = (Q_P : D_P) = \DD_P \) and \( Q_P = (Q_P)_p = \DD_P \), a contradiction.

In particular, if \( Q \in \overline{v}\text{-}\text{spec}(\DD) \) is strong, then the arguments above together with Theorem 5.3.3 show that \( Q \cap P \in \nu\text{-}\text{spec}(D) \setminus S(D) \) is strong. Conversely, if \( P \in \nu\text{-}\text{spec}(D) \setminus S(D) \), then Theorem 5.4.4 shows that there is a unique strong \( Q \in \overline{v}\text{-}\text{spec}(\DD) \) such that \( Q \cap D = P \).

**Theorem 5.4.4.** Let \( D \) be a seminormal Mori monoid, and let the sequence \( (D_i)_{i \geq 0} \) of Mori monoids be recursively defined by \( D_0 = D \) and \( D_{i+1} = \DD_i \) for all \( i \geq 0 \).

If \( k \in \mathbb{N} \) and \( Q \in S(D_k) \), then there exist strong prime ideals \( P_0, \ldots, P_k \in \nu\text{-}\text{spec}(D) \) such that \( P_0 = Q \cap D \not\subseteq P_1 \not\subseteq \ldots \not\subseteq P_k \).

**Proof.** 1. We use induction on \( k \).

\( k = 1 \): If \( Q \in S(\DD) \), then \( P_0 = Q \cap D \in \nu\text{-}\text{spec}(D) \) is strong and \( P_0 \notin S(D) \) by Theorem 5.4.3.3. Hence there exists some \( P_1 \in \nu\text{-}\text{spec}(D) \) such that \( P_0 \not\subseteq P_1 \), and \( P_1 \) is strong, since \( P_1 \notin S(D) \).

\( k \geq 2, k - 1 \rightarrow k \): Note that \( D_1 = \DD \). By the induction hypothesis, there exist strong prime ideals \( P'_0, \ldots, P'_{k-1} \in \overline{v}\text{-}\text{spec}(\DD) \) such that \( P'_0 = Q \cap D \not\subseteq P'_1 \not\subseteq \ldots \not\subseteq P'_{k-1} \), and we set \( P_i = P'_i \cap D \) for all \( i \in [0, k - 1] \). By Theorem 5.4.3.3 it follows that \( P_0 = Q \cap D \not\subseteq P_1 \not\subseteq \ldots \not\subseteq P_{k-1} \), and \( P_i \in \nu\text{-}\text{spec}(D) \setminus S(D) \) is strong for all \( i \in [0, k - 1] \). Hence there exists some \( P_k \in \nu\text{-}\text{max}(D) \) such that \( P_{k-1} \not\subseteq P_k \), and clearly \( P_k \) is strong.

□
Theorem 5.4.5. Let $D$ be a seminormal Mori monoids, let the sequence $(D_i)_{i \geq 0}$ of Mori monoids be recursively defined by $D_0 = D$ and $D_{i+1} = \tilde{D}_i$ for all $i \geq 0$. Then

$$\tilde{D} = \bigcup_{i \geq 0} D_i$$

is a Krull monoid.

**Proof.** $(D_i)_{i \geq 0}$ is an ascending sequence of Mori monoids. Hence

$$D^* = \bigcup_{i \geq 0} D_i \subseteq K$$

is a monoid. We set $v^* = v(D^*)$, $t^* = t(D^*)$ and $v_i = v(D_i)$, $t_i = t(D_i)$, and we obtain $t_i \leq t_{i+1} \leq t^*$ for all $i \geq 0$. In particular, if $J \in \mathcal{I}_{v^*}(D^*)$ or if $J \in \mathcal{I}_{v_i}(D_{i+1})$, then $J \cap D_i \in \mathcal{I}_{v_i}(D_i)$. It is now sufficient to prove the following three assertions.

I. $D^* \subseteq \tilde{D}$.

II. $D^*$ is a Mori monoid.

III. $S(D^*) = \emptyset$.

Indeed, by II and III it follows that $D^*$ is a Mori monoid satisfying $v^*\text{-max}(D^*) = \mathcal{R}(D^*)$. Hence $D^*_P$ is a dv-monoid for all $P \in v^*\text{-max}(D^*)$, and therefore $D^*$ is a Krull monoid by Theorem 5.2.5. In particular, $D^*$ is completely integrally closed, hence $\tilde{D} \subseteq D^* = D^*$, and therefore $D^* = \tilde{D}$ by I.

I. It clearly suffices to prove that $D_i \subseteq \tilde{D}$ for all $i \geq 0$, and we proceed by induction on $i$. For $i = 0$, there is nothing to do. Thus suppose that $i \geq 0$ and $D_i \subseteq \tilde{D}$. Since $\tilde{D}$ is completely integrally closed by Theorem 5.4.2, Theorem 5.3.3 implies that there is a chain $I_n,i \subseteq D_{i+1}$ such that $I_n,i \subseteq P_i$, and therefore $I_n,i \subseteq \tilde{D}_i$ for all $i \geq 0$. Then it follows that $I_{n,i} \cap D = I$ for all $n,i \geq 0$.

Let $k \in \mathbb{N}$ be such that there is no chain $I \subseteq P_0 \subseteq P_1 \subseteq \ldots \subseteq P_k$, where $P_0, \ldots, P_k \in v\text{-spec}(D)$, and suppose that there is some $i \geq k$ such that $n_i+1 = n_i$ for all $i \geq k$. Indeed, then it follows that $I = I_{0,0} \neq \{0\}$ and $n_0 = 0$. Then it follows that $I_{n,i} \cap D = I$ for all $n,i \geq 0$.

II. Let $(I_n)_{n \geq 0}$ be an ascending chain in $\mathcal{I}_{v^*}(D^*)$. For $i, n \geq 0$, we set $I_{n,i} = I_n \cap D_i$. For every $i \geq 0$, $(I_{n,i})_{n \geq 0}$ is an ascending sequence in $\mathcal{I}_{v_i}(D_i)$, and it terminates since $D_i$ is a Mori domain. Let $n_i \geq 0$ be minimal such that $I_{n,i} = I_{n+1,i}$ for all $n \geq n_i$. Then the sequence $(n_t)_{t \geq 0}$ is monotonically increasing, and since

$$I_n = \bigcup_{i \geq 0} I_{n,i} \text{ for all } n \geq 0,$$

it suffices to prove that there exists some $k \geq 0$ such that $n_{i+1} = n_i$ for all $i \geq k$. Indeed, then it follows that $I_k = I_{0,0} \neq \{0\}$ and $n_0 = 0$. Then it follows that $I_{n,i} \cap D = I$ for all $n,i \geq 0$.

III. Assume to the contrary that there is some $P^* \in S(D^*)$. For $i \geq 0$, set $P_i = P^* \cap D_i \in v_i\text{-spec}(D_i)$. Then $(D_i)_{P_i} \subseteq D^*_P \subseteq K$, $P^*_P \neq 0$, and $D^*_P$ is not a dv-monoid. Hence $(D_i)_{P_i} \neq 0$ is a dv-monoid, and therefore $P_i$ is strong. If $Q_i \in v_i\text{-max}(D_i)$ is such that $P_i \subset Q_i$, then $Q_i \in S(D_i)$, $P_i \subset Q_i \cap D$, and Theorem 5.4.4 implies that there is a chain $P_i \subset P_1 \subset \ldots \subset P_t$ in $v\text{-spec}(D)$. As $i \geq 0$ is arbitrary, this contradicts Theorem 3.2.7.2. \qed
CHAPTER 6

Ideal theory of polynomial rings

6.1. The content and the Dedekind-Mertens Lemma

Throughout this Section, let $D$ be a ring, $D[X]$ a polynomial ring, $d = d(D)$ and $v = v(D)$.

**Definition 6.1.1.** Let $R \supset D$ be an overring. For $D$-submodules $M, N \subset R$ we write (as usual in ring theory) $MN$ instead of $D(MN)$.

For a polynomial $g = b_0 + b_1X + \ldots + b_mX^m \in R[X]$, the $D$-module

$$c_D(g) = D(b_0, \ldots, b_m) = \sum_{j=0}^m Db_j \subset R$$

is called the $D$-content of $g$. If $J \subset R$ is a $D$-submodule, then $g \in J[X]$ if and only if $c_D(g) \subset J$.

Obviously, $c_D(af) = ac_D(f)$ and $c_D(fg) \subset c_D(f)c_D(g)$ for all $a \in R$ and $f, g \in R[X]$, but equality need not hold [indeed, if $D = R = \mathbb{Z}[2i]$ and $f = 2i + 2X$, then $f^2 = -4 + 8iX + 4X^2$, hence $c(f) = (2i, 2)$, $c(f^2) = (4)$, and $c(f)^2 = (4, 4i) \neq c(f^2)$].

The **Dedekind-Mertens number** of a non-zero polynomial $g \in R[X]$ with respect to $D$ is defined by

$$\mu_D(g) = \inf \{ k \in \mathbb{N} \mid c_D(f)^k c_D(g) = c_D(f)^{k-1} c_D(fg) \text{ for all } f \in R[X] \} \in \mathbb{N} \cup \{\infty\}.$$ 

If $f, g \in R[X]$, then $c_D(fg) \leq c_D(f)c_D(g)$ implies $c_D(f)^{k-1}c_D(fg) \leq c_D(f)^{k}c_D(g)$ for all $k \in \mathbb{N}$, and therefore

$$\mu_D(g) = \inf \{ k \in \mathbb{N} \mid c_D(f)^k c_D(g) \leq c_D(f)^{k-1} c_D(fg) \text{ for all } f \in R[X] \} \in \mathbb{N} \cup \{\infty\}.$$ 

We shall see in Theorem 6.1.2 that $\mu_D(g)$ only depends on the $D$-module $c_D(g)$ and not on the embedding ring $R$.

The classical **Dedekind-Mertens Lemma** asserts that $\mu_D(g) \leq \deg_D(g) + 1$ for all $g \in D[X]^*$. We shall prove a more general statement in Theorem 6.1.2.

**Theorem 6.1.2.** Let $R \supset D$ be an overring, $g \in R[X]$ and $\delta(g)$ the number of non-zero coefficients of $g$. For $M \in \max(D)$, we denote by $\rho_M(g)$ the minimal number of generators of the $D_M$-module $c_D(g)_M$; that is, $\rho_M(g) = \dim_{D/M}(c_D(g)_M/Mc_D(g)_M)$. Then

$$\mu_D(g) \leq \max\{ \rho_M(g) \mid M \in \max(D) \} \leq \delta(g) \leq \deg(g) + 1.$$ 

For the proof we need the following variant of Nakayama’s Lemma.

**Lemma 6.1.3.** Let $D$ be local with maximal ideal $M$.

1. Let $A, B$ be $D$-modules such that $A \subset B$ and $B/A$ is finitely generated. If $B = A + MB$, then $B = A$.

2. Let $L$ be a $D$-module and $A, B \subset L$ submodules. If $A$ is finitely generated and $A \subset B + MA$, then $A \subset B$. 

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Then it suffices to prove that

By Lemma 6.1.3 we obtain

for all $f$ and by 1. we obtain $A+B = B + M(A+B)$.

Proof of Theorem 6.1.2. For $f \in R[X]$, we set $C_f = c_D(f)$. If $f, g \in R[X]$, then we obviously have $C_f C_g \subset C_{fg}$ and therefore $C_f^k C_g \subset C_{fg}^{k-1}$ for all $k \in \mathbb{N}$.

It suffices to prove the result if $D$ is local with maximal ideal $M$. Indeed, suppose that this is done. Let $g \in R[X]$ and $k \in \mathbb{N}$ be such that $k \geq \rho_M(g)$ for all $M \in \text{max}(D)$. We must prove that $C_f^k C_g = C_{fg}^{k-1}$ for all $f \in R[X]$. For $f \in R[X]$ and $M \in \text{max}(D)$, let $f_M \in R_M[X]$ be the image of $f$ in $R_M[X]$. Then $c_{D_M}(f_M) = (C_f)_M$, and the local result implies $c_{D_M}(f_M)^k c_{D_M}(g_M) = c_{D_M}(f_M)^{k-1} c_{D_M}(f_M g_M)$, that is, $(C_f^k C_g)_M = C_{f_M}^k C_{g_M} = C_{f_M}^{k-1} C_{f_M g_M} = (C_{fg})_M$. Since this holds for all $M \in \text{max}(D)$, the assertion follows.

Assume now that $D$ is local, $M = D \setminus D^\times$, $R \supset D$ is an overring, and for $g \in R[X]$, we set $\rho(g) = \rho_M(g)$. We prove first:

A. If $g, g_1 \in R[X]$ and $C_{g-g_1} \subset MC_g$, then $C_g = C_{g_1}$ and $\mu_D(g) = \mu_D(g_1)$.

Proof of A. Since $g = g_1 + (g-g_1)$, we obtain $C_g \subset C_{g_1} + C_{g-g_1} \subset C_{g_1} + MC_{g_1}$ and therefore $C_g \subset C_{g_1}$ by Lemma 6.1.3. But $C_{g-g_1} = C_{g_1-g} \subset MC_{g_1} \subset MC_g$, hence we obtain also $C_{g_1} \subset C_g$ and therefore $C_g = C_{g_1}$.

By symmetry, it is now sufficient to prove that $\mu_D(g) \leq \mu_D(g_1)$, and for this we may assume that $k = \mu_D(g_1) < \infty$. If $f \in R[X]$, then

$$C_f^k C_g = C_f^k C_{g_1} = C_f^{k-1} C_{fg_1} = C_f^{k-1} C_{f_{g_1}} \subset C_f^{k-1} (C_{fg_1} + C_{f_{g_1}})$$

$$\subset C_f^{k-1} (C_{fg_1} + C_{f_{g_1}}) = C_f^{k-1} C_{fg_1} + MC_f C_g.$$  

By Lemma 6.1.3 we obtain $C_f^k C_g \subset C_{fg}^{k-1}$. □ [A.]

We prove Theorem 6.1.2 by induction on $\rho(g)$. If $g = 0$, then $\mu_D(g) = 0$. Thus we may assume that

$$g = \sum_{j=0}^m b_j X^j, \quad \text{where} \quad m \in \mathbb{N}_0, \ b_0, \ldots, b_m \in R \quad \text{and} \quad b_m \neq 0.$$  

$\rho(g) = 1$: Then $C_g = D b$ for some $b \in R$. For $j \in [0, m]$, there exists some $d_j \in D$ such that $b_j = d_j b$, and we assert that there is some $l \in [0, m]$ such that $d_j \notin M$ (indeed, otherwise we have $C_g \subset MC_g$ and consequently $C_g = 0$ by Lemma 6.1.3). Let $l \in [0, m]$ be such that $d_l \notin M$ and $d_j \in M$ for all $j \in [0, l-1]$. We must prove that $C_f C_g \subset C_{fg}$ for all $f \in R[X]$. Thus suppose that

$$f = \sum_{i=0}^n a_i X^i, \quad \text{where} \quad n \in \mathbb{N}_0, \ a_0, \ldots, a_n \in R \quad \text{and} \quad c_k = \sum_{i=0}^k a_{k-i} d_i b.$$  

Then $C_f C_g = C_f b = D(a_0 b, \ldots, a_n b)$. If $a_i = 0$ for all $i > n$ and $b_j = 0$ for all $j > m$, then

$$f_{g} = \sum_{k=0}^{m+n} c_k X^k, \quad \text{where} \quad c_k = \sum_{i=0}^k a_{k-i} d_i b \quad \text{for all} \quad k \in [0, m+n].$$  

It suffices to prove that $a_i b \in C_{f_{g}} + MC_f C_g$ for all $i \in [0, n]$. Indeed, once this is done, it follows that $C_f C_g \subset C_{f_{g}} + MC_f C_g$ and therefore $C_f C_g \subset C_{f_{g}}$ by Lemma 6.1.3.

We proceed by induction on $i$. Let $i \in [0, n]$ and suppose that $a_{\nu} b \in C_{f_{g}} + MC_f C_g$ for all $\nu \in [0, i-1]$. Then

$$c_{i+l} = a_i d_i b + \sum_{\nu=0}^{l-1} a_{i+l-\nu} d_{\nu} b + \sum_{\nu=l+1}^{l+i} a_{i+l-\nu} d_{\nu} b \in C_{f_{g}}.$$  

□
If \( \nu \in [0, l - 1] \), then \( d_\nu \in M \) and \( a_{i+l-\nu}d_\nu b \in MC_fC_g \). If \( \nu \in [l + 1, l + i] \), then \( i + l - \nu \in [0, i - 1] \) and \( a_{i+l-\nu}d_\nu b \in D(C_{fg} + MC_fC_g) = C_{fg} + MC_fC_g \) by the induction hypothesis. Hence it follows that \( a_i d_\nu b \in C_{fg} + MC_fC_g \), and since \( d_\nu \in D \setminus M = D^\times \), we obtain \( a_i b \in C_{fg} + MC_fC_g \).

\[ \rho(g) = k \geq 2, \ k - 1 \rightarrow k: \]

If
\[
g_1 = \sum_{j=0}^{m} b_j X^j, \quad \text{then} \quad C_{g-g_1} \subset MC_g, \quad \text{hence} \quad C_g = C_{g_1} \quad \text{and} \quad \mu_D(g) = \mu_D(g_1).
\]

Therefore we may assume that \( g = g_1 \). Since \( b_m \notin MC_g \), there exists a subset \( L \subset [0, m - 1] \) such that \( |L| = k - 1 \) and \( \{b_m\} \cup \{b_{\mu} \mid \mu \in L\} \) is a minimal generating set of \( C_g \). Then \( C_g = D b + E \), where \( E = D(\{b_{\mu} \mid \mu \in L\}) \), and for every \( j \in [0, m] \), there is a representation

\[
b_j = \lambda_j b_m + b'_j, \quad \text{where} \quad b'_j = \sum_{\mu \in L} \lambda_{j,\mu} b_{\mu} \in E,
\]

such that \( \lambda_j, \lambda_{j,\mu} \in D \) for all \( j \in [0, m] \) and \( \mu \in L \), \( \lambda_n = 1 \) and \( \lambda_{m,\mu} = 0 \) for all \( \mu \in L \), and if \( j \in L \), then \( \lambda_{j,j} = 1 \) and \( \lambda_j = \lambda_{j,\mu} = 0 \) for all \( \nu \in L \setminus \{j\} \). We set

\[
g_0 = \sum_{j=0}^{m} d_j b X^j = b_m X^m + \ldots \quad \text{and} \quad g_1 = \sum_{j=0}^{m-1} b'_j X^j.
\]

Then \( g = g_0 + g_1 \), \( C_{g_0} = b_m D \), \( C_{g_1} = E \), \( \rho(g_0) = 1 \), and \( \rho(g_1) = k - 1 \). By the induction hypothesis and since \( \rho(g_0) = 1 \), we have \( C_{f}^{k-1} C_{g_1} = C_{f}^{k-2} C_{f,g_1} \) and \( C_{f,g_0} = C_f C_{g_0} = b_m C_f \) for all \( f \in R[X] \), and we must prove that \( C_{f}^{k} C_g \subset C_{f}^{k-1} C_{f,g} \) for all \( f \in R[X] \). We proceed by induction on \( \deg(f) \). We may assume that \( f \neq 0 \),

\[
f = \sum_{i=0}^{n} a_i X^i = a_n X^n + f_1, \quad \text{where} \quad a_0, \ldots, a_n \in R, \ a_n \neq 0 \quad \text{and} \quad C_{f_1}^{k} C_g \subset C_{f_1}^{k-1} C_{f_1,g}.
\]

Then it follows that \( a_n b_m \in C_{f,g} \). We use the induction hypothesis to prove the following assertion.

**B.** \( C_{f,g_1} \subset C_{f,g} + b_m C_{f_1} \) and \( C_{f,g_1} \subset C_{f,g} + a_n C_{g_1} \)

**Proof of B.** Since \( C_{f,g_0} = C_{a_n X^n g_0 + f_1 g_0} \subset C_{a_n X^n g_0} + C_{f_1 g_0} \subset a_n b_m D + C_{f_1} b_m \subset C_{f,g} + C_{f_1,b_m} \), we obtain \( C_{f,g_1} = C_{f(a_n g_0) g_1} \subset C_{f,g} + C_{f_1 g} + b_m C_{f_1} \).

In the same way, \( C_{a_n X^n g} = C_{X^n (a_n g_0 + a_n g_1)} \subset C_{a_n g_0} + C_{a_n g_1} = a_n b_m D + a_n C_{g_1} \subset C_{f,g} + C_{f,g_1} \) and therefore \( C_{f,g_1} \subset C_f C_{a_n X^n g} \subset C_{f_1} C_{a_n X^n g} \subset C_f C_{a_n g} + a_n C_{g_1} \). \( \square \)

\( C_{f}^{k} C_g \) is the \( D \)-module generated by the set \( A \) of all elements \( \alpha = a_0^v \ldots a_{n-1}^v b_j \in R \), where \( v_0, \ldots, v_{n-1}, v \in \mathbb{N}_0 \), \( v_0 + \ldots + v_{n-1} + v = k \) and \( j \in [0, m] \).

- If \( v \neq 0 \) and \( j \in J \), then \( \alpha = a_0^v \ldots a_{n-1}^v a^{v-1} abd_j \in C_{f}^{k-1} C_{f,g} \).
- If \( v \neq 0 \) and \( j \notin J \), then \( \alpha = a_0^v \ldots a_{n-1}^v a^{v-1} ab \in C_{f}^{k-1} C_{g_1} \).
- If \( v = 0 \), then \( \alpha = a_0^v \ldots a_{n-1}^v b_j \in C_{f_1} C_g \subset C_{f_1}^{k-1} C_{f_1,g} \subset C_{f_1}^{k-1} C_{f_1,g} \subset C_{f_1}^{k-1} (C_{f_1} + a_n C_{g_1}) \) (by the induction hypothesis, B, and since \( C_{f_1} \subset C_{f} \)).

Putting the three cases together, we get

\[
C_{f}^{k} C_{f,g} \subset C_{f}^{k-1} C_{f,g} + C_{f}^{k-1} a_n C_{g_1} + C_{f}^{k-1} (C_{f,g} + a_n C_{g_1}) = C_{f}^{k-1} C_{f,g} + C_{f}^{k-1} a_n C_{g_1}.
\]

Using B and the induction hypothesis, it follows that

\[
C_{f}^{k-1} a_n C_{g_1} \subset a_n C_{f}^{k-2} C_{f,g} \subset a_n C_{f}^{k-2} (C_{f,g} + b_m C_f) \subset a_n C_{f}^{k-2} C_{f,g} + C_{f}^{k-1} a_n b_m \subset C_{f}^{k-1} C_{f,g},
\]

which completes the proof. \( \square \)
Corollary 6.1.4. Let $R \supset D$ be an overring.

1. For every $g \in R[X]$ there exists some $m \in \mathbb{N}$ such that $c_D(f)^m c_D(g) = c_D(f)^{m-1} c_D(fg)$ for all $f \in R[X]$.

2. Let $f, g \in R[X]$, and suppose that $c_D(f)$ is a finitely cancellative $D$-submodule of $R$ (that means, $c_D(f)M = c_D(f)N$ implies $M = N$ for all finitely generated $D$-submodules $M, N \subset R$). Then $c_D(fg) = c_D(f)c_D(g)$.

3. Let $D$ be a domain, $K = q(D)$ and $r$ a module system on $K$ such that $r \geq d$. If $f \in K[X]$ and $c_D(f)_r$ is $r$-finitely $r$-cancellative, then $c_D(fg)_r = [c_D(f)c_D(g)]_r$ for all $g \in K[X]$.

Proof. Obvious by Theorem 6.1.2. □

Theorem 6.1.5. Let $D$ be a domain and $K = q(D)$. Then the following assertions are equivalent:

(a) $D$ is integrally closed.

(b) For all $f, g \in K[X]$ we have $c_D(fg)_r = [c_D(f)c_D(g)]_r$. 

(c) For all $f, g \in K[X]$ we have $c_D(f)c_D(g) \subset c_D(fg)_r$.

(d) For all $f \in K[X]$ we have $fK[X] \cap D[X] = f[c_D(f)]^{-1}[X]$.

Proof. (a) $\Rightarrow$ (b) Since $D$ is integrally closed, we have $D_u = D$, and therefore $d_u$ is a finitely cancellative ideal system on $D$. Hence $c_D(fg)d_u = [c_D(f)c_D(g)]d_u$, and since $d_u \leq v$, the assertion follows.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (d) Let $f \in K[X]$. We must prove that, for all $g \in K[X]$, we have $fg \in D[X]$ if and only if $g \in c_D(f)^{-1}[X]$. If $g \in c_D(f)[X]$ and $fg \in D[X]$, then $c_D(f)c_D(g) \subset c_D(fg)_r \subset D$, hence $c_D(g) \subset c_D(f)^{-1}$ and therefore $g \in c_D(f)^{-1}[X]$. Conversely, if $g \in c_D(f)^{-1}[X]$, then $c_D(g) \subset c_D(f)^{-1}$ and therefore $c_D(fg) \subset c_D(f)c_D(g) \subset D$, which implies $fg \in D[X]$.

(d) $\Rightarrow$ (a) Let $u \in K$ be integral over $D$, and let $g \in D[X]$ be a monic polynomial such that $g(u) = 0$. Then $g = (X - u)h$, where $h \in K[X]$, and therefore $g \in (X - u)K[X] \cap D[X] = (X - u)\{1, u\}^{-1}[X]$. Hence $h \in \{1, u\}^{-1}[X]$, which implies that $uh \in D[X]$ and thus $u \in D$, since $h$ is monic. □

Theorem 6.1.6. Let $D$ be a domain and $K = q(D)$. Then the following assertions are equivalent:

(a) $D$ is local and integrally closed.

(b) If $f \in D[X]$, $u \in K^\times$, $f(u) = 0$ and $c_D(f)$ is invertible, then $u \in D$ or $u^{-1} \in D$.

(c) If $f \in D[X]$ be such that some coefficient of $f$ lies in $D^\times$ and $u \in K^\times$ is such that $f(u) = 0$, then $u \in D$ or $u^{-1} \in D$.

Proof. (a) $\Rightarrow$ (b) Let $f \in D[X]$ and $u = b^{-1}a \in K^\times$, where $a, b \in D^\times$, be such that $f(u) = 0$ and $c_D(f)$ is invertible. Then $f = (bX - a)h$ for some $h \in K[X]$, and

$$c_D(f) = c_D(bX - a)c_D(h) = (a, b)c_D(h) \supset c_D(f).$$

Hence $c_D(f) = (a, b)c_D(h)$, and therefore $(a, b)$ is invertible. Since $D$ is local, Theorem 6.1.4 implies $(a, b) = (b, a)$, and therefore $u \in D$ or $u^{-1} \in D$.

(b) $\Rightarrow$ (c) Let $f \in D[X]$ and some coefficient of $f$ lies in $D^\times$, then $c_D(f) = D$.

(c) $\Rightarrow$ (a) Let $u \in K^\times$ be integral over $D$, and let $f = X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0 \in D[X]$ be a monic polynomial of minimal degree such that $f(u) = 0$. If $u \notin D$, then $n \geq 2$, $u^{-1} \in D$ and $u^{n-1} + a_{n-1}u^{n-2} + \ldots + (a_1 + a_0u^{-1}) = 0$, which contradicts the minimality of $n$. Hence $D$ is integrally closed.

In order to prove that $D$ is local, we take some $M = \max(D)$ and prove that $D \setminus M \subset D^\times$. If $u \in D \setminus M$, then $M + Du = D$, and there exist elements $a \in M$ and $b \in D^\times$ such that $a + bu = 1$. If $a = 0$, then $u \in D^\times$ and we are done. Thus suppose that $a \neq 0$. Then $u^{-1}a$ is a zero of the polynomial
For \( uX - a(X - b) = uX^2 - X + ab \in D[X] \), and therefore \( u^{-1}a \in D \) or \( a^{-1}u \in D \). If \( a^{-1}u \in D \), then \( u \in aD \subset M \), a contradiction. If \( u^{-1}a \in D \), then \( a = ud \) for some \( d \in D \), hence \( 1 = u(d + b) \) and \( u \in D^\times \).

6.2. Nagata rings

**Remarks and Definition 6.2.1.** Let \( D \) be a ring and \( K = q(D) \) its total quotient ring.

1. We denote by \( \mathcal{F}(D) = \{ c^{-1}J \mid c \in D^*, \ J \subset D \} \) the set of all fractional ideals of \( D \). If \( I, J \in \mathcal{F}(D) \) and \( a \in D \), then \( aI, I + J, IJ \in \mathcal{F}(D) \). For \( I \in \mathcal{F}(D) \), we define
   \[
   I[X] = \left\{ \sum_{i=0}^{n} a_i X^i \mid a_0, \ldots, a_n \in I \right\} \subset K[X].
   \]

2. Let \( R \supset D \) be an overring such that \( R^* \subset D^* \), and assume that \( q(D) \subset q(R) \). For \( I \in \mathcal{F}(D) \), we denote by
   \[
   IR = R I = \{ x_1a_1 + \ldots + x_na_n \mid n \in \mathbb{N}, \ x_1, \ldots, x_n \in I, \ a_1, \ldots, a_n \in R \} \in \mathcal{F}(R)
   \]
   the the \( R \)-submodule of \( q(R) \) generated by \( I \). If \( I, J \in \mathcal{F}(D) \), then \( (IJ)R = (IR)(JR) \), and if \( I = D(a_1, \ldots, a_n) = D a_1 + \ldots + D a_n \), then \( IR = R(a_1, \ldots, a_n) = R a_1 + \ldots + R a_n \).

3. For a \( D[X] \)-submodule \( J \subset K[X] \), we call
   \[
   c_D(J) = \sum_{f \in J} c_D(f) \subset K
   \]
   the content of \( J \). By definition, \( c_D(J) \subset K \) is a \( D \)-submodule.

4. Let \( I \triangleleft D \) be an ideal. We identify the rings \( D[X]/I[X] \) and \((D/I)[X]\) by means of the canonical isomorphism. Explicitly, we set
   \[
   \sum_{i \geq 0} a_i X^i + I[X] = \sum_{i \geq 0} (a_i + I) X^i \quad \text{for every polynomial} \quad f = \sum_{i \geq 0} a_i X^i \in D[X].
   \]

For a multiplicatively closed subset \( T \subset D^* \), we identify the rings \( (T^{-1}D)[X] \) and \( T^{-1}D[X] \) by means of the canonical isomorphism. Explicitly, we set
   \[
   \sum_{i \geq 0} \frac{a_i}{t} X^i = \sum_{i \geq 0} a_i X^i / t \quad \text{for every polynomial} \quad f = \sum_{i \geq 0} a_i X^i \in D[X] \quad \text{and} \quad t \in T.
   \]

**Theorem 6.2.2.** Let \( D \) be a ring, \( K = q(D) \) and \( I, J \in \mathcal{F}(D) \).

1. \( ID[X] = I[X] = \{ f \in K[X] \mid c_D(f) \subset I \} \in \mathcal{F}(D[X]) \), \( c_D(I[X]) \subset I \), \( I[X] \cap K = I \), and \( (IJ)[X] = I[X]J[X] \).

2. \( I \) is finitely generated [principal] if and only if \( I[X] \) is finitely generated [principal]. More precisely, if \( I = D(a_1, \ldots, a_n) \) for some \( a_1, \ldots, a_n \in K \), then \( I[X] = D[X](a_1, \ldots, a_n) \), and if \( I[X] = D[X] f_1, \ldots, f_n \) for some \( f_1, \ldots, f_n \in K[X] \), then \( I = D(f_1, 0, \ldots, f_n(0)) \).

3. Let \( D \) be a domain and \( J^* \neq 0 \). Then \( (I : J)[X] = (I[X] : J[X]) \). In particular (for \( I = D \), \( J^{-1}[X] = J[X]^{-1} \)).

**Proof.** 1. By definition, \( I[X] = \{ f \in K[X] \mid c_D(f) \subset I \} \subset ID[X] \), \( c_D(I[X]) \subset I \), and \( I \subset I[X] \) implies \( I = c_D(I) \subset c_D(I[X]) \). Therefore \( I[X] \cap K = \{ a \in K \mid c_D(a) = aD \subset I \} = I \).

If \( f \in ID[X] \), then \( f = a_1 f_1 + \ldots + a_n f_n \), where \( n \in \mathbb{N}, \ a_1, \ldots, a_n \in I \) and \( f_1, \ldots, f_n \in D[X] \). For \( i \in [1, n] \), we have \( c_D(a_i f_i) = a_i c_D(f_i) \subset a_i D \subset I \), hence \( a_i f_i \in I[X] \) and \( f \in I[X] \). Consequently, \( (IJ)[X] = (IJ)D[X] = (ID[X])(JD[X]) = I[X]J[X] \).
2. Obviously, $I = D(a_1, \ldots , a_n)$ implies $I[X] = D[X](a_1, \ldots , a_n)$. Thus let $f_1, \ldots , f_n \in K[X]$ be such that $I[X] = D[X](f_1, \ldots , f_n)$. For all $i \in [1, n]$, $f_i \in I[X]$ implies $f_i(0) \in I$, and therefore $D(f_i(0), \ldots , f_n(0)) \subset I$. If $a \in I \subset I[X]$, then $a = f_1 g_1 + \ldots + f_n g_n$, for some $g_1, \ldots , g_n \in D[X]$, and therefore $a = (f_1 g_1 + \ldots + f_n g_n)(0) = f_1(0)g_1(0) + \ldots + f_n(0)g_n(0) \in D(f_1(0), \ldots , f_n(0))$.

3. Since $(I : J)[X]/J[X] = ((I : J)J[X] \subset I[X])$, we obtain $(I : J)[X] \subset (I[X] : J[X])$. Suppose now that $c \in I^*$ and $F \in (I[X] : J[X]) \subset K[X]$. Then $Fc \in I[X] \subset K[X]$ and therefore $F \in K[X]$. If $b \in J$, then $bF \in I[X]$ implies $I \supset c_D(bF) = bc_D(F)$, hence $Jc_D(F) \subset I$, $c_D(F) \subset (I : J)$ and consequently $F \in (I : J)[X]$.

**Theorem and Definition 6.2.3.** Let $D$ be a ring, $K = q(D[X])$ the total quotient ring of the polynomial ring $D[X]$ and $N = \{ f \in D[X] \mid c_D(f) = D \}$.

1. $N \subset D[X]^*$ is a multiplicatively closed subset.

   The ring $D(X) = N^{-1}D[X] \subset K$ is called the Nagata ring of $D$. If $D$ is a field, then $N = D[X]^*$, and $D(X)$ is just the field of rational functions (thus the terminology is consistent).

2. Let $J \subset D$ be an ideal, and let $\pi : D[X] \rightarrow D/J[X]$ be the canonical epimorphism. Then $JD(X) = N^{-1}J[X] \subset D(X)$, $JD(X) \cap D = J[X] \cap D = J$, and there is an isomorphism

$$\Phi : D(X)/JD(X) \rightarrow (D/J)(X), \quad \text{given by} \quad \Phi(f + JD(X)) = \frac{\pi(f)}{\pi(g)}.$$

3. If $P \in \text{spec}(D)$, then $P[X] \in \text{spec}(D[X])$, $PD(X) \in \text{spec}(D[X])$, and the natural embedding $j_P : D[X] \rightarrow PD[X] = (D \supset J)[X]$ induces an isomorphism $\iota_P : D[X]_{P[X]} \cong D_P(X)$.

4. max $D(X) = \{ PD(X) \mid P \in \text{max}(D) \}$.

**Proof.** 1. If $f \in N$ and $g \in D[X]^*$, then $c_D(fg) = c_D(f)c_D(g) = c_D(g) \neq 0$ by Corollary 6.1.4. Hence $fg \neq 0$, which implies $f \in N$. If $f, g \in N$, then $c_D(fg) = c_D(f)c_D(g) = D$, hence $fg \in N$, and $N$ is multiplicatively closed.

2. Clearly, $JD(X) = JN^{-1}D[X] = N^{-1}JD[X] = N^{-1}J[X] \subset D(X)$. If $a \in JD(X) \cap D$, then there is some $f \in N$ such that $af \in J[X]$, and therefore $c_D(af) = ac_D(f) = aD \subset J$, which implies $a \in J$. Hence $JD(X) \cap D \subset J[X] \subset J \subset JD(X) \cap D \subset JD(X) \cap D$, and thus equality holds.

There is an isomorphism

$$\Phi : D(X)/JD(X) = N^{-1}D[X]/N^{-1}J[X] \cong N^{-1}(D[X]/J[X]) = N^{-1}(D/J)[X] = \pi(N)^{-1}(D/J)[X],$$

given by

$$\Phi\left(\frac{f}{g} + JD(X)\right) = \frac{\pi(f)}{\pi(g)} \quad \text{for all} \quad f \in D[X] \quad \text{and} \quad g \in N.$$

Therefore it suffices to prove that $\pi(N) = \{ \pi(f) \mid f \in D[X], c_D(\pi(f)) = D/J \}$. If $f \in D[X]$, then $c_D/\pi(f) = c_D(f) + J$, and therefore $f \in N$ implies $c_D(\pi(f)) = D/J$. To prove the converse, let $f \in D[X]$ be such that $c_D/\pi(f) = D/J$. Then $c_D(f) + J = D$, and there exists some $u \in J$ such that $c_D(f) + uD = D$. If $n \in \mathbb{N}$ and $n > \text{deg}(f)$, then $c_D(f + uX^n) = c_D(f) + uD = D$, hence $f + uX^n \in N$ and $\pi(f) = \pi(f + uX^n) \in \pi(N)$.

3. Let $P \in \text{spec}(D)$. Then $D[X]/P[X] = (D/P)[X]$ is a domain. Hence $P[X] \in \text{spec}(D[X])$, and since $P[X] \cap N = \emptyset$, it follows that $PD(X) = N^{-1}P[X] \in \text{spec}(D(X))$. By definition

$$D_{P}(X) = N_{P}^{-1}D_P[X], \quad \text{where} \quad N_P = \{ F \in D_P[X] \mid c_{D_P}(F) = D_P \}.$$

If $f \in D[X] \setminus P[X]$, then $c_{D}(f) \not\in P$, hence $c_{D}(j_P(f)) = c_{D}(f)P = D_P$ and therefore $j_P(f) \in N_P$. Hence it follows that $j_P(D[X] \setminus P[X]) \subset N_P$, and therefore $j_P$ induces a ring homomorphism

$$\iota_P : D[X]_{P[X]} \rightarrow D_P(X), \quad \text{given by} \quad \iota_P\left(\frac{g}{f}\right) = \frac{g/1}{f/1} \quad \text{for all} \quad g \in D[X] \quad \text{and} \quad f \in D[X] \setminus P[X].$$
\( \iota_P \) is surjective: If \( z \in D_P(X) \), then there exist \( g \in D[X], \ f \in D[X] \setminus P[X] \) and \( s, t \in D \setminus P \) such that
\[
 z = \frac{g/s}{f/t} = \frac{tg/s}{sf/t} = \iota_P \left( \frac{tg}{sf} \right) \quad (\text{note that } sf \in D[X] \setminus P[X]).
\]

\( \iota_P \) is injective: If \( z \in \Ker(\iota_P) \subset D[X]_{P[X]} \), then
\[
 z = \frac{g}{f}, \quad \text{where } g \in D[X], \ f \in D[X] \setminus P[X] \quad \text{and} \quad \frac{g}{f} = 0 \in D_P(X), \quad \text{hence} \quad \frac{g}{f} = 0 \in D_P(X).
\]

Therefore there exists some \( s \in D \setminus P \) such that \( sg = 0 \), and as \( s \in D[X] \setminus P[X] \), this implies \( z = 0 \).

4. If \( P \in \max(D) \), then \( D(X)/PD(X) \cong (P/D)(X) \) is a field, and therefore \( PD(X) \in \max D(X) \).

Thus assume that \( M \in \max D(X) \). Then \( M = N^{-1}Q \), where \( Q \in \Spec D[X] \) is maximal such that \( Q \cap N = \emptyset \). It is now sufficient to prove that
\[
 J = \sum_{f \in Q} c_D(f) \neq D.
\]

Indeed, then there exists some \( P \in \max(D) \) such that \( J \subset P \), hence \( Q \subset P[X] \), and it follows that \( M = N^{-1}Q \subset N^{-1}P[X] = PD(X) \), and therefore \( M = PD(X) \).

Assume to the contrary that \( J = D \). Then there exist \( f_1, \ldots, f_m \in Q \) such that \( 1 = c(f_1) + \ldots + c(f_m) \).

Let \( k_1, \ldots, k_m \in \mathbb{N} \) be such that \( k_j > \deg(f_1 + X^{k_2}f_2 + \ldots + X^{k_{j-1}}f_{j-1}) \) for all \( j \in [2, m] \), and consider the polynomial \( f = f_1 + X^{k_2}f_2 + \ldots + X^{k_m}f_m \). Then \( c_D(f) = c_D(f_1) + \ldots + c_D(f_m) \), hence \( 1 \in c_D(f) \) and \( f \in Q \), a contradiction.

\[\textbf{Theorem 6.2.4.}\] Let \( K \) be a field, \( v \) be valuation of \( K \) and \( v^* \) the trivial extension of \( v \) to \( K(X) \). Then \( \mathcal{O}_{v^*} = \mathcal{O}_v(X) \).

**Proof.** By definition, \( \mathcal{O}_v(X) = N^{-1}\mathcal{O}_v[X] \), where
\[
 N = \{ f \in \mathcal{O}_v[X] \mid c_{\mathcal{O}_v}(f) = \mathcal{O}_v \} = \left\{ \sum_{i \geq 0} a_i X^i \in \mathcal{O}_v[X] \mid v(a_i) = 0 \text{ for some } i \geq 0 \right\},
\]
and therefore \( N = \{ f \in \mathcal{O}_v[X] \mid v^*(f) = 0 \} \). If \( f \in \mathcal{O}_v[X]^\ast \), then \( f = a_0f_0 \), where \( a \in \mathcal{O}_v^\ast \), \( f_0 \in N \) and \( v(a) = v^*(f_0) \). Therefore we obtain
\[
 \mathcal{O}_{v^*} = \left\{ \frac{a_0}{g_0} \mid a \in K, \ v(a) \geq 0, \ f_0, g_0 \in N \right\} = \left\{ \frac{f}{g_0} \mid f \in \mathcal{O}_v[X] \right\} = \mathcal{O}_v(X). \]

**Theorem und Definition 6.2.5.** Let \( D \) be a domain, \( K = q(D) \) and \( r \) be a finitary module system on \( K \) such that \( r \geq d = d(D) \) (then \( \{1\}_d = D \) implies \( \{1\}_r = D \cap D \)).

1. \( N_r = \{ f \in D[X] \mid c_D(f)_r = D_r \} \subset D[X] \) is a multiplicatively closed subset.

   The domain \( N_r(D) = N_r^{-1}D[X] \subset K[X] \) is called the \( r \text{-Nagata domain} \) of \( D \). Note that \( D(X) = N_q(D) \).

2. Let \( J \in \mathcal{F}(D) \) be a fractional ideal of \( D \). Then \( JN_r(D) = N_r^{-1}J[X] \) is a fractional ideal of \( N_r(D), \) and \( J \subset JN_r(D) \cap K \subset J_r \).

3. If \( I, J \in \mathcal{F}(D), \ J^* \neq \emptyset \) and \( I_r = I \), then \( (I : J)N_r(D) = (JN_r(D) : JN_r(D)) \). In particular, \( (JN_r(D))^{-1} = J^{-1}N_r(D) \).

4. \( \max N_r(D) = \{ PN_r(D) \mid P \in r_D^{-\max}(D) \} \). If \( P \in r_D^{-\max}(D) \) and \( M = PN_r(D) \), then \( N_r(D)_M = D[X]_{P[X]} = D_P(X) \).

5. If \( J \in \mathcal{F}(D) \), then
\[
 JN_r(D) \cap K = \bigcap_{P \in r_D^{-\max}(D)} J_P.
\]
6. If $J \in \mathcal{F}(D)$, then $(JJ^{-1})_r = D_r$ if and only if $JN_r(D) = 0$ is an invertible fractional ideal of $N_r(D)$. In particular, if $r$ is an ideal system on $D$ and $J \in \mathcal{F}_r(D)$, then $J$ is $r$-invertible if and only if $JN_r(D)$ is an invertible fractional ideal of $N_r(D)$.

7. Pic $N_r(D) = 0$. Every invertible fractional ideal of $N_r(D)$ is principal.

**Proof.** 1. Since $D_r = \{1\}$, is $r$-cancellative, we may apply Corollary 6.1.4. If $f, g \in N_r$, then $c_D(fg)_r = [c_D(f)c_D(g)]_r = c_D(f)_r c_D(g)_r = D_r$ and thus $fg \in N_r$.

2. Clearly, $N_r^{-1}J[X]$ is an $N_r(D)$-submodule of $K(X) = q(N_r(D))$, and if $a \in D^*$ and $aJ \subset D$, then $aN_r^{-1}J[X] \subset N_r(D)$. Hence $N_r^{-1}J[X]$ is a fractional ideal of $N_r(D)$, and $J \subset J[X] \subset N_r^{-1}J[X] \cap K$. If $a \in N_r^{-1}J[X] \cap K$, then there exists some $g \in N_r$ such that $ag \in J[X]$, hence $c_D(ag)_r \subset J$ and $c_D(a)_r = ac_D(g)_r = aD_r \subset J_r$, and therefore $a \in J_r$.

3. $N_r^{-1}(IJ)[X] = N_r^{-1}(I[X]J[X]) = (N_r^{-1}I[X])(N_r^{-1}J[X])$. Hence it follows that $(N_r^{-1}(I : J)[X])(N_r^{-1}J[X]) = N_r^{-1}((I : J)J)[X] \subset N_r^{-1}J[X]$.

and therefore $N_r^{-1}(I : J)[X] \subset (N_r^{-1}I[X] : N_r^{-1}J[X])$. If $J = \{0\}$, then equality holds.

Assume now that $I = I_r$, $b \in J^*$ and $F \in (N_r^{-1}I[X] : N_r^{-1}J[X])$. Since $J \subset N_r^{-1}J[X]$, we obtain $bF \in N_r^{-1}I[X]$, and therefore there exist some $f \in b^{-1}I[X] \subset K[X]$ and $g \in N_r$ such that $gF = f$. If $a \in J$, then $af = aFg \in N_r^{-1}I[X]$, and there exists some $h \in N_r$ such that $ah \in I[X]$. Hence it follows that $c_D(af)_r \subset I_r$ and $ac_D(af)_r = c_D(af)_r r, c_D(h)_r = c_D(ah)_r \subset I_r = I$. Since $a \in J$ was arbitrary, we obtain $Jc_D(f) \subset I$, and there exists some $d \in (I : J)$, and $F \in N_r^{-1}(I : J)[X]$.

4. For the proof of $\max N_r(D) = \{PN_r(D) \mid P \in r_D\text{-max}(D)\}$ we proceed in three steps:

- If $P \in r_D\text{-spec}(D)$, then $N_r^{-1}P[X] \in \text{spec} N_r(D)$.

  If $P \in r_D\text{-spec}(D)$ and $f \in P[X]$, then $c_D(f)_r \subset P = P_r \cap D_r$. Hence it follows that $c_D(f)_r \subset P_r \subset D_r$, $P[X] \cap N_r = \emptyset$, and $N_r^{-1}P[X] \in \text{spec} N_r(D)$.

- If $M \in \max N_r(D)$, and $P \in r_D\text{-max}(D)$, such that $P = PN_r(D)$.

  Suppose that $M \in \max N_r(D)$, say $M = N_r^{-1}Q$ for some $Q \in \text{spec} D[X]$ such that $Q \cap N_r = \emptyset$. We set

  $$J = \sum_{f \in Q} c_D(f)_r \subset D_r, \text{ and we assert that } J_r = \left( \bigcup_{f \in Q} c_D(f)_r \right)_r \neq D_r.$$  

  Assume the contrary. Since $r$ is finitary, there exist $f_1, \ldots, f_m \in Q$ such that $1 \in [c_D(f_1) \cup \ldots \cup c_D(f_m)]_r$.

  Let $k_2, \ldots, k_m \in \mathbb{N}$ be such that $k_j > \deg(f_j + X^{k_j}f_2 + \ldots + X^{k_j-1}f_{j-1})$ for all $j \in [2, m]$. Then we obtain $f = f_1 + X^{k_2}f_2 + \ldots + X^{k_m}f_m \in Q$, and $c_D(f)_r = [c_D(f_1) + \ldots + c_D(f_m)]_r = [c_D(f_1) \cup \ldots \cup c_D(f_m)]_r$, hence $c_D(f)_r \subset D_r$.

  As $J_r \neq D_r$, we obtain $J \subset J_r \subset D_r \subset D$, and there exists some $P \in r_D\text{-max}(D)$ such that $J \subset P$. If $f \in Q$, then $c_D(f)_r \subset J \subset P$, hence $f \in P[X]$, and therefore $Q \subset P[X]$. Hence it follows that $M = N_r^{-1}Q \subset N_r^{-1}P[X] = PN_r(D)$, and therefore $M = PN_r(D)$.

- If $P \in r_D\text{-max}(D)$, then $PN_r(D) \in \max N_r(D)$.

  If $P \in r_D\text{-max}(D)$, then $PN_r(D) \in \text{spec} N_r(D)$, and there exists some $M \in \max N_r(D)$ such that $PN_r(D) \subset M$. As we have just proved, $M = PN_r(D)$ for some $P^r \in r_D\text{-max}(D)$, and we obtain $P \subset PN_r(D) \cap D \subset P^r \cap D \subset P^r = P'$, hence $P = P'$ and $PN_r(D) = M$.

  If $P \in r_D\text{-max}(D)$, then $N_r(D)N_r^{-1}P[X] = N_r^{-1}D[X]N_r^{-1}P[X] = D[X]P[X] = DP(X)$ by Theorem 6.2.3.3 (note that in our case all rings are subrings of $K(X)$ the isomorphism $\iota_P$ given there is the identity map).

5. If $J \in \mathcal{F}(D)$, then Theorem 3.2.2 implies

$$JN_r(D) \cap K = \bigcap_{M \in \max N_r(D)} JN_r(D)_M \cap K = \bigcap_{P \in r_D\text{-max}(D)} JD_P(X) \cap K.$$
Hence it suffices to prove that $JD_P(X) \cap K = J_P$ for all $P \in r_d\text{-max}(D)$. If $P \in r_d\text{-max}(D)$, then clearly $J_P \subset JD_P(X) \cap K$. Thus suppose that $a \in JD_P(X) \cap K$. Since $JD_P(X) = JN_P^{-1}D_P[X] = N_P^{-1}J_P[X]$, where $N_P = \{ g \in D_P[X] \mid c_{D_P}(g) = D_P \}$, there exists some $g \in N_P$ such that $ag \in J_P[X]$. Hence $c_{D_P}(ag) = ac_{D_P}(g) \subset J_P$, and if $s \in c_{D_P}(g) \setminus P_P = D_P^*$, then $as \in J_P$ and therefore $a \in J_P$.

6. Suppose that $(JJ^{-1}) \not\subset D_r$. Then $(JJ^{-1}) \not= (JJ^{-1}) \cap D \not\subset D$, and therefore there exists some $P \in r_d\text{-max}(D)$ such that $(JJ^{-1}) \cap D = J_N(D)(JN_r(D))^{-1} \cap D \subset P_N(D) \cap D \subset P$, which implies that $(JJ^{-1}) \subset P_r \subset D_r$.

6. Let $J \subset N_r(D) = N_r^{-1}D[X]$ be an invertible ideal. Then $J = (f_1, \ldots, f_m)$ for some $m \in \mathbb{N}$ and $f_1, \ldots, f_m \in D[X]^*$. Let $k_1, \ldots, k_m \in \mathbb{N}$ be such that $k_j > \deg(f_1 + X^{k_1}f_2 + \ldots + X^{k_1-1}f_{j-1})$ for all $j \in [2, m]$. If $f = f_1 + X^{k_1}f_2 + \ldots + X^{k_{m-1}}f_m \in J$, then $c_D(f) = c_D(f_1) + \ldots + c_D(f_m)$, and we assert that $J = fN_r(D)$. By Theorem 3.2.2 it suffices to prove that $J_M = fN_r(D)_M$ for all $M \in \max N_r(D)$. Let $M \in \max N_r(D)$ and $P \in r_d\text{-max}(D)$ such that $M = P_N(D)$. Then $N_r(D)_M = D[X]P[X]$, and by Theorem 4.1.4 there exists some $j \in [1, m]$ such that $J_M = f_jN_r(D)_M = f_jD[X]P[X]$.

Assume to the contrary that $g \notin P[X]$, then $c_D(fh) = c_D(f_jg) \subset c_D(f_j)P \subset c_D(f_j)P \subset c_D(f)P = c_D(h)P = c_D(f)P$. Hence we obatin $c_D(fj)_P \subset c_D(f)_P = c_D(f)P = c_D(fh)_P \subset c_D(fj)P \subset c_D(fj)_P$, and therefore $c_D(fj)_P = \{0\}$ by Lemma 6.1.3. But this implies that $f_j = 0$, a contradiction. \hfill \Box

### 6.3. Kronecker domains

**Definition 6.3.1.** Let $K$ be a field. A subring $R \subset K(X)$ is called a Kronecker domain if $X \in R^\times$ and $f(0) \in fR$ for all $f \in K[X]$.

**Theorem 6.3.2.** Let $K$ be a field and $R \subset K(X)$ a Kronecker domain.

1. If $f = a_0 + a_1X + \ldots + a_nX^n \in K[X]$, then $fR = Ra_0 + \ldots + Ra_n$.

2. $R$ is a Bezout domain, and $K(X) = q(R)$. In particular, $R$ is a GCD-domain, $t(R) = d(R)$.

3. Let $R \subset Y \subset K(X)$ be a valuation domain. Then $V = Y \cap K$ is a valuation domain of $K$, and $Y = V(X)$.

**Proof.** 1. Clearly, $X \in R$ implies $fR \subset a_0R + \ldots + a_nR$. For the reverse inclusion we prove that $a_i \in fR$ for all $i \in [0, n]$ by induction on $i$.

\[ i = 0: a_0 = f(0) \in fR. \]

\[ i \in [1, n], \ i - 1 \to i: \text{If } a_0, \ldots, a_{i-1} \in fR, \text{ then } f' = X^{-i}[f - (a_0 + a_1X + \ldots + a_{i-1}X^{i-1})] \in fR, \]

and therefore $a_i = f'(0) \in f'R \subset fR$.

2. We prove that every ideal of $R$ generated by two elements is a principal ideal. Thus let $\alpha, \beta \in R$ and $f, g \in K[X]^*$ such that $\alpha = \frac{f}{h}$ and $\beta = \frac{g}{h}$. If $n \geq \deg(f)$, then $fR + gR = fR + X^n gR = (f + X^n g)R$ by 1., and therefore $\alpha R + \beta R = (\alpha + X^n \beta)R$.

In order to prove that $K(X) = q(R)$, it suffices to prove that $K[X] \subset q(R)$. If $f \in K[X]$, then $h = (1 + Xf)^{-1} \in R$, and therefore $f = X^{-1}(h^{-1} - 1) \in q(R)$. 


3. If \( x \in K \setminus V \), then \( x \in K(X) \setminus Y \), hence \( x^{-1} \in K \cap Y = V \), and therefore \( V \) is a valuation domain of \( K \). Let \( y : K(X) \to \Gamma \cup \{ \infty \} \) be a valuation such that \( \mathcal{O}_y = Y \). Then \( y(X) = 0 \), and if \( f = a_0 + a_1 X + \ldots + a_n X^n \in K[X] \), then \( a_i \in f R \subset f Y \) and therefore \( y(a_i) \geq y(f) \) for all \( i \in [0, n] \). On the other hand, \( y(f) \geq \min \{ y(a_i X^i) \mid i \in [0, n] \} \) if \( \min \{ y(a_i) \mid i \in [0, n] \} \) \( y(f) \). Hence equality holds, and we obtain \( y(K) = y(K[X]) \). Since \( \Gamma = y(K(X)) = q(y(K[X])) = q(y(K^\infty)) = y(K^\infty) \), it follows that \( v = y \mid K : K \to \Gamma \cup \{ \infty \} \) is a valuation such that \( \mathcal{O}_v = V \), and \( y = v^* \), the trivial extension of \( v \) to \( K(X) \). Hence \( Y = V(X) \) by Theorem 6.2.4.

**Definition 6.3.3.** Let \( D \) be a domain, \( K = q(D) \) and \( r \) a finitary module system on \( K \) such that \( r \geq d(D) \). Then

\[
\mathcal{K}_r(D) = \left\{ \frac{f}{g} \mid f \in D[X], \ g \in D[X]^*, \ c_D(f) \subset c_D(g)_r \right\} \subset K(X)
\]

is called the \( r \)-Kronecker function domain of \( D \).

**Theorem 6.3.4.** Let \( D \) be a domain, \( K = q(D) \) and \( r \) a finitary module system on \( K \) such that \( r \geq d(D) \).

1. \( \mathcal{K}_r(D) \) is a Kronecker domain of \( K(X) \), and if \( f \in K[X] \) and \( g \in K[X]^* \), then \( \frac{f}{g} \in \mathcal{K}_r(D) \) if and only if \( c_D(f) \subset c_D(g)_r \).

2. There is a surjective monoid homomorphism

\[
\varepsilon : K(X) \to \Lambda_r(K), \text{ given by } \varepsilon \left( \frac{f}{g} \right) = c_D(g)^{-1}_{r} c_D(f)_r \text{ for all } f \in K[X] \text{ and } g \in K[X]^*.
\]

3. \( \varepsilon^{-1} \left( \Lambda^+_r(K) \right) = \mathcal{K}_r(D), \) \( \varepsilon^{-1} \left( 1 \right) = K_r(D)^\times \), and \( \varepsilon \) induces monoid isomorphisms

\[
K(X)/K_r(D)^\times \cong \Lambda_r(K) \text{ and } K_r(D)/K_r(D)^\times \cong \Lambda^+_r(K).
\]

**Proof.** 1. Let \( f, g \in K[X], \ g \neq 0 \) and \( a \in D^* \) such that \( af, ag \in D[X] \). If \( c_D(f) \subset c_D(g)_r \), then \( c_D(af) = ac_D(f) \subset ac_D(g)_r = c_D(af)_r \), and therefore \( \frac{f}{g} = \frac{af}{ag} \in K_r(D) \) by definition. Conversely, assume that \( \frac{f}{g} \in K_r(D) \), and let \( f_1, g_1 \in D[X] \) be such that \( g_1 \neq 0 \), \( c_D(f_1) \subset c_D(g_1)_r \), and \( \frac{f}{g} = \frac{f_1}{g_1} \). Then \( c_D(f_1)_r \subset c_D(g_1)_r \), \( f g_1 = f_1 g \), and since \( r_a \) is finitely cancellative, we obtain

\[
\left[ c_D(f) c_D(g_1) \right]_r = c_D(f g_1)_r = c_D(f_1 g)_r = \left[ c_D(f_1) c_D(g) \right]_r \subset \left[ c_D(g_1) c_D(g) \right]_r
\]

and therefore \( c_D(f) \subset c_D(f)_r \subset c_D(g)_r \).

Next we prove that \( K_r(D) \subset (K(X) \setminus V) \) is a subring. Suppose that \( \alpha, \beta \in K_r(D) \), say \( \alpha = \frac{f}{h} \) and \( \beta = \frac{g}{h} \), where \( f, g, h \in K[X] \), \( h \neq 0 \) and \( c_D(f) \cup c_D(g) \subset c_D(h)_r \). Then \( \alpha + \beta = \frac{f + g}{h}, \ \alpha \beta = \frac{f g}{h^2} \), \( c_D(f + g) \subset c_D(f) + c_D(g) \subset c_D(h)_r \), and \( c_D(f g) \subset c_D(f) c_D(g) \subset c_D(h)^2 = c_D(h^2)_r \), which implies \( \alpha + \beta \in K_r(D) \) and \( \alpha \beta \in K_r(D) \).

Clearly, \( X \in K_r(D), \ X^{-1} \in K_r(D), \) and if \( f \in K[X] \), then \( c_D(f(0)) = Df(0) \subset c_D(f) \subset c_D(f)_r \), hence \( \frac{f(0)}{f} \in K_r(D) \) and therefore \( f(0) \in fK_r(D) \). Hence \( K_r(D) \) is a Kronecker domain.
2. If $f, f_1 \in K[X]$ and $g, g_1 \in K[X]^\ast$ are such that $\frac{f}{g} = \frac{f_1}{g_1}$, then $fg_1 = f_1g$, and as $r_a$ is finitely cancellative, we obtain $\{c_D(f)\} = c_D(f_1) = c_D(f_1) = c_D(f)$. Therefore $c_D(g)_r = [c_D(f_1) = c_D(f)]_r$, and hence $c_D(g) = [c_D(f_1)] = c_D(f)$.

3. By Theorem 4.4.3.2 (b), $\tau_r$ is bijective. By 2., $\tau$ induces a commutative diagram

\[
\begin{array}{ccc}
\tau_r: K & \longrightarrow & K(X) \\
\downarrow & & \downarrow \varepsilon \\
D & \longrightarrow & K_r(D)
\end{array}
\]

where the upward arrows are inclusions. If $t^* = t(K_r(D)), then t^* = \varepsilon t$ by Theorem 2.6.2, and by Theorem 6.3.2.2, $\varphi$ is the set of all $t$-valuation monoids $Y$ such that $K_r(D) \subseteq Y \subseteq K(X)$, and by Theorem 3.4.10 the assignment $Y \mapsto \varepsilon(Y)$ defines a bijective map $\varepsilon: \mathcal{Y} \rightarrow \mathcal{W}$. Hence $\varepsilon = \varepsilon \circ \varepsilon: \mathcal{Y} \rightarrow \mathcal{W}$ is bijective. If $Y \in \mathcal{Y}$, then $K_r(D)^{\times} = \varepsilon^{-1}(1) \subseteq Y$, and $\varepsilon(Y) = \varepsilon^{-1}(\varepsilon(Y)) = (\varepsilon | K)^{-1} \varepsilon(Y) = Y \cap K$. If $V \in \mathcal{V}$, then $Y = V^{-1}(V) \in \mathcal{V}$, $V = Y \cap K$, and therefore $Y = V(X) = \delta(V)$ by Theorem 6.3.2.3. \hfill \Box

### 6.4. $v$-ideals and $t$-ideals in polynomial domains

**Throughout this section, let $D$ be a domain and $K = q(D).**

We use $t$ and $v$ for the corresponding operations both for $D$ and $D[X]$. 

**Definition 6.4.1.**

1. An ideal $J \triangleleft D[X]$ is called **almost principal** if there exist $f \in J \setminus D$ and $r \in D^\ast$ such that $J \subseteq r^{-1}fD[X]$.
2. For a $D[X]$-submodule $J \subseteq K[X]$, we call

\[
c_D(J) = \sum_{f \in J} c_D(f)
\]

the **content** of $J$. By definition, $c_D(J) \subset K$ is a $D$-submodule, and $J \subseteq c_D(J)[X]$.

**Theorem 6.4.2.**

1. Let $J \triangleleft D[X]$ be an ideal. Then $JK = JK[X] = fK[X]$ for some $f \in J$, and the following assertions are equivalent:
   (a) $f \in D^\ast$.
   (b) $J \cap D^\ast \neq \emptyset$.
   (c) $JK = K[X]$.

   In particular, if $J$ is almost principal and $f \in J \setminus D$ and $r \in D^\ast$ are such that $J \subseteq r^{-1} fD[X]$, then $JK = fK[X] \neq K[X]$, and $J \cap D^\ast = \emptyset$.
2. Let $q$ be an ideal system on $D[X]$ such that $q \geq d(D[X])$, $S \subset D[X]^\ast$ a set of polynomials of bounded degree and $J = S_q \triangleleft D[X]$. If $JK \neq K[X]$, then $J$ is almost principal.
3. If \( f \in D[X] \), then \( (fK[X] \cap D[X])K = fK[X] \).

4. If \( \{0\} \neq J \subset D[X] \), then \( J \) is a prime ideal such that \( J \cap D^* = \emptyset \) if and only if \( J = fK[X] \cap D[X] \) for some irreducible polynomial \( f \in K[X] \).

5. The following assertions are equivalent:
   (a) For every fractional ideal \( F \in \mathcal{F}(D[X]) \) such that \( F \subset K[X] \) there exists some \( s \in D^* \) such that \( sF \subset D[X] \).
   (b) Every fractional ideal \( F \in \mathcal{F}(D[X]) \) is of the form \( F = hB \), where \( h \in K(X) \) and \( B \subset D[X] \) is an ideal satisfying \( B \cap D^* \neq \emptyset \).
   (c) For every \( f \in D[X]^* \) we have \( fK[X] \cap D[X] = r^{-1}fB \), where \( r \in D^* \) and \( B \subset D[X] \) is an ideal satisfying \( B \cap D^* \neq \emptyset \).
   (d) Every non-zero ideal \( J \subset D[X] \) such that \( JK \neq K[X] \) is almost principal.

6. The equivalent conditions in 5. are fulfilled in the following cases:
   - \( D \) is noetherian or \( D[X] \) is \( q \)-noetherian for some ideal system \( q \geq d(D[X]) \).
   - If \( \overline{D} \) denotes the integral closure of \( D \), then there exists some \( c \in D^* \) such that \( c\overline{D} \subset D \).

Proof. 1. Clearly, \( JK = \{cg \mid c \in K, g \in J\} = JK[D[X] = JK[K[X] = f'[K[X] for some \( f' \in JK \).
If \( f' = cf \), where \( f \in J \) and \( c \in K^\times \), then \( JK = f'K[X] = fK[X] \).

(a) \( \Rightarrow \) (b) \( f \in J \cap D^* \).

(b) \( \Rightarrow \) (c) If \( c \in J \cap D^* \), then \( 1 = cc^{-1} \in JK \), and therefore \( JK = K[X] \).

(c) \( \Rightarrow \) (a) If \( JK = K[X] = fK[X] \), then \( f \in K[X]^\times \cap J \subset K^\times \cap D[X] = D^* \).
In particular, if \( f \in D \setminus J \) and \( r \in D^* \) are such that \( J \subset r^{-1}fD[X] \), then \( fD[X] \subset J \subset r^{-1}fD[X] \)
implies \( JK = fK[X] \), and by the above we obtain \( JK \neq K[X] \) and \( J \cap D^* = \emptyset \).

2. Since \( JK \neq K[X] \), there exists some polynomial \( f \in J \setminus D \) such that \( JK = fK[X] \). We set \( f = X^t(a_nX^n + \ldots + a_1X + a_0) \), where \( t, n \in \mathbb{N}_0 \), \( t + n = \deg(f) > 0 \), \( a_0, \ldots, a_n \in D \) and \( a_0a_n \neq 0 \). Let \( m \in \mathbb{N}_0 \) be such that \( \deg(h) \leq m + \deg(f) \) for all \( h \in S \). It suffices to prove that \( a_0^{-1}h \subset fD[X] \) for all \( h \in S \). Indeed, if this is done, then it follows that \( a_0^{-1}S \subset fD[X] \) and \( J = S \subset (a_0^{-1}S)^{-1}fD[X] \).

Thus let \( h \in S \subset fK[X] \), say \( h = fg \), where \( g \in K[X] \). Then \( \deg(g) = \deg(h) - \deg(f) \leq m \),
and we set \( g = b_mX^m + b_{m-1}X^{m-1} + \ldots + b_0 \), where \( b_0, \ldots, b_m \in K \). Then
\[ h = fg = X^t \sum_{i=0}^{n+m} c_iX^i \in D[X] \, \text{, where } c_i = \sum_{i=0}^l a_{i-1}b_i \text{ for all } l \in [0, m] \, \text{ (with } a_i = 0 \text{ for } i > n) \.

We use induction on \( l \) to prove that \( a_0^{-1}b_l \subset D \) for all \( l \in [0, m] \). Clearly, \( a_0b_0 = c_0 \in D \). Thus let \( l \in [1, m] \), and suppose that \( a_0^{-1}b_j \subset D \) for all \( j \in [0, l-1] \). Then
\[ a_0^{-1}c_l = a_0^{-1}b_l + \sum_{i=0}^{l-1} a_{i-1}a_0^{-1}b_i \subset D \, \text{, and therefore } a_0^{-1}b_l \subset D \.

Hence it follows that \( a_0^{-1}g \subset fD[X] \) and \( a_0^{-1}h \subset fD[X] \).

3. If \( f \in D[X] \), then \( fK[X] = fD[X]K \subset (fK[X] \cap D[X])K \subset fK[X] \).

4. Suppose that \( \{0\} \neq J \subset D[X] \). If \( J = fK[X] \cap D[X] \) for some irreducible polynomial \( f \in K[X] \),
then \( J \cap D^* = \emptyset \) by 1., and \( J \) is a prime ideal of \( D[X] \), since \( fK[X] \) is a prime ideal of \( K[X] \).

To prove the converse, let \( J \) be a prime ideal such that \( J \cap D^* = \emptyset \). Then \( JK = fK[X] \) for some \( f \in J \) by 1., and since \( JK = D^*^{-1}J \) and \( J \cap D^* = \emptyset \), it follows that \( JK \) is a prime ideal of \( K[X] \), and \( J = JK \cap D[X] = fK[X] \cap D[X] \).

5. (a) \( \Rightarrow \) (b) Let \( F \in \mathcal{F}(D[X]) \) be a fractional ideal and \( v \in D[X]^* \) such that \( C = \{vF \subset D[X] \). If \( C = \{0\} \), then \( J = \{0\} \) and the assertion follows with \( h = 0 \) and \( B = D[X] \). If \( C \cap D^* \neq \emptyset \), then the assertion follows with \( h = v^{-1} \) and \( B = C \).
We may now assume that \( C \neq \{0\} \) and \( C \cap D^* = \emptyset \). Then \( CK \subseteq K[X] \) is a non-zero ideal, and thus \( CK = fK[X] \) for some \( f \in D[X] \setminus D \). Consequently, \( E = f^{-1}C \subseteq K[X] \) is a fractional ideal, and by (a) there exists some \( s \in D^* \) such that \( B = sE \subseteq D[X] \). Since \( fK[X] = CK = fs^{-1}BK = fBK \), we obtain \( BK = K[X] \) and therefore \( B \cap D^* \neq \emptyset \). As \( F = v^{-1}C = v^{-1}fs^{-1}B \), the assertion follows with \( h = v^{-1}fs^{-1} \in K(X) \).

(b) \( \Rightarrow \) (c) Let \( f \in D[X]^* \). By assumption, \( fK[X] \cap D[X] = hB' \), where \( h \in K(X) \), \( B' \subset D[X] \) and \( B' \cap D^* \neq \emptyset \). Hence \( B'K = K[X] \), and \( fK[X] = (fK[X] \cap D[X])K = hB'K = hK[X] \) (by 3.). Therefore we obtain \( h = r^{-1}af \) for some \( a, r \in D^* \), and \( fK[X] \cap D[X] = r^{-1}afB' = r^{-1}fB \), where \( B = aB' \subset D[X] \), and \( B \cap D^* \supseteq a(B' \cap D^*) \neq \emptyset \).

(c) \( \Rightarrow \) (d) Let \( \{0\} \neq J \subset D[X] \) be such that \( JK \neq K[X] \). By 1. there exists some \( f \in J \setminus D \) such that \( JK = fK[X] \). By (c) there exist \( r \in D^* \) and \( B \subset D[X] \) such that \( fK[X] \cap D[X] \subset r^{-1}fB \), and therefore \( J \subset fK[X] \cap D[X] \subset r^{-1}fB \subset r^{-1}fD[X] \).

(d) \( \Rightarrow \) (a) Let \( F \in \mathcal{F}(D[X]) \) be a fractional ideal such that \( F \subset K[X] \), and let \( f \in D[X]^* \) be such that \( J = fF \subset D[X] \). If \( f \not\in D \), we are done. Thus suppose that \( f \notin D \). Then \( J \subset J' = fK[X] \cap D[X] \), and \( J'K = fK[X] \neq K[X] \). By (d) there exists some \( f' \in J' \setminus D \) and some \( r \in D^* \) such that \( J' \subset r^{-1}f'D[X] \), and therefore \( f'K[X] = J'K = fK[X] \) by 3. Hence \( f' = b^{-1}af \) for some \( a, b \in D^* \), and if \( s = br \in D^* \), then \( sf = brf = bf^{-1}J \subset bf^{-1}rJ' \subset bf^{-1}f'D[X] = adD[X] \subset D[X] \).

6. If \( D \) is noetherian, then \( D[X] \) is noetherian, and if \( D[X] \) is \( q \)-noetherian for some ideal system \( q \geq d(D[X]) \), then (d) follows by 2.

If \( D \) is integrally closed, we verify (c). Let \( f \in D[X]^* \). If \( f \in D \), then \( fK[X] \cap D[X] = D[X] \) and (c) holds with \( r = f \) and \( B = D[X] \). If \( f \notin D \), then \( fK[X] \cap D[X] = fc_{D}(f)^{-1}I[X] \) by Theorem 6.1.5. If \( 0 \neq r \in c_{D}(f) \), then (c) holds with \( B = rc_{D}(f)^{-1}I[X] \).

Assume finally that \( \overline{D} \) is the integral closure of \( D \) and there is some \( c \in D^* \) such that \( c\overline{D} \subset D \). Then (d) holds for \( \overline{D} \), and we verify it for \( D \). Let \( J \subset D[X] \) be a non-zero ideal such that \( JK \neq K[X] \). Then \( \overline{J} = J\overline{D}[X] \) is a non-zero ideal of \( \overline{D}[X] \) and \( \overline{J}K = JK[X] \neq K[X] \). Hence there exist some \( \overline{f} \in \overline{J} \setminus \overline{D} \) and \( \overline{r} \in \overline{D}^* \) such that \( \overline{r}\overline{J} \subset \overline{f}\overline{D}[X] \). Then \( f = e\overline{f} \in J \setminus D, \ r = e\overline{r} \in D^*, \) and \( rJ \subset c\overline{r}\overline{J} \subset (c\overline{f})c\overline{D}[X] \subset fD[X] \).

**Theorem 6.4.3.**

1. The assignment \( I \mapsto I[X] \) defines injective monoid homomorphisms \( j: \mathcal{F}(D) \to \mathcal{F}(D[X]) \),

\[
j_L = j \mid \mathcal{F}_L(D) : \mathcal{F}_L(D) \to \mathcal{F}_L(D[X]), \quad j_v = j \mid \mathcal{F}_v(D) : \mathcal{F}_v(D) \to \mathcal{F}_v(D[X]),
\]

and it induces group monomorphisms \( j_v^* = j_v \mid \mathcal{F}_v(D)^\times : \mathcal{F}_v(D)^\times \to \mathcal{F}_v(D[X])^\times \),

\[
j_L^* = j_L \mid \mathcal{F}_L(D)^\times : \mathcal{F}_L(D)^\times \to \mathcal{F}_L(D[X])^\times \text{ and } j_L^* = j_L \mid \mathcal{F}_L(D)^\times : \mathcal{F}_L(D)^\times \to \mathcal{F}_L(D[X])^\times.
\]

2. Let \( I \in \mathcal{F}(D)^* \) be a non-zero fractional ideal.

(a) \( I \) is invertible [finitely generated, a principal ideal] if and only if \( I[X] \) is invertible [finitely generated, a principal ideal].

(b) \( I[X] \), and if \( I \in \mathcal{F}_v(D) \), then \( I \) is \( v \)-invertible [\( v \)-finitely generated] if and only if \( I[X] \) is \( v \)-invertible [\( v \)-finitely generated].

(c) \( I[X] \), and if \( I \in \mathcal{F}_t(D) \), then \( I \) is \( t \)-invertible [\( t \)-finitely generated] if and only if \( I[X] \) is \( t \)-invertible [\( t \)-finitely generated].

In particular, \( j_v^* \) induces a group monomorphism \( j^*: C_v(D) \to C_v(D[X]) \), mapping \( \text{Pic}(D) \) into \( \text{Pic}(D[X]) \) and \( \mathcal{C}(D) \) into \( \mathcal{C}(D[X]) \).

**Proof.** By Theorem 6.2.2, the assignment \( I \mapsto I[X] \) defines an injective monoid homomorphism \( j: \mathcal{F}(D) \to \mathcal{F}(D[X]) \). If \( I \in \mathcal{F}(D) \), then \( I^{-1}[X] = I[X]^{-1} \), and \( I \) is finitely generated [a principal ideal].
if and only if $I[X]$ is finitely generated [a principal ideal]. Hence $j' = j|\mathcal{F}(D)\times : \mathcal{F}(D)\times \to \mathcal{F}(D[X])\times$ is a group monomorphism.

If $I \in \mathcal{F}(D)$, then $I[X]_v = (I[X]^{-1})^{-1} = (I^-1)^{-1}[X] = I_v[X]$. To prove the corresponding result for the $t$-operation, let $\mathcal{F}(I)$ denote the set of all finitely generated fractional ideals $J \in \mathcal{F}(D)$ such that $J \subset I$. If $J \in \mathcal{F}(I)$, then $J[X] \in \mathcal{F}(D[X])$ is also finitely generated, hence $J_t = J_v$, $J[X]_t = J[X]_v$, and we obtain

$$I_t[X] = \bigcup_{J \in \mathcal{F}(I)} J_t[X] = \bigcup_{J \in \mathcal{F}(I)} J_v[X] = \bigcup_{J \in \mathcal{F}(I)} J[X]_v = \left( \bigcup_{J \in \mathcal{F}(I)} J[X] \right)_t = I[X]_t$$

(note that the union is taken over a directed family).

Next we prove that a fractional $t$-ideal $I \in \mathcal{F}_t(D)$ is $t$-finitely generated if and only if $I[X]$ is $t$-finitely generated (note that a fractional $v$-ideal is $v$-finitely generated if and only if it is $t$-finitely generated). If $I \in \mathcal{F}_t(D)$, then $I = J_t$ for some $J \in \mathcal{F}(I)$, and therefore $I[X] = J_t[X] = J[X]_t \in \mathcal{F}_t(D[X])$. Conversely, assume that $I[X] \in \mathcal{F}_t(D[X])$. Then $I[X] = E_t$ for some finite set $E \subset I[X]$. Since

$$I[X] = \bigcup_{J \in \mathcal{F}(I)} J_t[X] \quad \text{(directed union)},$$

there exists some $J \in \mathcal{F}(I)$ such that $E \subset J_t[X]$, which implies $I[X] = E_t = J_t[X]$, and therefore $I = I[X] \cap K = J_t[X] \cap K = J_t \in \mathcal{F}_t(D)$.

We have proved that $j(\mathcal{F}_v(D)) \subset \mathcal{F}_v(D[X])$ and $j(\mathcal{F}_t(D)) \subset \mathcal{F}_t(D[X])$, and we assert that the injective maps $j_v = j|\mathcal{F}_v(D) : \mathcal{F}_v(D) \to \mathcal{F}_v(D[X])$ and $j_t = j|\mathcal{F}_t(D) : \mathcal{F}_t(D) \to \mathcal{F}_t(D[X])$ are monoid homomorphisms. Indeed, if $I_1, I_2 \in \mathcal{F}_v(D)$, then $I_1 \cdot I_2)[X] = (I_1 I_2)_v[X] = I_1[X]_v \cdot I_2[X]_v$, and the same argument holds for $t$ instead of $v$. Hence $j_v$ and $j_t$ induce group monomorphisms $j'_v : \mathcal{F}_v(D)^\times \to \mathcal{F}_v(D[X])^\times$ and $j'_t : \mathcal{F}_t(D)^\times \to \mathcal{F}_t(D[X])^\times$. Since $\mathcal{F}(D)^\times \subset \mathcal{F}_v(D)^\times \subset \mathcal{F}_v(D)^\times$ are subgroups, we obtain $j' = j'_v|\mathcal{F}(D)^\times$ and $j'_t = j'_v|\mathcal{F}_t(D)^\times$ by definition. In particular, if $I \in \mathcal{F}(D)^\times$ is invertible $[I_v]$ is $v$-invertible, $I_t$ is $t$-invertible, then $I[X]$ is invertible $[I[X]_v]$ is $v$-invertible, $I[X]_t$ is $t$-invertible].

If $I \in \mathcal{F}(D)^\times$ and $I[X]$ is invertible, then $D[X] = I[X]I[X]^{-1} = I[X]I^{-1}[X]$, and therefore $D = (II^{-1})[X] \cap D = II^{-1}$. Hence $I$ is invertible.

If $I_v[X]$ is $v$-invertible, then $D[X] = (I_v[X]I_v[X]^{-1})_v = (I_v I_v^{-1})_v[X] = I_v$, and therefore $D = (I_v I_v^{-1})_v[X] \cap D = (I_v I_v^{-1})_v$. Hence $I_v$ is $v$-invertible. The same argument holds for $t$ instead of $v$.

If $I \in \mathcal{F}_v(D)^\times$, then $I[X]$ is principal if and only if $I[X]$ is principal. Hence $j'_v$ induces a group monomorphism $j^* : \mathcal{C}_v(D) \to \mathcal{C}_v(D[X])$. For $I \in \mathcal{F}_v(D)^\times$, we denote by $[I] \in \mathcal{C}_v(D)$ the class of $I$, and for $J \in \mathcal{F}_v(D[X])^\times$ we denote by $[J] \in \mathcal{C}_v(D[X])$ the class of $J$. If $\epsilon = [I] \in \mathcal{F}_v(D)^\times$, then $j^*(\epsilon) = [I[X]]$. If $\epsilon \in \Pic(D)$, then $I \in \mathcal{F}(D)^\times$, hence $I[X] \in \mathcal{F}(D[X])^\times$ and $j^*(\epsilon) = [I[X]] \in \Pic(D[X])$. If $\epsilon \in \mathcal{C}_v(D)$, then $I \in \mathcal{F}_v(D)^\times$, hence $I[X] \in \mathcal{F}_v(D[X])^\times$ and $j^*(\epsilon) = [I[X]] \in \mathcal{C}_v(D[X])$.

\[ \square \]

**Theorem 6.4.4.** The following assertions are equivalent:

(a) $D$ is integrally closed.

(b) If $J < D[X]$ and $J \cap D^\times \neq \emptyset$, then $J_v = \mathcal{C}_D(J)_v[X]$.

(c) If $J \in \mathcal{I}_v(D[X])$ and $J \cap D^\times \neq \emptyset$, then $J \cap D \in \mathcal{I}_v(D)$, and $J = (J \cap D)[X]$.

(d) If $J \in \mathcal{I}_v(D[X])$ and $J \cap D^\times \neq \emptyset$, then $J \cap D \in \mathcal{I}_v(D)$, and $J = (J \cap D)[X]$.

(e) If $J < D[X]$ and $J \cap D^\times \neq \emptyset$, then $J_t = \mathcal{C}_D(J)_t[X]$.

(f) If $f, g \in D[X]^\times$ and $a \in D^\times$ are such that $\mathcal{C}_D(fg) \subset aD$, then $\mathcal{C}_D(fD)(g) \subset aD$.

**Proof.** (a) $\Rightarrow$ (b) Suppose that $J < D[X]$ and $J \cap D^\times \neq \emptyset$. Then $J \subset \mathcal{C}_D(J)[X]$ and therefore $J_v \subset \mathcal{C}_D(J)[X]_v = \mathcal{C}_D(J)_v[X]$. For the proof of the reverse inclusion, observe that $J_v$ is the intersection of all fractional principal ideals containing $J$. Hence it suffices to prove:
If $h \in K(X)$ and $J \subset hD[X]$, then $c_J(D)[X_v] \subset hD[X]$.

Let $h = g^{-1}b \in K(X)$, where $g, b \in D[X]$ are coprime in $K[X]$, and suppose that $J \subset hD[X]$. Then it clearly suffices to prove that $c_J(D)[X] \subset hD[X]$. We have $gJ \subset bD[X]$, and if $c \in J \cap D^*$, then $cg = bq$ for some $q \in D[X]$, and as $b$ and $g$ are coprime in $K[X]$, we obtain $b \in D^*$. For all $q \in J$, we obtain $g\subset bD$, and therefore, by Theorem 6.15, $c_J(D) \subset c_J(D)[X] = c_J(D)^v \subset bD$, hence $g\subset bD[X] = hD[X]$.

(b) $\Rightarrow$ (c) If $J \in \mathcal{I}_v(D[X])$ and $J \cap D^* \neq \emptyset$, then $J = c_J(D)[X]$ by (b), and thus it follows that $c_J(D)[X_v] = J \cap D \in \mathcal{I}_v(D)$.

(c) $\Rightarrow$ (d) Let $J \in \mathcal{I}_v(D[X])$ be such that $J \cap D^* \neq \emptyset$, and denote by $\mathcal{F}(J)$ the set of all finitely generated ideals $B \subset J$ such that $B_v \cap D^* \neq \emptyset$. Then

$$J = J_t = \bigcup_{B \in \mathcal{F}(J)} B_v \quad \text{implies} \quad J \cap D = \bigcup_{B \in \mathcal{F}(J)} B_v \cap D \quad \text{and} \quad (J \cap D)[X] = \bigcup_{B \in \mathcal{F}(J)} (B_v \cap D)[X].$$

If $B \in \mathcal{F}(J)$, then $B \cap D \in \mathcal{I}_v(D)$, and $B_v = (B_v \cap D)[X]$. Since all unions are directed, it follows that $J \cap D \in \mathcal{I}_v(D)$ and $J \cap D = (J \cap D)[X]$.

(d) $\Rightarrow$ (e) Suppose that $J \subset D[X]$ and $J \cap D^* \neq \emptyset$. By (d) we have $J_t = (J_t \cap D)[X]$, and

$$c_D(J_t) = c_B(J_t)[X_v] \subset c_D(J_t)[X] = (J_t \cap D)[X] = J_t.$$

(e) $\Rightarrow$ (f) Let $f, g \in D[X]^*$ and $a \in D^*$ be such that $c_D(fg) \subset aD$, and set $J = D[X](a, g) \subset D[X]$. Then $J \cap D^* \neq \emptyset$, and therefore $J_t = c_D(J_t)[X]$ by (e). Since $fJ = D[X](fa, fg) \subset aD[X]$, we obtain $f \subset c_D(J_t)[X] = fJ_t \subset aD[X]$, and therefore $c_D(f) \subset aD$.

(f) $\Rightarrow$ (a) Let $u \in K$ be integral over $D$ and $f \in D[X]$ a monic polynomial such that $f(u) = 0$. Then $f = (X - u)g$ for some monic polynomial $g \in K[X]$. Let $t \in D^*$ be such that $tu \in D$ and $tg \in D[X]$. Then $h = t^2f = t(X - u)(tg) \in t^2D[X]$, hence $c_D(h) \subset t^2D$, and therefore $c_D(t(X - u))c_D(tg) \subset t^2D$. Since $tu \in c_D(t(X - u))$ and $t \in c_D(tg)$, we obtain $t^2u \in t^2D$ and therefore $u \in D$. □

**Theorem 6.4.5.** Let $D$ be integrally closed. Then the group monomorphism $j^* : C_v(D) \rightarrow C_v(D[X])$ (see Theorem 6.4.3) is an isomorphism, $j^*(\text{Pic}(D)) = \text{Pic}(D[X])$ and $j^*(C(D)) = C(D[X])$.

**Proof.** By Theorem 6.4.3 it suffices to prove that $C_v(D[X]) \subset j^*(C_v(D))$, $\text{Pic}(D[X]) \subset j^*(\text{Pic}(D))$ and $C_v(D[X]) \subset j^*(C_v(D))$. For $I \in \mathcal{F}_v(D[X])$ we denote by $[I] \in C_v(D)$ the class of $I$, for $I \in \mathcal{F}_v(D[X])$, we denote by $[J] \in C_v(D(X))$ the class of $J$.

Let $c = [F] \in C_v(D[X])$, where $F \in \mathcal{F}_v(D[X])$. By Theorem 6.4.2 it follows that $F = hB$ for some ideal $B \subset D[X]$ such that $B \cap D^* \neq \emptyset$. Then $B \subset C_v(D[X])$ and $c = [B]$. By the Theorems 6.4.4 and 6.4.3 it follows that $B \cap D \subset \mathcal{F}_v(D^\times)$ and $B = (B \cap D)[X]$. Hence we obtain $[B \cap D] \in C_v(D)$ and $c = j^*([B \cap D])$.

If $c \in \text{Pic}(D[X])$, then $F \in \mathcal{F}_v(D[X]), B = (B \cap D)[X] \subset \mathcal{F}_v(D[X])$, hence $B \cap D \subset \mathcal{F}_v(D^\times)$, $[B \cap D] \in C(D)$ and $c = j^*([B \cap D]) \in j^*(C(D))$.

If $c \in \text{Pic}(D[X])$, then $F \in \mathcal{F}_v(D[X]), B = (B \cap D)[X] \subset \mathcal{F}_v(D[X])$, hence $B \cap D \subset \mathcal{F}_v(D^\times), [B \cap D] \in \text{Pic}(D)$ and $c = j^*([B \cap D])$. □

**Theorem 6.4.6.** Each of the following assertions hold for $R = D$ if and only if it holds for $R = D[X]$.

1. $R$ is integrally closed.
2. $R$ is completely integrally closed (equivalently, every non-zero $v$-ideal is $v$-invertible).
3. $R$ is a $v$-domain (equivalently, every $v$-finitely generated non-zero $v$-ideal is $v$-invertible).
4. $R$ is a Krull domain (equivalently, every non-zero $t$-ideal is $t$-invertible).
5. $R$ is a PVMD (equivalently, every $t$-finitely generated non-zero $t$-ideal is $t$-invertible).
6. \( R \) is factorial (equivalently, \( R \) is a Krull domain and \( \mathcal{C}(R) = 0 \)).

7. \( R \) is a GCD-domain (equivalently, \( R \) is a PVMD and \( \mathcal{C}(R) = 0 \)).

**Proof.** A. We prove first: If \( D[X] \) is integrally closed, then \( D \) is integrally closed.

Let \( D[X] \) be integrally closed and \( x \in K \) integral over \( D \). Then \( x \in K(X) \) is integral over \( D[X] \), hence \( x \in D[X] \cap K = D \). \( \square[A] \)

B. Let \( r \in \{ v, t \} \). To prove 2. 3. 4. and 5., it suffices to show the equivalence of the following two assertions:

(a) Every (\( r \)-finitely generated) non-zero \( r \)-ideal of \( D \) is \( r \)-invertible.

(b) Every (\( r \)-finitely generated) non-zero \( r \)-ideal of \( D[X] \) is \( r \)-invertible.

**Proof.** If every \( r \)-finitely generated \( r \)-ideal of \( D \) is \( r \)-invertible, then \( r \) is \( r \)-noetherian, hence \( D \) is \( r \)-closed and thus integrally closed by Theorem 4.3.2. In the same was, if every \( r \)-finitely generated \( r \)-ideal of \( D[X] \) is \( r \)-invertible, then \( D[X] \) is integrally closed, and therefore \( D \) is integrally closed by A.

Hence for the proof of B we may assume that \( D \) is integrally closed.

(a) \( \Rightarrow \) (b) Let \( F \subset D[X] \) be an (\( r \)-finitely generated) non-zero \( r \)-ideal. By Theorem 6.4.2 \( F = hB \) for some \( h \in K(X) \) and \( B \subset D[X] \) such that \( B \cap D^* \neq \emptyset \). Then \( B \) is an (\( r \)-finitely generated) non-zero \( r \)-ideal. By Theorem 6.4.4 \( B \cap D \) is an \( r \)-ideal, and \( B = (B \cap D)[X] \). If \( B \) is \( r \)-finitely generated, then \( B \cap D \subset D \), \( D \), and therefore \( F \) is \( r \)-invertible.

(b) \( \Rightarrow \) (a) Let \( I \subset D \) be an (\( r \)-finitely generated) non-zero \( r \)-ideal. By Theorem 6.4.3, \( I[X] \) is an (\( r \)-finitely generated) non-zero \( r \)-ideal and as \( I[X] \) is \( r \)-invertible by assumption, it follows that \( I \) is \( r \)-invertible.

C. The assertions 6. and 7. follow by B and Theorem 6.4.5.

D. Finally we prove: If \( D \) is integrally closed, then \( D[X] \) is integrally closed.

**Proof.** Let \( D \) be integrally closed. By Corollary 4.4.5

\[
D = \bigcap_{V \in \mathcal{V}} V \quad \text{and therefore} \quad D[X] = \bigcap_{V \in \mathcal{V}} V[X],
\]

where \( \mathcal{V} \) is the set of all valuation domains \( V \) such that \( D \subset V \subset K \). Therefore it suffices to prove that \( V[X] \) is integrally closed for all \( V \in \mathcal{V} \).

If \( V \in \mathcal{V} \), then every \( t \)-ideal of \( V \) is principal, hence \( V \) is a PVMD, and by B, \( V[X] \) is a PVMD. Hence \( V[X] \) is integrally closed. \( \square \)

**Theorem 6.4.7.** Let \( D \) be a Mori domain, and suppose that either \( D \) integrally closed or \( D \) contains an uncountable subfield. Then \( D[X] \) is a Mori domain.

**Proof.** CASE 1: \( D \) is integrally closed.

We prove that every \( J \in \mathcal{I}(D[X])^* \) is \( t \)-finitely generated. If \( J \in \mathcal{I}(D[X])^* \), then Theorem 6.4.2 implies that \( J = hB \) for some \( h \in K(X) \) and \( B \subset J[X] \) such that \( B \cap D^* \neq \emptyset \). By Theorem 6.4.4 we obtain \( B \cap D \in \mathcal{I}(D) \) and \( B = (B \cap D)[X] \in \mathcal{I}_{t,t}(D) \), since \( D \) is a Mori domain. By Theorem 6.4.3 it follows that \( B \) and therefore also \( J \) is \( t \)-finitely generated.

CASE 2: \( D \) contains an uncountable field \( \Delta \).

Assume to the contrary that \( D[X] \) is not \( t \)-noetherian. Then there exists a sequence \( (g_n)_{n \geq 0} \) in \( D[X] \) such that \( (g_0, \ldots, a_{n-1}) \subset (g_0, \ldots, a_n) \) for all \( n \geq 1 \), and therefore

\[
(D[X]:\{g_0, \ldots, a_n\}) \subset (D[X]:\{g_0, \ldots, a_{n-1}\}).
\]

For \( n \in \mathbb{N} \), let \( h_n \in K(X) \) be such that \( h_ng_i \in D[X] \) for all \( i \in [0, n-1] \) and \( h_ng_n \notin D[X] \). Since \( K[X] \) is noetherian, there exists some \( m \in \mathbb{N} \) such that, for all \( n \geq m \),

\[
(K[X]:\{g_0, \ldots, a_n\}) = (K[X]:\{g_0, \ldots, a_{n-1}\}).
\]
For $n \geq m$ we have $h_n \in (K[X]:\{g_0, \ldots, a_{n-1}\})$, hence $h_n g_n \in K[X] \setminus D[X]$, and by the subsequent Lemma 6.4.8 the set $C_n = \{c \in \Delta \mid (h_n g_n)(c) \in D\}$ is finite. Hence there exists some $c \in \Delta$ such that for all $n > m$ we have $h_n(c)g_n(c) \notin D$, and $h_n(c)g_i(c) \in D$ for all $i \in [m, n-1]$. Consequently, 

$$(D:\{g_0(c), \ldots, g_{n-1}(c)\}) \subsetneq (D:\{g_0(c), \ldots, g_n(c)\})$$

and $(\{g_0(c), \ldots, g_n(c)\}_{c\in\Delta})_{n\geq m}$ is a properly ascending sequence of $v$-ideals of $D$, a contradiction. 

Lemma 6.4.8. Let $D$ be a domain, $K = q(D)$, $\Delta \subset D$ a subfield, $g \in K[X]$ a polynomial such that $\deg(g) = d \in \mathbb{N}$. If $c_0, \ldots, c_d \in \Delta$ are distinct such that $g(c_i) \in D$ for all $i \in [0, d]$, then $g \in D[X]$. 

Proof. If $g = a_0 + a_1 X + \ldots + a_d X^d$, then $(a_0, \ldots, a_d)$ is a solution of the system of equations

$$
\begin{pmatrix}
1 & c_0 & c_0^2 & \ldots & c_0^d \\
1 & c_1 & c_1^2 & \ldots & c_1^d \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & c_d & c_d^2 & \ldots & c_d^d
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
\vdots \\
a_d
\end{pmatrix}
= 
\begin{pmatrix}
g(c_0) \\
g(c_1) \\
\vdots \\
g(c_d)
\end{pmatrix}
\in D^{d+1}
$$

with a determinant in $\Delta^\times \subset D^\times$. Hence $a_0, \ldots, a_d \in D$. 

□