# CHARACTERIZATION OF PRÜFER-LIKE MONOIDS AND DOMAINS BY GCD-THEORIES

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Abstract. We combine ideal-theoretic and divisor-theoretic methods to characterize various classes of Prüfer-like monoids and domains by the gcd-properties of certain semigroups of invertible ideals.

## 1. Introduction

One of the main themes of multiplicative ideal theory during the last decades was the characterization and investigation of various classes of integral domains defined by the invertibility properties of certain classes of ideals. In this field, Prüfer domains form the classical antetype, and there is a wealth of generalizations and variations of this concept in the literature.

In this paper, we combine ideal-theoretic and divisor-theoretic methods to arrange some known characterizations of Prüfer-like domains in a new way and to present several new ones. One of the basic ideas in our investigations is to address the gcd-properties of certain semigroups of invertible (integral) ideals and to combine this viewpoint with the concept of gcd-theories.

Although the theory of integral domains is our main concern, the paper is written in the language of (commutative cancellative) monoids in order to point out the purely multiplicative character of the theory. The main results are the Theorems 3.4 and 3.5 and the subsequent theorems and corollaries. In Section 2 we gather the necessary results from the theory of monoid homomorphisms and ideal systems.

## 2. Monoids and homomorphisms

Throughout this paper, by a *monoid* $D$ we mean (deviating from the usual terminology) a commutative multiplicative semigroup with unit element $1 \in D$ and a zero element $0 \in D$ (satisfying $1x = x$ and $0x = 0$ for all $x \in D$), and we always assume that $D^\bullet = D \setminus \{0\}$ is cancellative (that is, for all $a, b \in D$ and $c \in D^\bullet$, if $ac = bc$, then $a = b$). We set $\mathbf{0} = \{0\}$, denote by $D^\times$ the group of invertible elements of $D$, and we call $D$ *reduced* if $D^\times = \{1\}$. A subset $J \subset D$ is called an *ideal* if $DJ = J$, and it is called a *principal ideal* if $J = aD$ for some $a \in D$.

For a monoid $D$, we denote by $\mathsf{q}(D) = D^{\bullet -1}D$ its total quotient monoid. If $K = \mathsf{q}(D)$, then $K^\bullet = K^\times$ is a quotient group of $D^\bullet$. The most important examples we have in mind

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are the multiplicative monoids of integral domains. If $D$ is (the multiplicative monoid of) an integral domain, then $K$ is (the multiplicative monoid of) its quotient field.

*Let $D$ be a monoid and $K = \mathsf{q}(D)$ its total quotient monoid.*

For subsets $X, Y \subset K$, we set $(X:Y) = \{z \in K \mid zY \subset X\}$, $X^{-1} = (D:X)$, we call $X$ $D$-fractional if $X^{-1} \cap D^\bullet \neq \emptyset$, and we denote by $\mathcal{F}(D)$ the set of all $D$-fractional subsets of $K$.

Throughout, we use the language of ideal systems as developed in the monograph "Ideal Systems" [10], and all undefined notions are as there. In particular, we view an ideal system on $D$ as a map $r \colon \mathcal{F}(D) \to \mathcal{F}(D)$ (see [10, Ch. 11]). For an ideal system $r$ on $D$, we denote by

- $\mathcal{F}_r(D) = \{X_r \mid X \in \mathcal{F}(D)\} = \{A \in \mathcal{F}(D) \mid A_r = A\}$ be the semigroup of all fractional $r$-ideals, equipped with the $r$-multiplication defined by $(A,B) \mapsto (AB)_r$,
- $\mathcal{F}_{r,\mathsf{f}}(D) = \{E_r \mid E \subset K \text{ finite}\} \subset \mathcal{F}_r(D)$ the subsemigroup of all $r$-finite (that is, $r$-finitely generated) fractional $r$-ideals of $D$,
- $\mathcal{I}_r(D) = \{J \in \mathcal{F}_r(D) \mid J \subset D\}$ the subsemigroup of all (integral) $r$-ideals,

For an ideal system $r$ on $D$, we denote the associated finitary ideal system by $r_\mathsf{f}$ (it is denoted by $r_s$ in [10]). It is given by

$$X_{r_\mathsf{f}} = \bigcup_{E \subset X \text{ finite}} E_r \quad \text{for every} \quad X \in \mathcal{F}(D)\,,$$

and it satisfies $\mathcal{F}_{r,\mathsf{f}}(D) = \mathcal{F}_{r_\mathsf{f},\mathsf{f}}(D)$. The ideal system $r$ is called *finitary* if $r = r_\mathsf{f}$.

For any two ideal systems $r$ and $q$ on $D$ we write $r \le q$ if $\mathcal{F}_q(D) \subset \mathcal{F}_r(D)$. Note that $r \le q$ holds if and only if $X_r \subset X_q$ [equivalently, $X_q = (X_r)_q$] for all $X \in \mathcal{F}(D)$. If $r \le q$, then $\mathcal{F}_q(D) \cap \mathcal{F}_{r,\mathsf{f}}(D) \subset \mathcal{F}_{q,\mathsf{f}}(D)$ [indeed, if $I \in \mathcal{F}_q(D) \cap \mathcal{F}_{r,\mathsf{f}}(D)$, then $I = E_r$ for some finite subset $E \subset I$, and $I = I_q = (E_r)_q = E_q$].

We denote by $s = s(D)$ the system of semigroup ideals, given by $X_s = DX$ for all $X \in \mathcal{F}(D)$, by $v = v(D)$ the ideal system of multiples ("Vielfachenideale"), given by $X_v = (X^{-1})^{-1}$ for all $X \in \mathcal{F}(D)$, and by $t = t(D) = v_\mathsf{f}$ the associated finitary system of $v$. The systems $s$ and $t$ are finitary, the system $v$ usually not. For every ideal system $r$ on $D$ we have $s \le r_\mathsf{f} \le r \le v$ and $r_\mathsf{f} \le t$. We shall frequently use that $\mathcal{F}_v(D) = \{A^{-1} \mid A \in \mathcal{F}(D)\}$ (see [10, Theorem 11.8]).

If $D$ is an integral domain, then the (Dedekind) ideal system $d = d(D)$ of usual ring ideals is given by $X_d = {}_D\langle X\rangle$ for all $X \in \mathcal{F}(D)$ (that is, $X_d$ is the fractional $D$-ideal generated by $X$). $d$ is a finitary ideal system, and there is a one-to-one correspondence between ideal systems $r \ge d$ and star operations on $D$, given as follows:

> If $* \colon \mathcal{F}_d(D)^\bullet \to \mathcal{F}_d(D)^\bullet$ is a star operation on $D$ and $r^* \colon \mathcal{F}(D) \to \mathcal{F}(D)$ is defined by $X_{r^*} = {}_D\langle X\rangle^*$ for $X \in \mathcal{F}(D)^\bullet$ and $X_{r^*} = \{0\}$ if $X \subset \{0\}$, then $r^*$ is an ideal system satisfying $r^* \ge d$. Conversely, if $r \ge d$ is an ideal system, and if we

define $*_r$ by $J^{*_r} = J_r$ for all $J \in \mathcal{F}_d(D)^\bullet$, then $*_r$ is a star operation, and by the very definition we have $r^{*_r} = r$ and $*_{r^*} = *$.

Let $r$ be an ideal system on $D$. A fractional $r$-ideal $A \in \mathcal{F}_r(D)$ is called *$r$-invertible* if $(AA^{-1})_r = D$ [equivalently, $(AB)_r = D$ for some subset $B \subset K$]. By definition, a fractional $r$-ideal is $r$-invertible if and only if it is an invertible element of the semigroup $\mathcal{F}_r(D)$. Thus we denote by $\mathcal{F}_r(D)^\times$ the group of all $r$-invertible fractional $r$-ideals. If $q$ is an ideal system such that $r \le q$, then $\mathcal{F}_r(D)^\times \subset \mathcal{F}_q(D)^\times$ is a subgroup (consequently, if $A,\, B \in \mathcal{F}_r(D)^\times$, then $(AB)_r = (AB)_q$, and this holds in particular, if $q = v$). If $r$ is finitary, then $\mathcal{F}_r(D)^\times = \mathcal{F}_{r,\mathfrak{f}}(D)^\times$ (that is, if $A \in \mathcal{F}_r(D)$ is $r$-invertible, then both $A$ and $A^{-1}$ are $r$-finite). This may fail if $r$ is not finitary; then it may occur that $\mathcal{F}_{r,\mathfrak{f}}(D)^\times \subsetneq \mathcal{F}_r(D)^\times \cap \mathcal{F}_{r,\mathfrak{f}}(D)$ (it is well known that not every $v$-domain is a P$v$MD).

For a non-empty subset $X \subset D$ we denote by $\mathrm{GCD}(X)$ the set of all greatest common divisors of $X$. If $d \in \mathrm{GCD}(X)$, then $\mathrm{GCD}(X) = dD^\times$, and if $D$ is reduced, we write (as usual) $d = \gcd(X)$ instead of $\mathrm{GCD}(X) = \{d\}$. If $X = \{a_1, \ldots, a_n\} \subset D$, we write $\mathrm{GCD}(a_1, \ldots, a_n)$ resp. $\gcd(a_1, \ldots, a_n)$ instead of $\mathrm{GCD}(X)$ resp. $\gcd(X)$. Note that $\mathrm{GCD}(\emptyset) = \{0\}$.

$D$ is called a

- GCD-*monoid* if $\mathrm{GCD}(a,b) \ne \emptyset$ for all $a, b \in D$ [equivalently, $\mathrm{GCD}(X) \ne \emptyset$ for every finite subset $X \subset D$].
- *complete* GCD-*monoid* if $\mathrm{GCD}(X) \ne \emptyset$ for every subset $X \subset D$.

**Lemma 2.1.** *Let $D$ be a monoid, $X \subset D$ and $d \in D$.*

1. *If $X_v = dD$, then $d \in \mathrm{GCD}(X)$.*
2. *If $d \in \mathrm{GCD}(X)$ and $\mathrm{GCD}(bX) \ne \emptyset$ for all $b \in D$, then $X_v = dD$.*
3. *$D$ is a* GCD-*monoid if and only if every $v$-finite (fractional) $v$-ideal of $D$ is principal [equivalently, $D$ is a $v$-Bezout monoid].*
4. *$D$ is a complete* GCD-*monoid if and only if every $v$-ideal of $D$ is principal.*
5. *If $D$ is a* GCD-*monoid, then*

$$X_v = \bigcap_{\substack{d \in D \\ X \subset dD}} dD\,, \quad \text{and} \quad d \in \mathrm{GCD}(X) \quad \text{if and only if} \quad X_v = dD\,.$$

*Proof.* [10, Theorem 11.5 and Exercise 11.9]. □

If $D$ is a reduced GCD-monoid, then any two elements $a,\, b \in D$ have a unique least common multiple $\mathrm{lcm}(a,b)$. If $a = 0$ or $b = 0$, then $\mathrm{lcm}(a,b) = 0$, and if $a,\, b \in D^\bullet$, then $\mathrm{lcm}(a,b) = ab\gcd(a,b)^{-1}$.

We recall the relations between GCD-monoids and lattice-ordered (abelian) groups. A partially ordered group $(\Gamma, \le)$ is called *lattice-ordered* if $\sup(X)$ exists for every non-empty finite subset $X \subset \Gamma$ [equivalently, $\inf(X)$ exists for every non-empty finite subset $X \subset \Gamma$]. A lattice-ordered group $(\Gamma, \le)$ is called *completely lattice-ordered* if every non-empty subset of $\Gamma$ which is bounded below possesses an infimum [equivalently, every non-empty subset of $\Gamma$ which is bounded above possesses a supremum]. The following simple Lemma 2.2 specifies the connection between GCD-monoids and lattice-ordered groups.

**Lemma 2.2.** *Let $G$ be a reduced monoid and $L = \mathsf{q}(G)$. For $x, y \in L^\bullet$, we define $x \le y$ if $x^{-1}y \in G$.*

1. *$(L^\bullet, \le)$ is a partially ordered group, $G = \{z \in L^\bullet \mid 1 \le z\}$, and if $a, b \in G$, then $a \,|\, b$ holds if and only if $b \le a$.*
2. *$G$ is a* GCD*-monoid if and only if $(L^\bullet, \le)$ is a lattice-ordered group. If this is the case, then $G$ is complete if and only if $(L^\bullet, \le)$ is completely lattice-ordered. For every (finite) subset $X \subset G$, we have*

$$\inf(X) = \gcd(X) \quad \textit{and} \quad \sup(X) = \operatorname{lcm}(X)\,.$$

*Proof.* 1. Obvious.

2. Let $G$ be a GCD-monoid, $a, b \in L^\bullet$ and $c \in G^\bullet$ such that $ac, bc \in G$. Then $c^{-1}\gcd(ac, bc) \in L^\bullet$ is a common upper bound of $a$ and $b$. Now the assertions follow by [8, § 15] and [10, Exercises 10.1 and 10.2]. □

A (*monoid*) *homomorphism* $\varphi\colon D \to G$ is always assumed to satisfy $\varphi(1) = 1$, $\varphi(0) = 0$ and $\varphi(D^\bullet) \subset G^\bullet$. For every homomorphism $\varphi\colon D \to G$ there exists a unique homomorphism $\mathsf{q}(\varphi)\colon \mathsf{q}(D) \to \mathsf{q}(G)$ satisfying $\mathsf{q}(\varphi) \mid D = \varphi$, and we call $\mathsf{q}(\varphi)$ the *quotient homomorphism* of $\varphi$.

A monoid homomorphism $\varphi\colon D \to G$ is called a *divisor homomorphism* if, for all $a, b \in D$, $\varphi(a) \,|\, \varphi(b)$ implies $a \,|\, b$.
If $K = \mathsf{q}(D)$, $L = \mathsf{q}(G)$, and $\phi = \mathsf{q}(\varphi)\colon K \to L$ is the quotient homomorphism of $\varphi$, then $\varphi$ is a divisor homomorphism if and only if $\phi^{-1}(G) = D$.

In the case of integral domains, the following Theorem goes back to F. Lucius [16]. A preliminary version valid for monoids is in [10, Exercise 18.10], the subsequent proof is new.

**Theorem 2.3.** *Let $\varphi\colon D \to G$ be a cofinal divisor homomorphism and $\phi\colon K \to L$ its quotient homomorphism.*

1. *For every subset $X \subset K$ we have $X^{-1} = \phi^{-1}[\phi(X)^{-1}]$.*
2. *Let $a \in L$ and $X \subset K$ be such that $aG = \phi(X)_v$. Then $aG = [aG \cap \phi(K)]_v$ and $\phi^{-1}(aG) = X_v$.*
3. *Let $X, Y \subset K$ and $a \in L$ be such that $\phi(X)_v = aG$ and $\phi(Y)_v = a^{-1}G$. Then $(XY)_v = D$.*
4. *The following assertions are equivalent:*
   (a) *For every $a \in G$ there exists a [finite] subset $X \subset D$ such that $aG = \varphi(X)_v$ (and thus $a \in \mathrm{GCD}(\varphi(X))$).*
   (b) *For every $a \in L$ there exists a [finite] subset $X \subset K$ such that $aG = \phi(X)_v$.*

*Proof.* 1. Let $X \subset K$. If $a \in X^{-1}$, then $aX \subset D$ implies $\phi(a)\phi(X) = \phi(aX) \subset G$ and thus $\phi(a) \in \phi(X)^{-1}$. Conversely, if $a \in \phi^{-1}[\phi(X)^{-1}]$, then $\phi(aX) = \phi(a)\phi(X) \subset G$ and therefore $aX \subset D$, whence $a \in X^{-1}$.

2. We have $\phi(X)_v = \phi\big[\phi^{-1}\big(\phi(X)\big)\big]_v \subset \phi[\phi^{-1}(aG)]_v = [aG \cap \phi(K)]_v \subset aG = \phi(X)_v$ and therefore $aG = [aG \cap \phi(K)]_v$. Using 1., we obtain, for every $z \in L$:

$$\begin{aligned} z \in \phi^{-1}(aG)^{-1} &\iff z\phi^{-1}(aG) \subset D \iff \phi(z)\phi[\phi^{-1}(aG)] \subset G \\ &\iff \phi(z)[aG \cap \phi(K)]_v \subset G \iff \phi(z)aG \subset G \\ &\iff \phi(z) \in a^{-1}G = \phi(X)^{-1} \iff z \in \phi^{-1}[\phi(X)^{-1}] = X^{-1}\,. \end{aligned}$$

3. Since $\phi(XY)_v = [\phi(X)_v\phi(Y)_v]_v = G$, it follows by 2. that $(XY)_v = \phi^{-1}(G) = D$.

4. (a) $\Rightarrow$ (b) Let $a \in L$ and $c \in G^\bullet$ such that $ca \in G$. By (a), there exists some $u \in D^\bullet$ such that $\varphi(u) \in cG$ and thus $\varphi(u)a \in G$. Then there exists some $X_0 \subset D$ such that $\varphi(u)aG = \varphi(X_0)_v$, and therefore $aG = \varphi(u)^{-1}\varphi(X_0)_v = \phi(u^{-1}X_0)_v$.

(b) $\Rightarrow$ (a) If $a \in G$ and $X \subset K$ is such that $aG = \phi(X)_v$, then 2. implies that $X \subset \phi^{-1}(aG) \subset D$. □

**Definition 2.4.** A divisor homomorphism $\varphi\colon D \to G$ is called a GCD-*theory* [ *of finite type*] if $G$ is a GCD-monoid, and for every $a \in G$ there exists a [finite] subset $X \subset D$ such that $aG = \varphi(X)_v$. A GCD-theory of finite type is also called a *quasi divisor theory.*

## 3. Prüfer-like monoids and domains

*Let $D$ be a monoid and $K = \mathsf{q}(D)$ its total quotient monoid.*

We recall the definition of several classes of monoids (and domains) which are defined by invertibility properties of their ideals and which we are going to characterize by means of GCD-theories and related concepts.

**Definition 3.1.** Let $D$ be a monoid, and let $r$ and $q$ be ideal systems on $D$ such that $r \le q$.

1. $D$ is called an $(r,q)$-*Dedekind monoid* if $\mathcal{F}_q(D)^\bullet \subset \mathcal{F}_r(D)^\times$ [that is, every non-zero fractional $q$-ideal is $r$-invertible].

   $D$ is called an $r$-*Dedekind monoid* it it is an $(r,r)$-Dedekind monoid.
2. $D$ is called an $(r,q)$-*Prüfer monoid* if $\mathcal{F}_{q,\mathfrak{f}}(D)^\bullet \subset \mathcal{F}_r(D)^\times$ [that is, every non-zero fractional $q$-finite $q$-ideal is $r$-invertible].

   $D$ is called an $r$-*Prüfer monoid* if and only if it is an $(r,r)$-Prüfer monoid.

For any property **P** of monoids we say that an integral domain $D$ is a **P**-*domain* if its multiplicative monoid is a **P**-monoid.

Let assumptions be as in Definition 3.1. Clearly, $D$ is an $(r,q)$-Dedekind monoid if every (integral) non-zero $q$-ideal is $r$-invertible, and it is an $(r,q)$-Prüfer monoid if every (integral) non-zero $q$-finite $q$-ideal is $r$-invertible. If $D$ is an $r$-Dedekind [ $r$-Prüfer ] monoid, then $D$ is an $(r,q)$-Dedekind [ $(r,q)$-Prüfer ] monoid, and if $D$ is an $(r,q)$-Dedekind [ $(r,q)$-Prüfer ] monoid, then $D$ is a $q$-Dedekind [ $q$-Prüfer ] monoid. If $r$ is finitary, then $D$ is an $(r,q)$-Prüfer monoid if and only if it is an $(r,q_{\mathfrak f})$-Prüfer monoid. In particular, $D$ is a $(t,v)$-Prüfer monoid if and only if $D$ is a $t$-Prüfer monoid.

For integral domains, the concepts of Definition 3.1 were introduced in [2], the monoid case was investigated in [11]. The definitions of $r$-Prüfer and $r$-Dedekind monoids (resp. domains) coincide with those given in [10, §17 and §23]. The definition of $r$-Dedekind domains given in [2] coincides with ours only if $r$ is finitary.

A $v$-Dedekind monoid is a completely integrally closed monoid [10, Theorem 14.1], a $t$-Dedekind monoid is a Krull monoid [10, Theorem 23.4], and an $(r,v)$-Prüfer monoid is an $r$-GCD-monoid [10, Def. 17.6]. Consequently, a $v$-Dedekind domain is a completely integrally closed domain, a $t$-Dedekind domain is a Krull domain, and a $d$-Dedekind domain is just a Dedekind domain. A $v$-Prüfer domain is a $v$-domain (that is, a regularly integrally closed domain in the sense of [4, Ch. VII, §1, Ex. 30, 31] ), a $t$-Prüfer domain is a P$v$MD (that is, a pseudo-Prüfer domain in the sense of [4, Ch. VII, §2, Ex. 19] ), and a $d$-Prüfer domain is just a Prüfer domain. A $(d,v)$-Prüfer domain is a GGCD-domain (generalized GCD-domain, see [10, Def. 17.6] ). A $(d,v)$-Dedekind domain is a pseudo-Dedekind domain (introduced in [17] under the name "Generalized Dedekind domains" and thorough investigated in [3] ), and a $(t,v)$-Dedekind domain is a pre-Krull domain (introduced in [18]).

**Definition 3.2.** Let $D$ be a monoid and $r$ an ideal system on $D$. For $A,\,B\in\mathcal F_r(D)^\times$, we define $A\le B$ if $B\subset A$, and we consider the following sets of $r$-ideals:

- $\mathcal I_r^*(D)=\mathcal I_r(D)\cap\mathcal F_r(D)^\times$, $\mathcal I_r^*(D)_0=\mathcal I_r^*(D)\cup\{\mathbf 0\}$, $\mathcal F_r(D)_0^\times=\mathcal F_r(D)^\times\cup\{\mathbf 0\}$,
- $\Lambda_r(D)=\{(C^{-1}A)_r\mid C,\,A\in\mathcal F_{v,\mathfrak f}(D)\cap\mathcal F_r(D)^\times\}\cup\{\mathbf 0\}\subset\mathcal F_r(D)_0^\times$ and
- $\Lambda_r^+(D)=\{(C^{-1}A)_r\mid C,\,A\in\mathcal F_{v,\mathfrak f}(D)\cap\mathcal F_r(D)^\times,\ A\subset C\}\cup\{\mathbf 0\}\subset\mathcal I_r^*(D)_0$.

By the very definition, we have $\mathcal I_{v,\mathfrak f}(D)\cap\mathcal F_r(D)_0^\times\subset\Lambda_r^+(D)\subset\mathcal F_r(D)_0^\times\subset\mathcal F_v(D)$, and

$$\Lambda_r^+(D)=\{(C^{-1}A)_r\mid C,\,A\in\mathcal F_{v,\mathfrak f}(D)\cap\mathcal F_r(D)^\times,\ C\subset A\subset D\}\cup\{\mathbf 0\}$$

(since for every $B\in\mathcal F_r(D)$ there is some $b\in D^\bullet$ such that $bB\subset D$).

$\mathcal I_r^*(D)_0$ is a reduced monoid with total quotient monoid $\mathsf q(\mathcal I_r^*(D)_0)=\mathcal F_r(D)_0^\times$, and $\Lambda_r^+(D)\subset\mathcal I_r^*(D)_0$ is a submonoid with quotient monoid $\mathsf q(\Lambda_r^+(D))=\Lambda_r(D)$ [ indeed, if $A,\,C\in\mathcal F_{v,\mathfrak f}(D)\cap\mathcal F_r(D)^\times$, let $a\in D^\bullet$ be such that $aA,\,aC\in\mathcal I_{v,\mathfrak f}(D)\cap\mathcal F(D)_0^\times\subset\Lambda_r^+(D)$, and observe that $(C^{-1}A)_r=((aC)^{-1}(aA))_r$ ]. The group $\Lambda_r(D)^\bullet$ and the monoid $\Lambda_r^+(D)$ are modifications of the Lorenzen $r$-group and the Lorenzen $r$-monoid (see [10, §19]).

The *canonical divisor homomorphism* $\partial\colon D^\bullet\to\mathcal I_r(D)$ is defined by $\partial(a)=aD$. It is easily checked that $\partial$ is indeed a divisor homomorphism, it satisfies $\partial(D^\bullet)\subset\mathcal I_r^*(D)$, and its quotient homomorphisms induces the exact sequence

$$1 \;\to\; D^\times \;\to\; \mathsf q(D)^\bullet \;\overset{\mathsf q(\partial)}{\to}\; \mathcal F_r(D)^\times \;\to\; \mathcal C_r(D) \;\to\; \mathbf 0\,,$$

where $\mathcal C_r(D)$ denotes the $r$-class group.

$(\mathcal F_r(D)^\times, \le)$ resp. $(\Lambda_r(D)^\bullet, \le)$ are partially ordered abelian groups which are lattice-ordered [ completely lattice-ordered ] if and only if $\mathcal I_r^*(D)_0$ resp. $\Lambda_r^+(D)$ are [ complete ] GCD-monoids.

**Lemma 3.3.** *Let $D$ be a monoid, $r$ an ideal system on $D$ and $X, Y \in \Lambda_r^+(D)$. Then there exist $A, B, C \in \mathcal F_{v,\mathsf f}(D) \cap \mathcal F_r(D)^\times$ such that $A \cup B \subset C$, $X = (C^{-1}A)_r$ and $Y = (C^{-1}B)_r$.*

*Proof.* By definition, $X = (C_1^{-1}A_1)_r$ and $Y = (C_2^{-1}A_2)_r$, where $A_i$, $C_i \in \mathcal F_{v,\mathsf f}(D) \cap \mathcal F_r(D)^\times$ and $A_i \subset C_i$ for $i \in \{1,2\}$. Therefore it follows that $X = [(C_1C_2)_r^{-1}(C_2A_1)_r]_r$ and $Y = [(C_1C_2)_r^{-1}(C_1A_2)_r]_r$, whereupon $(C_1C_2)_r$, $(C_2A_1)_r$, $(C_1A_2)_r \in \mathcal F_{v,\mathsf f}(D) \cap \mathcal F_r(D)^\times$ and $(C_2A_1)_r \cup (C_1A_2)_r \subset (C_1C_2)_r$. □

**Theorem 3.4.** *Let $D$ be a monoid, $r$ an ideal system on $D$ and $\Gamma \subset \mathcal I_r^*(D)_0$ a submonoid containing all principal ideals such that, for all $A, B \in \Gamma^\bullet$, $A \subset B$ implies $(B^{-1}A)_r \in \Gamma$ [ the main examples are $\Gamma = \mathcal I_r^*(D)_0$ and $\Gamma = \Lambda_r^+(D)$ ].*

1. *If $\Omega \subset \Gamma$ and $C \in \Gamma$, then $C = \gcd(\Omega)$ if and only if $C = \big(\bigcup_{J\in\Omega} J\big)_v$. In particular, $\Gamma$ is a GCD-monoid if and only if $(A \cup B)_v \in \Gamma$ holds for all $A, B \in \Gamma^\bullet$.*
2. *If $X \subset D$, $I = X_v \in \Gamma$ and $X^\# = \{aD \mid a \in X\} = \partial(X) \subset \Gamma$, then $I = \gcd(X^\#)$.*
3. *Let $\Gamma$ be a GCD-monoid. Then:*
   - (a) *If $A, B \in \Gamma$, then $(A \cup B)_v = \gcd(A,B)$ and $A \cap B = \operatorname{lcm}(A,B)$.*
   - (b) *$\mathcal I_{v,\mathsf f}(D) \subset \Gamma$ and $D$ is an $(r,v)$-Prüfer monoid.*
   - (c) *$\Gamma$ is complete if and only if $\mathcal I_v(D) \subset \Gamma$.*
   - (d) *The canonical divisor homomorphism $\partial\colon D \to \Gamma$ is a GCD-theory. Moreover, for all $I \in \Gamma$ and all subsets $X \subset D$ we have $\gcd(\partial(X)) = I$ if and only if $I = X_v$.*

*Proof.* Note that $\Gamma$ is reduced, and $\Gamma \subset \mathcal I_v^*(D)_0 \subset \mathcal I_v(D)$. For $A, B \in \Gamma$ we have $B \mid A$ (that is, $A \in B\Gamma$) if and only if $A \subset B$.

1. Let $\Omega \subset \Gamma$ and $Z = \bigcup_{J\in\Omega} J$. Suppose first that $C = \gcd(\Omega)$. Then $J \subset C$ for all $J \in \Omega$, hence $Z \subset C$ and therefore $Z_v \subset C$. On the other hand, if $a \in D$ is such that $Z \subset aD$, then $J \subset aD$ for all $J \in \Omega$, hence $C \subset aD$, and since $Z_v$ is the intersection of all principal ideals of $D$ containing $Z$, it follows that $C = Z_v$.

For the converse, suppose that $C = Z_v$. For all $I \in \Gamma$ we have $C \subset I$ if and only if $J \subset I$ for all $J \in \Omega$, that is, $I$ is a common divisor of $\Omega$ if and only if $I$ divides $C$. Hence $C = \gcd(\Omega)$.

2. By 1., since $I = \big(\bigcup_{a\in X} aD\big)_v = \big(\bigcup_{J\in X^\#} J\big)_v$.

3.(a) We may assume that $A,\, B \in \Gamma^\bullet$. We have $(A \cup B)_v = \gcd(A,B)$ by 1., and $[AB(A^{-1} \cup B^{-1})]_v = (A \cup B)_v$. Hence it follows that

$$(AB)_v = [(A^{-1} \cup B^{-1})^{-1}(A \cup B)]_v = [(A \cap B)(A \cup B)]_v\,,$$

and consequently $A \cap B = (AB)_v (A \cup B)_v^{-1} = (AB)_v \gcd(A,B)^{-1} = \mathrm{lcm}(A,B)$.

(b) If $I \in \mathcal{I}_{v,\mathsf{f}}(D)$, then $I = E_v$ for some finite subset $E \subset D$, and $I = \gcd(\partial(E)) \in \Gamma$. Since $\mathcal{I}_{v,\mathsf{f}}(D)^\bullet \subset \Gamma^\bullet \subset \mathcal{I}_r^*(D)$, if follows that every $v$-finite $v$-ideal is $r$-invertible, and thus $D$ is an $(r,v)$-Prüfer monoid.

(c) If $\mathcal{I}_v(D) \subset \Gamma$, then $\Gamma$ is complete by (a). If $\Gamma$ is complete and $I \in \mathcal{I}_v(D)$, then $I = \gcd(\partial(I)) \in \Gamma$.

(d) By 2., $\partial$ is a GCD-theory. Let $X \subset D$ and $I \in \Gamma$ be given. If $I = X_v$, then $I = \gcd(\partial(X))$ by 2. Conversely, if $I = \gcd(\partial(X))$, then $I\Gamma = \partial(X)_{v(\Gamma)}$, and by Theorem 2.3.3 it follows that $X_v = \partial^{-1}(I\Gamma) = \{a \in D \mid aD \in I\Gamma\} = \{a \in D \mid aD \subset I\} = I$. □

**Theorem 3.5.** *Let $r$ be an ideal system on a monoid $D$.*

1. *$D$ is an $(r,v)$-Prüfer monoid if and only if $\Lambda_r^+(D)$ is a* GCD-*monoid* [ *equivalently, $\Lambda_r(D)^\bullet$ is a lattice-ordered group* ].
2. *$D$ is an $(r,v)$-Dedekind monoid if and only if $\mathcal{I}_r^*(D)_0$ is a complete* GCD-*monoid* [ *equivalently, $\mathcal{F}_r(D)^\times$ is a complete lattice-ordered group* ].
3. *If $D$ is an $r$-Prüfer monoid, then $\mathcal{I}_r^*(D)_0$ and $\Lambda_r^+(D)$ are* GCD-*monoids.*

*Proof.* The assertions concerning lattice-ordered groups follow by Lemma 2.2.

1. If $\Lambda_r^+(D)$ is a GCD-monoid, then $D$ is an $(r,v)$-Prüfer monoid by Theorem 3.4.3(a). Thus let $D$ be an $(r,v)$-Prüfer monoid and $A,\, B \in \Lambda_r^+(D)$. By Theorem 3.4.1, we must prove that $(A \cup B)_v \in \Lambda_r^+(D)$, and we may assume that $A,\, B \in \Lambda_r^+(D)^\bullet$. By Lemma 3.3, there exist $U,\, V,\, C \in \mathcal{F}_{v,\mathsf{f}}(D) \cap \mathcal{F}_r(D)^\times$ such that $A = (C^{-1}U)_r$, $B = (C^{-1}V)_r$ and $U \cup V \subset C$. Then $(A \cup B)_v = [C^{-1}(U \cup V)_v]_v$, and since $U,\, V \in \mathcal{F}_{v,\mathsf{f}}(D)^\bullet$, it follows that $(U \cup V)_v \in \mathcal{F}_{v,\mathsf{f}}(D)^\bullet \subset \mathcal{F}_r(D)^\times$. Hence we obtain $(A \cup B)_v = [C^{-1}(U \cup V)_v]_r \in \Lambda_r^+(D)$.

2. If $D$ is an $(r,v)$-Dedekind monoid and $\Omega \subset \mathcal{I}_r^*(D)_0$, then

$$C = \Big(\bigcup_{J \in \Omega} J\Big)_v \in \mathcal{I}_v(D) \subset \mathcal{I}_r^*(D)_0\,,$$

and $C = \gcd(\Omega)$ by Theorem 3.4.1. Hence $\mathcal{I}_r^*(D)_0$ is a complete GCD-monoid. Conversely, if $\mathcal{I}_v^*(D)_0$ is a complete GCD-monoid, then $\mathcal{I}_v(D) \subset \mathcal{I}_v^*(D)$ by Theorem 3.4.3(b), and thus $D$ is an $(r,v)$-Dedekind monoid.

3. Let $D$ be an $r$-Prüfer monoid. Then $D$ is an $(r,v)$-Prüfer monoid, and thus $\Lambda_r^+(D)$ is a GCD-monoid by 1. If $A,\, B \in \mathcal{I}_r^*(D)$, then $(A \cup B)_r \in \mathcal{I}_r^*(D)$ by [11, Theorem 5.1(h)], hence $(A \cup B)_v = (A \cup B)_r$, and thus $\mathcal{I}_r^*(D)$ is a GCD-monoid by Theorem 3.4.1. □

**Corollary 3.6.** *For a monoid $D$, the following assertions are equivalent*:

(a) *$D$ is a $v$-Prüfer monoid.*

(b) $\mathcal{I}_v^*(D)_0$ *is a* GCD-*monoid* [*equivalently,* $\mathcal{F}_v(D)^\times$ *is a lattice-ordered group*].

(c) $\Lambda_v^+(D)$ *is a* GCD-*monoid* [*equivalently,* $\Lambda_v(D)^\bullet$ *is a lattice-ordered group*].

*Proof.* (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) By Theorem 3.5.3.

(b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) By Theorem 3.4.3(b). □

**Corollary 3.7.** *Let $D$ be a monoid and $r$ an ideal system on $D$ such that $D$ is an $r$-Prüfer monoid. Then $\mathcal{F}_r(D)^\times$ and $\Lambda_r(D)^\bullet$ are lattice-ordered groups, the canonical divisor homomorphisms $\partial\colon D \to \Lambda_r^+(D)$ and $\partial\colon D \to \mathcal{I}_r^*(D)_0$ are* GCD-*theories, and for all $A,\ B \in \mathcal{F}_r(D)^\times$ [for all $A,\ B \in \Lambda_r(D)$] we have $\sup(A,B) = A \cap B$ and $\inf(A,B) = (A \cup B)_v$.*

*Proof.* By Theorem 3.5.3, Theorem 3.4.3(a) and Lemma 2.2. □

**Theorem 3.8.** *Let $D$ be a monoid and $r$ a finitary ideal system on $D$. Then $D$ is an $(r,v)$-Prüfer monoid if and only if $\mathcal{I}_r^*(D)_0$ is a* GCD-*monoid* [*equivalently,* $\mathcal{F}_r(D)^\times$ *is a lattice-ordered group*].

*Proof.* If $\mathcal{I}_r^*(D)_0$ is a GCD-monoid, then $D$ is an $(r,v)$-Prüfer monoid by Theorem 3.4.3(b). Thus let $D$ be an $(r,v)$-Prüfer monoid and $A,\ B \in \mathcal{I}_r^*(D)$. Since $r$ is finitary, it follows that $A$ and $B$ are $r$-finite, since they are $r$-invertible. Consequently, $(A \cup B)_r$ is $r$-finite, hence $(A \cup B)_v$ is $v$-finite and thus $r$-invertible, whence $(A \cup B)_v \in \mathcal{I}_r^*(D)$. By Theorem 3.4.1, $\mathcal{I}_r^*(D)_0$ is a GCD-monoid. □

**Corollary 3.9.** *For a monoid $D$, the following assertions are equivalent*:

(a) *$D$ is a $t$-Prüfer monoid.*

(b) $\mathcal{I}_t^*(D)_0$ *is a* GCD-*monoid* [*equivalently,* $\mathcal{F}_t(D)^\times$ *is a lattice-ordered group*].

(c) $\Lambda_t^+(D)$ *is a* GCD-*monoid* [*equivalently,* $\Lambda_t(D)^\bullet$ *is a lattice-ordered group*].

*Proof.* (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) By Theorem 3.5.3.

(c) $\Rightarrow$ (a) By Theorem 3.5.1.

(b) $\Rightarrow$ (a) By Theorem 3.8, $D$ is a $(t,v)$-Prüfer monoid, and thus it is a $t$-Prüfer monoid. □

**Remark 3.10.** Several special cases of the preceding theorems and corollaries are well known in the case of an integral domain $D$.

- $D$ is a PvMD if and only if $\mathcal{F}_t(D)^\times$ is a lattice-ordered group [13], [9], [19].
- $D$ is a G-GCD domain if and only if $\mathcal{F}_d(D)^\times$ is a lattice-ordered group [1].
- $D$ is a $v$-domain if and only if $\mathcal{F}_v(D)^\times$ is a lattice-ordered group [2].
- $D$ is an $r$-Prüfer domain if and only if $\mathcal{I}_r^*(D)$ is a GCD-monoid and $(A \cup B)_r = (A \cup B)_v$ for all $A,\ B \in \mathcal{I}_r^*(D)$ [2].

- $D$ is a pseudo-Dedekind domain if and only if $\mathcal{F}_d(D)^\times$ is a lattice-ordered group [3].
- If $D$ is a pre-Krull domain, then $\mathcal{F}_t(D)^\times$ is a lattice-ordered group [2].

We close with a fresh proof of the characterization of $v$-Prüfer and $t$-Prüfer monoids by means of GCD-theories. More details may be found in [10, Ch. 20] and (using the language of valuation theory) in [7].

**Theorem 3.11.** *Let $D$ be a monoid.*

1. *$D$ possesses a* GCD*-theory if and only if $D$ is a $v$-Prüfer monoid.*
2. *$D$ possesses a* GCD*-theory of finite type if and only if $D$ is a $t$-Prüfer monoid.*

*Proof.* 1. If $D$ is a $v$-Prüfer monoid, then $\mathcal{I}_v^*(D)_0$ and $\Lambda_v^+(D)$ are GCD-monoids by Corollary 3.6, and thus the canonical divisor homomorphisms $\partial\colon D \to \mathcal{I}_v^*(D)_0$ and $\partial\colon D \to \Lambda_v^+(D)$ are GCD-theories by Theorem 3.4.3(d).

Let now $\varphi\colon D \to G$ be a GCD-theory, $K = \mathsf{q}(D)$, $L = \mathsf{q}(G)$ and $\phi = \mathsf{q}(\varphi)\colon K \to L$. Let $J \in \mathcal{F}_{v,\mathsf{f}}(D)^\bullet$, say $J = E_v$ for some finite set $E \subset K$. By Theorem 2.3, it follows that $\phi(E)_v = aG$ for some $a \in L$, there exists a subset $X \subset K$ such that $\phi(X)_v = a^{-1}G$, and $(JX)_v = D$, whence $J$ is $v$-invertible.

2. If $D$ is a $t$-Prüfer monoid, then $\mathcal{I}_t^*(D)_0$ is a GCD-monoid by Corollary 3.9, and thus the canonical divisor homomorphism $\partial\colon D \to \mathcal{I}_t^*(D)_0$ is a GCD-theory by Theorem 3.4.3(d). If $J \in \mathcal{I}_t^*(D)$, then $J = E_v$ for some finite subset $E \subset D$ and $J\mathcal{I}_t^*(D)_0 = \phi(E)_v$. Hence $\partial$ is a GCD-theory of finite type.

Let now $\varphi\colon D \to G$ be a GCD-theory of finite type, $K = \mathsf{q}(D)$, $L = \mathsf{q}(G)$ and $\phi = \mathsf{q}(\varphi)\colon K \to L$. Let $J \in \mathcal{F}_{t,\mathsf{f}}(D)^\bullet$, say $J = E_t$ for some finite set $E \subset K$. By Theorem 2.3, it follows that $\phi(E)_v = aG$ for some $a \in L$, there exists a finite subset $X \subset K$ such that $\phi(X)_v = a^{-1}G$, and $D = (JX)_v = (EX)_v = (EX)_t$, whence $J$ is $t$-invertible. □

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