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IDEAL SEMIGROUPS OF NOETHERIAN DOMAINS AND PONIZOVSKI DECOMPOSITIONS

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1. Introduction

While the Picard group and the divisor class group of an integral domain are classical objects of interest in commutative algebra, the semigroup of all ideal classes has only recently attracted some interest. An older paper by E.C. Dade, O. Taussky and H. Zassenhaus [4] seems to have fallen into oblivion. The main interest in recent investigations was the question whether the class semigroup of an integral domain is a Clifford semigroup (in this case we call the domain *Clifford regular*).

In [13], it was reproved that orders in quadratic number fields are Clifford regular, while every algebraic number field of higher degree contains orders which are not Clifford regular. It was also proved there that a Clifford regular integrally closed domain is a Prüfer domain. In [2], it was proved that every valuation domain is Clifford regular, and S. Bazzoni [1] succeeded in characterizing all Clifford regular Prüfer domains. More generally, the Clifford regularity of Mori domains and t-class semigroup analogs are investigated in [9].

In this note, we continue the investigations of [4] and put them into the context of the structure theory of commutative semigroups as presented in [7] and [8]. Thereby we use the notion of lattices as in [12] (also called complete modules in [10]). The (partial) Ponizovski factors turn out to be the appropriate semigroup-theoretic notion to describe the multiplicative structure of lattices over one-dimensional (and in particular over Dedekind) domains.

In Section 2 we refer and complement the basics from the structure theory of commutative semigroups as far as this is needed for our purposes. In Section 3 we describe the multiplicative semigroup of lattices over one-dimensional domains, and in Section 4 we apply these results to determine the structure of the corresponding class semigroups. In Section 5 we use the concept of Dedekind's complementary modules to present some criteria for the existence of groups inside the lattice semigroup. In the context of orders in algebraic number fields, these criteria were proved in [6].

We denote by $\mathbb{N}$ the set of positive integers, we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and for $a,\, b \in \mathbb{Z}$ with $a \le b$, we set $[a,b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$.

2. Ponizovski decompositions of commutative semigroups

Throughout this section, let S be a commutative semigroup.

Our main reference for the theory of commutative semigroups is [7] (undefined notions are used as there). We use multiplicative notation. If S contains a unit element, we denote it by 1 and set $S^1 = S$. If S does not contain a unit element, we denote by $S^1 = S \cup \{1\}$ the semigroup built from S by adjoining a unit element. If S contains a zero element, we denote it by 0 (it satisfies $0x = 0$ for all $x \in S^1$). For subsets $A,\, B \subset S$ and $a \in S$ we set $AB = \{xy \mid x \in A,\, y \in B\}$ and $aB = \{ay \mid y \in B\}$.

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A subset $I \subset S$ is called an *ideal* if $SI \subset I$. For $a \in S$, the principal ideal S^1a is the smallest ideal containing a. If $I \subset S$ is an ideal, we define the *Rees quotient* to be the semigroup $S/I = (S \setminus I) \cup \{0\}$, where the product xy is defined as in S if x, y and xy all belong to $S \setminus I$, and $xy = 0$ otherwise.

For a congruence relation $\mathcal{C}$ on S, we denote by $S/\mathcal{C}$ the quotient semigroup, and for $a \in S$ we denote by $[a]_\mathcal{C} \in S/\mathcal{C}$ the congruence class of a. For $a, b \in S$ we define *Green's relation* $\mathcal{H}$ and the *archimedean relation* $\mathcal{N}$ by

$$a\mathcal{H}b \ \text{ if } \ aS^1 = bS^1\,, \quad \text{and} \quad a\mathcal{N}b \ \text{ if } \ a^n \in bS^1 \ \text{ and } \ b^m \in aS^1 \ \text{ for some } \ m, n \in \mathbb{N}\,.$$

$\mathcal{H}$ and $\mathcal{N}$ are congruence relations on S. For $a \in S$, the congruence class $[a]_\mathcal{N}$ is called the *archimedean component* of a. The semigroup S is called *archimedean* if it consists of a single archimedean component. If S is any semigroup and $a \in S$, then $[a]_\mathcal{N}$ is the largest archimedean subsemigroup of S containing a. Every archimedean component and every ideal of S is composed of $\mathcal{H}$-classes.

Let $\mathsf{E}(S)$ denote the set of all idempotents of S, endowed with the Rees order $\le$, defined by $e \le f$ if $ef = e$. Every archimedean component of S contains at most one idempotent. An $\mathcal{H}$-class $[a]_\mathcal{H} \in S/\mathcal{H}$ is a subgroup of S if and only if it contains an idempotent, and the set $\big\{[e]_\mathcal{H} \bigm| e \in \mathsf{E}(S)\big\}$ is the set of all maximal subgroups of S (see [7, Corollary I.4.5]). For $a \in S$, we define $\mathsf{E}(a) = \{e \in \mathsf{E}(S^1) \mid ae = a\}$, whence $\mathsf{E}(a) = \{e \in \mathsf{E}(S^1) \mid a \in Se\}$.

An element $a \in S$ is called *regular* if a lies in a subgroup of S. It is easily checked that a is regular if and only if there exist elements $b \in S$ and $e \in \mathsf{E}(S)$ such that $ab = e$, $ae = a$ and $be = b$. Indeed, e is the unit element and b is the inverse of a in the subgroup containing a. In particular, e and b are uniquely determined by a. We call b the *inverse* and e the *idempotent* of a.

Lemma 2.1. *If $a \in S$ is regular and e is the idempotent of a, then $e = \min \mathsf{E}(a)$.*

Proof. Let $b \in S$ be such that $ab = e$ and $f \in \mathsf{E}(a)$. Then $e = ab = fab = fe \le f$. □

The semigroup S is called *regular* or a *Clifford semigroup* if every element of S is regular. Thus S is a Clifford semigroup if and only if S is the disjoint union of its maximal subgroups.

An element $a \in S$ is called *π-regular* if there exists some $n \in \mathbb{N}$ such that a^n is regular. The semigroup S is called *π-regular* if every element of S is π-regular.

It is well known that S is regular if and only if every $\mathcal{H}$-class contains an idempotent, and S is π-regular if and only if every archimedean component contains an idempotent (see [7, Corollaries I.4.5 and III.3.2]). An element $a \in S$ is [π-]regular if and only if its $\mathcal{H}$-class $[a]_\mathcal{H} \in S/\mathcal{H}$ is [π-]regular. Consequently, S is [π-]regular if and only if $S/\mathcal{H}$ is [π-]regular. Note that $S/\mathcal{H}$ is regular if and only if it is a semilattice.

The following lemma gives more information on the structure of π-regular semigroups.

Lemma 2.2. *Let $a \in S$ and $n \in \mathbb{N}$ be such that a^n lies in a maximal subgroup G of S. Then $a^m \in G$ for all $m \ge n$. In particular, if a^n is regular, then a^m is regular for all $m \ge n$.*

Proof. We have $G = [e]_\mathcal{H}$ for some $e \in \mathsf{E}(S)$, and it suffices to prove that $a^{n+1} \in G$. Since $a \in [e]_\mathcal{N}$ and $a^n \in [e]_\mathcal{H}$, we obtain $a^n = eu$, $e^m = e = av$ and $e = a^n t$ for some $m \in \mathbb{N}$ and $u, v, t \in S^1$. Hence it follows that $a^{n+1} = eau \in eS^1$ and $e = a^{n+1}vt \in a^{n+1}S^1$, whence $a^{n+1} \in [e]_\mathcal{H}$. □

In the following we define the Ponizovski factors of a semigroup not only for complete semigroups as in [7] but under more general assumptions. This enables us to characterize complete semigroups by the structure of their Ponizovski factors.

Definition 2.3. For an idempotent $e \in \mathsf{E}(S^1)$, we define the *Ponizovski factor* P_e and the *partial Ponizovski factor* P_e^* by

$$P_e = Se \Big/ \Big(\bigcup_{\substack{f \in \mathsf{E}(S) \\ f < e}} Sf \Big) \quad \text{and} \quad P_e^* = P_e \setminus \{0\} = Se \setminus \Big(\bigcup_{\substack{f \in \mathsf{E}(S) \\ f < e}} Sf \Big) .$$

If $a \in S$ is regular, then $e = \min \mathsf{E}(a)$ by Lemma 2.1. In general however, $\mathsf{E}(a)$ need not have a minimum. We call S *almost complete* if for every $a \in S$ the set $\mathsf{E}(a)$ has a minimum. As usual, we call S *complete* if it is π-regular and almost complete. Note that S is (almost) complete if and only if $S/\mathcal{H}$ is (almost) complete. The (partial) Ponizovski factors are composed of $\mathcal{H}$-classes, and if $\rho \colon S \to S/\mathcal{H}$ denotes the canonical homomorphism, then $\rho(P_e^*) = P_{\rho(e)}^*$ for all $e \in \mathsf{E}(S^1)$.

If $a \in S$ and $e \in \mathsf{E}(S^1)$, then $a \in P_e^*$ if and only if $e = \min \mathsf{E}(a)$. Consequently, S is almost complete if and only if it is the union of its partial Ponizovsky factors $\{P_e^* \mid e \in \mathsf{E}(S^1)\}$. If $e,\, f \in \mathsf{E}(S^1)$ and $e \neq f$, then $P_e^* \cap P_f^* = \emptyset$, and if $e \in \mathsf{E}(S)$, then $[e]_{\mathcal{H}} \subset P_e^*$. In particular, $P_e^* \neq \emptyset$ for all $e \in \mathsf{E}(S)$, and if $1 \notin S$, then $P_1^* = \emptyset$ if and only if $S = \mathsf{E}(S)S$.

By definition, the Ponizovski factors P_e are semigroups with zero. If S is almost complete, then it is a subdirect product of its Ponizovski factors. Indeed, if for $e \in \mathsf{E}(S^1)$ the canonical projections $\pi_e \colon S \to P_e$ are defined by

$$\pi_e(x) = x \quad \text{if} \quad x \in P_e^* \,, \quad \text{and} \quad \pi_e(x) = 0 \quad \text{otherwise} \,,$$

then the family $\{\pi_e \mid e \in \mathsf{E}(S^1)\}$ separates the points of S, that is, if $x,\, y \in S$ and $\pi_e(x) = \pi_e(y)$ for all $e \in \mathsf{E}(S^1)$, then $x = y$.

It is now easy to characterize Clifford semigroups and complete semigroups by their partial Ponizovski factors.

Theorem 2.4. *S is a Clifford semigroup if and only if S is almost complete and all non-empty partial Ponizovski factors are groups.*

Proof. Observe that, for every $e \in \mathsf{E}(S)$, the partial Ponizovski factor P_e^* is a group if and only if $P_e^* = [e]_{\mathcal{H}}$. □

The semigroup S is called *elementary* if there exists a nilsemigroup $N \subset S$ such that either $N = S$ or $S \setminus N$ is a group. It is well known that the Ponizovski factors of a complete semigroup are elementary (see [7, Proposition IV.4.5]). With a slight additional condition, the converse is also true.

Theorem 2.5. *An almost complete semigroup S is complete if and only if all Ponizovski factors are elementary semigroups and, for every $a \in S$, the set $\{\min \mathsf{E}(a^n) \mid n \in \mathbb{N}\}$ is finite.*

Proof. Let first S be complete. If $a \in S$, then there exist some $n \in \mathbb{N}$ and $e \in \mathsf{E}(S)$ such that $a^n \in [e]_{\mathcal{H}}$. By Lemma 2.2, it follows that $a^m \in [e]_{\mathcal{H}}$ and thus $e = \min \mathsf{E}(a^m)$ for all $m \geq n$. Thus we must prove that all Ponizovski factors are elementary.

Let $e \in \mathsf{E}(S^1)$ and $a \in P_e^*$. Let $n \in \mathbb{N}$ and $f \in \mathsf{E}(S)$ be such that $a^n \in [f]_{\mathcal{H}} \subset P_f$. If $e = 1 \notin S$, then $P_1 = S/\mathsf{E}(S)S$ and thus $a^n = 0$ in P_1. If $e \in S$, then $a = ae$ and $f = a^n u$ for some $u \in S^1$, whence $ef = ea^n u = a^n u = f \leq e$. If $f = e$, then $a \in [e]_{\mathcal{H}}$, and if $f < e$, then $a^n = 0$ in P_e.

Assume now that S is almost complete, all Ponizovski factors are elementary semigroups and all sets $\{\min \mathsf{E}(a^n) \mid n \in \mathbb{N}\}$ for $a \in S$ are finite. Let $a \in S$. Then there exist some $n \in \mathbb{N}$ and $e \in \mathsf{E}(S^1)$ such that $e = \min \mathsf{E}(a^m)$ for all $m \geq n$. Hence it follows that $a^m \in P_e^*$ for all $m \geq n$. In particular, a^n is not nilpotent in P_e and thus lies in a subgroup $G \neq \{0\}$ of P_e. Hence G is a subgroup of S, and thus a^n is regular. □

3. Semigroups of ideals and lattices

Our standard references for the ideal theory of commutatitve rings are [3] and [11]. For an integral domain A, we set $A^\bullet = A \setminus \{0\}$, we denote by $A^\times$ the group of invertible elements of A, by $\mathcal{F}(A)$ the multiplicative semigroup of all non-zero fractional ideals of A, by $\mathcal{F}(A)^\times$ the subgroup of all A-invertible fractional ideals and by $\mathcal{E}(A)$ the set of all fractional ideals of A which are overrings of A.

Throughout this section, let R be a noetherian integral domain,
K its field of quotients and $L \supset K$ a finite extension field.

By an *R-lattice* in L we mean a finitely generated R-submodule of L which contains a K-basis of L, and we denote by $\mathcal{F}_L(R)$ the set of all R-lattices in L. Since $\mathcal{F}_K(A) = \mathcal{F}(A)$, the concept of R-lattices generalizes that of fractional ideals. A finitely generated R-module $\mathfrak{a} \subset L$ lies in $\mathcal{F}_L(R)$ if and only if for every $z \in L$ there exists some $q \in R^\bullet$ such that $qz \in \mathfrak{a}$. By an *R-order* in L we mean a subring $\Lambda \subset L$ which is an R-lattice in L. We denote by $\mathcal{E}_L(R)$ the set of all R-orders in L, whence $\mathcal{E}(R) = \mathcal{E}_K(R)$. If $\mathfrak{a}$ and $\mathfrak{b}$ are R-lattices in L, then

$$\mathfrak{a}+\mathfrak{b}\,, \quad \mathfrak{a}\cap\mathfrak{b}\,, \quad \mathfrak{a}\mathfrak{b} = {}_R\big(\{ab \mid a \in \mathfrak{a},\, b \in \mathfrak{b}\}\big) \quad \text{and} \quad (\mathfrak{a}:\mathfrak{b}) = \{z \in L \mid z\mathfrak{b} \subset \mathfrak{a}\}$$

are also R-lattices in L, and $R(\mathfrak{a}) = (\mathfrak{a}:\mathfrak{a})$ is an R-order in L (usually called the ring of endomorphism or the ring of multipliers of $\mathfrak{a}$). If $\Lambda \in \mathcal{E}_L(R)$, then $\mathcal{F}(\Lambda) \subset \mathcal{F}_L(R)$ is a subsemigroup. An R-lattice $\mathfrak{a}$ is called *Λ-invertible* if $\mathfrak{a} \in \mathcal{F}(\Lambda)^\times$, that is, if $\Lambda\mathfrak{a} = \mathfrak{a}$ and $\mathfrak{a}\,(\Lambda:\mathfrak{a}) = \Lambda$.

$\mathcal{F}_L(R)$ is a multiplicative semigroup, and if $\Lambda \in \mathcal{E}_L(R)$, then $\mathcal{F}(\Lambda) \subset \mathcal{F}_L(R)$ is a subsemigroup. In the sequel, we shall need the following variant of Nakayama's lemma.

Lemma 3.1. *Let $A \subset B$ be commutative rings, let $\mathfrak{a}$, $\mathfrak{b} \subset B$ be A-submodules such that $\mathfrak{b}$ is finitely generated, $\mathrm{Ann}_A(\mathfrak{b}) = 0$, $\mathfrak{a}\mathfrak{b} = \mathfrak{b}$ and $\mathfrak{a}^2 \subset \mathfrak{a}$. Then $\mathfrak{a} \subset B$ is a subring.*

Proof. Let $\mathfrak{b} = {}_A(b_1, \ldots, b_m)$. Then $\mathfrak{a}\mathfrak{b} = \mathfrak{b}$ implies

$$b_j = \sum_{\mu=1}^m a_{j,\mu} b_\mu \quad \text{with} \quad a_{j,\mu} \in \mathfrak{a}\,, \quad \text{whence} \quad \sum_{\mu=1}^m (\delta_{j,\mu} - a_{j,\mu}) b_\mu = 0 \quad \text{for all} \quad j \in [1, m]\,.$$

Thus $\det(\delta_{j,\mu} - a_{j,\mu})$ annihilates $\mathfrak{b}$, which implies $0 = \det(\delta_{j,\mu} - a_{j,\mu}) \equiv 1 \bmod \mathfrak{a}$. Hence $1 \in \mathfrak{a}$ and $\mathfrak{a} \subset B$ is a subring. □

Now we are ready to interpret the semigroup-theoretical notions in the language of ideal theory. Note that $\mathcal{F}_L(R)$ contains a unit element if and only if either $L = K$ or $R = K$ (in the first case R and in the second case L is a unit element).

Theorem 3.2. *Let $\Lambda \in \mathcal{E}_L(R)$ and $\mathfrak{a} \in \mathcal{F}_L(R)$.*

1. $\mathcal{E}_L(R) = \mathsf{E}\big(\mathcal{F}_L(R)\big)$ *is the set of idempotents of* $\mathcal{F}_L(R)$*, and for* Λ_1, $\Lambda_2 \in \mathcal{E}_L(R)$ *we have* $\Lambda_1 \le \Lambda_2$ *if and only if* $\Lambda_1 \supset \Lambda_2$.
2. $\mathcal{F}(\Lambda) = \Lambda\mathcal{F}_L(R)$.
3. $\mathsf{E}(\mathfrak{a}) = \{\Lambda' \in \mathcal{E}_L(R) \mid \Lambda'\mathfrak{a} = \mathfrak{a}\}$*, and* $R(\mathfrak{a}) = \min \mathsf{E}(\mathfrak{a})$*. In particular,* $\mathcal{F}_L(R)$ *is almost complete.*
4. *The partial Ponizovski factor* P_Λ^* *consists of all* $\mathfrak{c} \in \mathcal{F}_L(R)$ *with* $R(\mathfrak{c}) = \Lambda$*, and*
$$\mathcal{F}_L(R) = \bigcup_{\Lambda' \in \mathcal{E}_L(R)} P_{\Lambda'}^*\,.$$
5. *If* $\mathfrak{a}$ *is* Λ*-invertible, then* $\mathfrak{a}$ *is regular and* $\Lambda = R(\mathfrak{a})$.
6. $\mathfrak{a}$ *is regular if and only if* $\mathfrak{a}$ *is* $R(\mathfrak{a})$*-invertible.*

Proof. 1. Clearly, every $\Lambda' \in \mathcal{E}_L(R)$ is idempotent, and if $\Lambda_1,\, \Lambda_2 \in \mathcal{E}_L(R)$, then $\Lambda_1 \le \Lambda_2$ if and only if $\Lambda_1 = \Lambda_1\Lambda_2 \supset \Lambda_2$. By Lemma 3.1, every idempotent of $\mathcal{F}_L(R)$ lies in $\mathcal{E}_L(R)$.

2., 3. and 4. follow from the definitions observing 1. Note that $\mathcal{F}_L(R) = \mathcal{E}_L(R)\mathcal{F}_L(R)$, and thus $P_1^* = \emptyset$ if $L \ne K$.

5. and 6. If $\mathfrak{a}$ is Λ-invertible, then $\mathfrak{a}$ lies in the subgroup $\mathcal{F}_L(R)^\times$. Hence $\mathfrak{a}$ is regular, and by Lemma 2.1 it follows that $\Lambda = \min \mathsf{E}(\mathfrak{a}) = R(\mathfrak{a})$.

Conversely, if $\mathfrak{a}$ is regular, then there exist some $\Lambda' \in \mathcal{E}_L(R)$ and $\mathfrak{b} \in \mathcal{F}_L(R)$ such that $\Lambda'\mathfrak{a} = \mathfrak{a}$, $\Lambda'\mathfrak{b} = \mathfrak{b}$ and $\mathfrak{a}\mathfrak{b} = \Lambda'$. Hence $\mathfrak{a} \in \mathcal{F}(\Lambda')^\times$, and thus $\mathfrak{a}$ is regular. □

The following Main Theorem and the preceding auxiliary lemma on local domains are essentially true by [4]. We present them with shorter proofs.

Lemma 3.3. *Let A be a local noetherian domain with quotient field Q, $A' \in \mathcal{E}(A)$ and $\mathfrak{q} \in \mathcal{F}(A)$ such that $\mathfrak{q}A' = A'$. Then there exist some $N \in \mathbb{N}$, $\mu \in Q^\times$ and $A_1 \in \mathcal{E}(A)$ such that $A \subset A_1 \subset A'$, $\mu^{-1}\mathfrak{q}A_1 = A_1$ and, for all $n \in \mathbb{N}$, $(\mu\mathfrak{q})^n = A_1$ if and only if $n \ge N$.*

If $\mathfrak{p}$ denotes the maximal ideal of A, $\mathfrak{f} = \mathrm{Ann}_A(A'/A)$ and $l_A(A'/\mathfrak{f})$ is the length of the A-module $A'/\mathfrak{f}$, then

$$N \le \max\{1,\, \dim_{A/\mathfrak{p}}(A_1/\mathfrak{p}A_1) - 1\} \ \le\ \max\{1,\, l_A(A'/\mathfrak{f}) - 1\}\,.$$

Proof. Being a finitely generated A-module, A' is semilocal, say $\max(A') = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_r\}$. Then $\mathfrak{q} \not\subset \mathfrak{P}_i$ for all $i \in [1,r]$, hence $\mathfrak{q} \not\subset \mathfrak{P}_1 \cup \ldots \cup \mathfrak{P}_r$ and thus $\mathfrak{q} \cap A'^\times \ne \emptyset$. Let $\mu \in \mathfrak{q} \cap A'^\times$ and $\mathfrak{q}_1 = \mu^{-1}\mathfrak{q}$. Then $\mathfrak{q}_1 \in \mathcal{F}(A)$, $\mathfrak{q}_1 \subset A'$ and $1 \in \mathfrak{q}_1$. Hence $A \subset \mathfrak{q}_1 \subset \mathfrak{q}_1^2 \subset \ldots \subset A'$ is an ascending chain of submodules of the noetherian A-module A'. Let

$$N = \min\{n \in \mathbb{N} \mid \mathfrak{q}_1^n = \mathfrak{q}_1^{n+1}\} \quad \text{and} \quad A_1 = \mathfrak{q}_1^N\,.$$

Since $\mathfrak{q}_1^N = \mathfrak{q}_1^{2N}$, Lemma 3.1 implies that $A_1 \in \mathcal{E}(A)$. By definition we have $A \subset A_1 \subset A'$, $\mathfrak{q}_1^n = A_1$ for all $n \ge N$ and $\mathfrak{q}_1A_1 = \mathfrak{q}_1^{N+1} = A_1$. If $N \ge 2$, then $\mathfrak{q}_1^{N-1} \ne A_1$, and

$$A/\mathfrak{p} = A + \mathfrak{p}A_1/\mathfrak{p}A_1 \subsetneq \mathfrak{q}_1 + \mathfrak{p}A_1/\mathfrak{p}A_1 \subsetneq \ldots \subsetneq \mathfrak{q}_1^{N-1} + \mathfrak{p}A_1/A_1 \subsetneq A_1/\mathfrak{p}A_1$$

is and ascending chain of vector spaces over $A/\mathfrak{p}$ showing that $N+1 \le \dim_{A/\mathfrak{p}}(A_1/\mathfrak{p}A_1)$ and giving the first estimate for N.

It remains to prove the second estimate for N (which is independent of the intermediate domain A_1). If $\mathfrak{f} = A$, then $A = A'$ and there is nothing to do. Otherwise $\mathfrak{f} \subset \mathfrak{p}$ and thus $\dim_{A/\mathfrak{p}}(A_1/\mathfrak{p}A_1) = l_A(A_1/\mathfrak{p}A_1) \le l_A(A'/\mathfrak{f})$. □

Theorem 3.4. *Let R be one-dimensional and $\mathfrak{a} \in \mathcal{F}_L(R)$. Then there exists some $N \in \mathbb{N}$ such that $\mathfrak{a}^n$ is regular in $\mathcal{F}_L(R)$ for all $n \ge N$.*

If R is a Dedekind domain, then $N \le \max\{1,\, [L\!:\!K] - 1\}$.

Proof. Let $\Lambda = R(\mathfrak{a})$ and $\overline{\Lambda}$ the integral closure of Λ in L. By the Krull-Akizuki theorem, $\overline{\Lambda}$ is a Dedekind domain, and thus $\mathfrak{a}\overline{\Lambda}$ is $\overline{\Lambda}$-invertible. If $\mathfrak{a} = {}_\Lambda(a_1, \ldots, a_r)$, then there exist $b_1, \ldots, b_r \in (\overline{\Lambda}\!:\!\mathfrak{a}\overline{\Lambda})$ such that $a_1b_1 + \ldots + a_rb_r = 1$. We define

$$\Lambda' = \Lambda\big[\,\{a_ib_j \mid i,\, j \in [1,m]\}\,\big] \in \mathcal{E}_L(R) \quad \text{and} \quad \mathfrak{b} = {}_{\Lambda'}(b_1, \ldots, b_m)\,.$$

Then $(\mathfrak{a}\Lambda')\mathfrak{b}' = \Lambda'$, and thus $\mathfrak{a}\Lambda'$ and $\mathfrak{b}'$ are inverse elements in $\mathcal{F}(\Lambda')$. For every $\mathfrak{p} \in \max(\Lambda)$, the ring $\Lambda'_\mathfrak{p}$ is a finitely generated $\Lambda_\mathfrak{p}$-module, hence semilocal, and thus $\mathfrak{b}'_\mathfrak{p} = b_\mathfrak{p}\Lambda'_\mathfrak{p}$ for some $b_\mathfrak{p} \in L^\times$. Since $\Lambda'_\mathfrak{p} = \mathfrak{b}'_\mathfrak{p}$ for almost all $\mathfrak{p} \in \max(\Lambda)$, we may assume that $b_\mathfrak{p} = 1$ for almost all $\mathfrak{p} \in \max(\Lambda)$. Then we obtain

$$\mathfrak{b} = \bigcap_{\mathfrak{p} \in \max(\Lambda)} b_\mathfrak{p}\Lambda_\mathfrak{p} \in \mathcal{F}(\Lambda)^\times\,, \quad (\mathfrak{b}\Lambda')_\mathfrak{p} = b_\mathfrak{p}\Lambda'_\mathfrak{p} = \mathfrak{b}'_\mathfrak{p} \quad \text{and thus} \quad \mathfrak{b}\Lambda' = \mathfrak{b}'\,.$$

We set $\mathfrak{q} = \mathfrak{a}\mathfrak{b} \in \mathcal{F}(\Lambda)$, and we obtain $\mathfrak{q}\Lambda' = \Lambda'$. For every $\mathfrak{p} \in \max(\Lambda)$, we have $\mathfrak{q}_\mathfrak{p}\Lambda'_\mathfrak{p} = \Lambda'_\mathfrak{p}$, and we apply Lemma 3.3. There exist $n_\mathfrak{p} \in \mathbb{N}$, $\mu_\mathfrak{p} \in L^\times$ and $\Lambda_1(\mathfrak{p}) \in \mathcal{E}(\Lambda_\mathfrak{p})$ such that $\Lambda_\mathfrak{p} \subset \Lambda_1(\mathfrak{p}) \subset \Lambda'_\mathfrak{p}$, $\mu_\mathfrak{p}^{-1}\mathfrak{q}_\mathfrak{p}\Lambda_1(\mathfrak{p}) = \Lambda_1(\mathfrak{p})$ and $(\mu_\mathfrak{p}^{-1}\mathfrak{q}_\mathfrak{p})^n = \Lambda_1(\mathfrak{p})$ for all $n \ge n_\mathfrak{p}$. Moreover, we have the estimate

$$n_\mathfrak{p} \le \max\bigl\{1,\, \dim_{\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}}\bigl(\Lambda_1(\mathfrak{p})/\mathfrak{p}\Lambda_1(\mathfrak{p})\bigr) - 1\,\bigr\}\,.$$

For almost all $\mathfrak{p} \in \max(\Lambda)$ we have $\mathfrak{q}_\mathfrak{p} = \Lambda_\mathfrak{p}$ and $\Lambda_\mathfrak{p} = \Lambda'_\mathfrak{p} = \Lambda_1(\mathfrak{p})$, hence we may also assume that $\mu_\mathfrak{p} = 1$ and $n_\mathfrak{p} = 1$ for all but finitely many $\mathfrak{p} \in \max(\Lambda)$. Thus we obtain

$$\Lambda_1 = \bigcap_{\mathfrak{p}\in\max(\Lambda)} \Lambda_1(\mathfrak{p}) \in \mathcal{E}_L(R)\,, \quad \Lambda \subset \Lambda_1 \subset \Lambda' \quad \text{and} \quad \Lambda_{1\mathfrak{p}} = \Lambda_1(\mathfrak{p}) \quad \text{for all} \quad \mathfrak{p} \in \max(\Lambda)\,.$$

We define

$$\mathfrak{d} = \bigcap_{\mathfrak{p}\in\max(\Lambda)} \mu_\mathfrak{p}^{-1}\Lambda_\mathfrak{p} \in \mathcal{F}_L(R)\,.$$

Then $\mathfrak{d} \in \mathcal{F}(\Lambda)^\times$, and we set $N = \max\{n_\mathfrak{p} \mid \mathfrak{p} \in \max(\Lambda)\}$. If $n \ge N$, then

$$(\mathfrak{d}\mathfrak{q})^n = \bigcap_{\mathfrak{p}\in\max(\Lambda)} (\mu_\mathfrak{p}^{-1}\mathfrak{q}_\mathfrak{p})^n = \bigcap_{\mathfrak{p}\in\max(\Lambda)} A_1(\mathfrak{p}) = A_1 = (\mathfrak{d}\mathfrak{a}\mathfrak{b})^n = \mathfrak{a}^n(\mathfrak{d}\mathfrak{b})^n\,.$$

But $\mathfrak{d}\mathfrak{b} \in \mathcal{F}(\Lambda)^\times$, and therefore $\mathfrak{a}^n = \Lambda_1\bigl(\Lambda\!:\!(\mathfrak{d}\mathfrak{b})^n\bigr) \in \mathcal{F}(\Lambda_1)^\times$.

Let finally R be a Dedekind domain. We must prove that

$$\dim_{\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}}\bigl(\Lambda_{1\mathfrak{p}}/\mathfrak{p}\Lambda_{1\mathfrak{p}}\bigr) \le [L\!:\!K] \quad \text{for all} \quad \mathfrak{p} \in \max(\Lambda)\,.$$

Let $\mathfrak{p} \in \max(\Lambda)$ and $\wp = \mathfrak{p} \cap R$. Then $\wp\Lambda_{1\mathfrak{p}} \subset \mathfrak{p}\Lambda_{1\mathfrak{p}}$ and thus

$$\dim_{\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}}\bigl(\Lambda_{1\mathfrak{p}}/\mathfrak{p}\Lambda_{1\mathfrak{p}}\bigr) \le \dim_{R_\wp/\wp R_\wp}\bigl(\Lambda_{1\mathfrak{p}}/\mathfrak{p}\Lambda_{1\mathfrak{p}}\bigr) \le \dim_{R_\wp/\wp R_\wp}\bigl(\Lambda_{1\mathfrak{p}}/\wp\Lambda_{1\mathfrak{p}}\bigr) = k \quad (\text{say})\,.$$

Let $u_1, \ldots, u_k \in \Lambda_{1\mathfrak{p}}$ be such that $u_1 + \wp\Lambda_{1\mathfrak{p}}, \ldots, u_k + \wp\Lambda_{1\mathfrak{p}}$ are linearly independent over $R_\wp/\wp R_\wp$. By Nakayama's lemma, $u_1, \ldots, u_k$ is a minimal system of generators for ${}_{R_\wp}(u_1, \ldots, u_k)$ over $R_\wp$. Since $R_\wp$ is a discrete valuation domain, $(u_1, \ldots, u_k)$ are linearly independent over K and thus $k \le [L\!:\!K]$. □

Theorem 3.5. *Let R be one-dimensional. Then $\mathcal{F}_L(R)$ is complete. If R is a Dedekind domain and $L = K(\alpha)$ for some $\alpha \in L$, then $\mathcal{F}_L(R)$ is a Clifford semigroup if and only if $[L\!:\!K] \le 2$.*

Proof. By Theorem 3.2.5 and Theorem 3.4, $\mathcal{F}_L(R)$ is complete, and if R is a Dedekind domain and $[L\!:\!K] \le 2$, then $\mathcal{F}_L(R)$ is a Clifford semigroup.

Let now R be a Dedekind domain, $L = K(\alpha)$ and $d = [L\!:\!K] \ge 3$. We may assume that α is integral over R, and we adopt the construction given in [13] and [4] to our situation. It suffices to construct an R-lattice $\mathfrak{a} \in \mathcal{F}_L(R)$ which is not regular. Let $c \in R^\bullet \setminus R^\times$ and $\mathfrak{a} = R + \alpha R + c\alpha^2 R[\alpha]$. Then it is easily checked that $\mathfrak{a} \in \mathcal{F}_L(R)$, $\mathsf{R}(\mathfrak{a}) = R + cR[\alpha]$ and $\mathfrak{a}^{d-1} = R[\alpha] \subsetneq \mathsf{R}(\mathfrak{a})$, whence $\mathfrak{a}$ is not regular. □

4. Class semigroups

Throughout this section, let R be a noetherian integral domain, K its field of quotients and $L \supset K$ a finite extension field.

Two R-lattices $\mathfrak{a}$, $\mathfrak{b} \in \mathcal{F}_L(R)$ are called *arithmetically equivalent*, $\mathfrak{a} \sim \mathfrak{b}$, if $\mathfrak{a} = \lambda\mathfrak{b}$ for some $\lambda \in L^\times$. Arithmetical equivalence is a congruence relation on the semigroup $\mathcal{F}_L(R)$. Let $\mathcal{S}_L(R) = \mathcal{F}_L(R)/\!\sim$ denote the semigroup of equivalence classes $[\mathfrak{a}] = [\mathfrak{a}]_\sim$ of R-lattices in L. By definition, $\mathcal{S}(R) = \mathcal{S}_K(R)$ is the ideal class semigroup considered in [13], [1] or [9]. For $K = L$, the following proposition is proved in [4, Corollary 1.3.11]

Proposition 4.1. *Let $\mathcal{H}_\sim$ denote Green's relation on $\mathcal{S}_L(R)$. For any $\mathfrak{a}$, $\mathfrak{b} \in \mathcal{F}_L(R)$ we have*

$$\mathfrak{a}\,\mathcal{H}\,\mathfrak{b} \quad \text{in} \quad \mathcal{F}_L(R) \quad \text{if and only if} \quad [\mathfrak{a}]\,\mathcal{H}_\sim\,[\mathfrak{b}] \quad \text{in} \quad \mathcal{S}_L(R)\,.$$

In particular, there is an isomorphism

$$\Phi\colon \mathcal{F}_L(R)/\mathcal{H} \;\to\; \mathcal{S}_L(R)/\mathcal{H}_\sim\,, \quad \text{given by} \quad \Phi\big([\mathfrak{a}]_\mathcal{H}\big) = \big[[\mathfrak{a}]\big]_{\mathcal{H}_\sim}\,,$$

and $\mathcal{F}_L(R)$ is $[\pi$-$]$regular resp. $($almost$)$ complete if and only if $\mathcal{S}_L(R)$ has this property.

Proof. Obviously, $\mathfrak{a}\,\mathcal{H}\,\mathfrak{b}$ implies $[\mathfrak{a}]\,\mathcal{H}_\sim\,[\mathfrak{b}]$. Thus assume that $[\mathfrak{a}]\,\mathcal{H}_\sim\,[\mathfrak{b}]$. Then there exist $\mathfrak{u}$, $\mathfrak{v} \in \mathcal{F}_L(R)$ such that $[\mathfrak{a}] = [\mathfrak{b}][\mathfrak{u}] = [\mathfrak{b}\mathfrak{u}]$ and $[\mathfrak{b}] = [\mathfrak{a}][\mathfrak{v}] = [\mathfrak{a}\mathfrak{v}]$. Hence $\mathfrak{a} = \mathfrak{b}(\mathfrak{u}\lambda)$ and $\mathfrak{b} = \mathfrak{a}(\mathfrak{v}\mu)$ for some λ, $\mu \in L^\times$, whence $\mathfrak{a}\,\mathcal{H}\,\mathfrak{b}$. $\square$

In the following proposition, we make the connection between $\mathcal{F}_L(R)$ and $\mathcal{S}_L(R)$ even more explicit.

Proposition 4.2. *Let $\Lambda \in \mathcal{E}_L(R)$ and $\mathfrak{a} \in \mathcal{F}_L(R)$.*

1. $\mathsf{E}\big(\mathcal{S}_L(R)\big) = \big\{[\Lambda'] \bigm| \Lambda' \in \mathcal{E}_L(R)\big\}$, *and for Λ_1, $\Lambda_2 \in \mathcal{E}_L(R)$ we have $[\Lambda_1] \le [\Lambda_2]$ if and only if $\Lambda_1 \supset \Lambda_2$..*
2. $[\mathfrak{a}]$ *is regular in $\mathcal{S}_L(R)$ if and only if $\mathfrak{a}$ is regular in $\mathcal{F}_L(R)$.*
3. $\mathcal{S}(\Lambda) = [\Lambda]\mathcal{S}_L(R)$.
4. *We have $[\Lambda] \in \mathsf{E}\big([\mathfrak{a}]\big)$ if and only if $\Lambda \in \mathsf{E}(\mathfrak{a})$, and $\big[\mathsf{R}(\mathfrak{a})\big] = \min \mathsf{E}\big([\mathfrak{a}]\big)$.*
5. *The partial Ponizovski factor $P^*_{[\Lambda]}$ of $[\Lambda]$ in $\mathcal{S}_L(R)$ is given by*
$$P^*_{[\Lambda]} = \big\{[\mathfrak{c}] \bigm| \mathfrak{c} \in P^*_\Lambda\big\}\,,$$
and $\mathcal{S}_L(R)$ is the union of its partial Ponizovski factors. In particular, $P^*_{[\Lambda]}$ is a group if and only if P^*_Λ is a group. If K is a global field, then $P^*_{[\Lambda]}$ is finite.

Proof. 1. Obviously, $\Lambda' \in \mathcal{E}_L(R)$ implies $[\Lambda'] \in \mathsf{E}\big(\mathcal{S}_L(R)\big)$. Thus let $\mathfrak{a} \in \mathcal{F}_L(R)$ be such that $[\mathfrak{a}] \in \mathcal{S}_L(R)$ is idempotent. Then $[\mathfrak{a}^2] = [\mathfrak{a}]$ implies $\mathfrak{a}^2 = \lambda\mathfrak{a}$ for some $\lambda \in L^\times$ and thus $(\lambda^{-1}\mathfrak{a})^2 = \lambda^{-1}\mathfrak{a}$. Hence $\lambda^{-1}\mathfrak{a} \in \mathcal{E}_L(R)$, and $[\mathfrak{a}] = [\lambda^{-1}\mathfrak{a}]$.

If Λ_1, $\Lambda_2 \in \mathcal{E}_L(R)$, then $\Lambda_1 \le \Lambda_2$ if and only if $[\Lambda_1]_\mathcal{H} \le [\Lambda_2]_\mathcal{H}$. Hence the assertion follows by Theorem 3.2.1 and Proposition 4.1.

2. By Proposition 4.1, since $\mathfrak{a}$ is regular if and only if $[\mathfrak{a}]_\mathcal{H}$ is regular.

3. Obvious.

4. Clearly, $\Lambda \in \mathsf{E}(\mathfrak{a})$ implies $[\Lambda] \in \mathsf{E}\big([\mathfrak{a}]\big)$. Thus let $[\Lambda] \in \mathsf{E}\big([\mathfrak{a}]\big)$. Then $[\Lambda\mathfrak{a}] = [\mathfrak{a}]$, hence $\Lambda\mathfrak{a} = \lambda\mathfrak{a}$ for some $\lambda \in L^\times$, and therefore $\lambda^2\mathfrak{a} = \Lambda^2\mathfrak{a} = \Lambda\mathfrak{a} = \lambda\mathfrak{a}$, whence $\lambda\mathfrak{a} = \mathfrak{a}$ and $\Lambda \in \mathsf{E}(\mathfrak{a})$. The equality $\big[\mathsf{R}(\mathfrak{a})\big] = \min \mathsf{E}\big([\mathfrak{a}]\big)$ is now obvious.

5. The structure of the partial Ponizovski factor $P^*_{[\Lambda]}$ follows from 1. and 4., and its finiteness in the case of global fields is a special case of the Jordan-Zassenhaus theorem (see [12, Theorem (26.4)]). $\square$

5. Complementary lattices

Throughout this section, let R be a Dedekind domain,
K its field of quotients and $L \supset K$ a finite separable extension field.

In this section we use Dedekind's concept of complementary modules to investigate whether a single partial Ponizovski factor P^*_Λ of $\mathcal{F}_L(R)$ is a group. Observe that P^*_Λ is a group if and only if every ideal $\mathfrak{a}$ of Λ with $\Lambda = \mathsf{R}(\mathfrak{a})$ is Λ-invertible. We have to assume that R is a Dedekind domain and L is separable over K. In this case, the integral closure $\overline{R}$ of R in L is the smallest idempotent of $\mathcal{F}_L(R)$. Hence the group $\mathcal{F}(\overline{R})^\times = \overline{R}\mathcal{F}_L(R)$ is the kernel of $\mathcal{F}_L(R)$ (see [7, Proposition IV.4.5]).

We denote by $t\colon L \to K$ the trace. Since $L \supset K$ is separable, the induced bilinear form $(x,y) \mapsto t(x,y)$ on L is non-degenerated, and we use the duality theory of R-lattices as derived in [5, Section 3]. For $\mathfrak{a} \in \mathcal{F}_L(R)$, the *complementary lattice* is defined by $\mathfrak{a}' = \{x \in L \mid t(x\mathfrak{a}) \subset R\}$. Then $\mathfrak{a}' \in \mathcal{F}_L(R)$, and for any $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c} \in \mathcal{F}_L(R)$, we make use of the relations

$$\mathfrak{a}'' = \mathfrak{a}\,, \quad (\mathfrak{a}\mathfrak{b})' = (\mathfrak{a}' \!:\! \mathfrak{b}) \quad \text{and} \quad \big((\mathfrak{a} \!:\! \mathfrak{b}) \!:\! \mathfrak{c}\big) = (\mathfrak{a} \!:\! \mathfrak{b}\mathfrak{c})\,.$$

For an R-order $\Lambda \in \mathcal{E}_L(R)$, we call Λ' the *codifferent*, its Λ-inverse $\mathfrak{D}_\Lambda = (\Lambda : \Lambda')$ the *different* and $\mathfrak{f}_\Lambda = (\Lambda \!:\! \overline{R})$ the *conductor* of Λ.

Lemma 5.1. *Let $\Lambda \in \mathcal{E}_L(R)$.*
1. $\mathsf{R}(\Lambda') = \Lambda$.
2. $\overline{R}\,\mathfrak{D}_\Lambda \subset \mathfrak{f}_\Lambda \mathfrak{D}_{\overline{R}}$.
3. $\overline{R}' = \Lambda' \mathfrak{f}_\Lambda$.

Proof. Note that $\Lambda \subset \overline{R} \subset \overline{R}' \subset \Lambda'$, and that $\mathfrak{f}_\Lambda$ is the greatest $\overline{R}$-module contained in Λ.
1. $\mathsf{R}(\Lambda') = (\Lambda' \!:\! \Lambda') = (\Lambda\Lambda')' = \Lambda$.
2. From $\overline{R}'\,\mathfrak{D}_\Lambda = \overline{R}'(\Lambda \!:\! \Lambda') \subset \Lambda$ we obtain $\overline{R}'\,\mathfrak{D}_\Lambda \subset \mathfrak{f}_\Lambda$ and $\overline{R}\,\mathfrak{D}_\Lambda = \mathfrak{D}_{\overline{R}}\,\overline{R}'\,\mathfrak{D}_\Lambda \subset \mathfrak{f}_\Lambda\,\mathfrak{D}_{\overline{R}}$.
3. From $\overline{R} = (\mathfrak{f}_\Lambda \!:\! \mathfrak{f}_\Lambda) = \big((\Lambda \!:\! \overline{R}) \!:\! (\Lambda \!:\! \overline{R})\big) = \big(\Lambda \!:\! \overline{R}(\Lambda \!:\! \overline{R})\big) = (\Lambda \!:\! \mathfrak{f}_\Lambda)$ we infer $\overline{R}' = \Lambda' \mathfrak{f}_\Lambda$. □

Theorem 5.2. *For and R-order $\Lambda \in \mathcal{E}_L(R)$, the following assertions are equivalent:*
- (a) *The partial Ponizovski factor P_Λ^* of $\mathcal{F}_L(R)$ is a group (that is, every fractional ideal $\mathfrak{a} \in \mathcal{F}(\Lambda)$ satisfying $\mathsf{R}(\mathfrak{a}) = \Lambda$ is Λ-invertible).*
- (b) *Λ' is Λ-invertible.*
- (c) *$\mathsf{R}(\mathfrak{D}_\Lambda) = \Lambda$.*
- (d) *$\overline{R}\,\mathfrak{D}_\Lambda = \mathfrak{f}_\Lambda\,\mathfrak{D}_{\overline{R}}$.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (b) By definition, $\mathfrak{a} = \Lambda'(\Lambda : \Lambda') \in \mathcal{F}_L(R)$ is an ideal of Λ and thus $\mathfrak{a}^2 \subset \mathfrak{a}$. By assumption, $\Lambda = \mathsf{R}(\mathfrak{D}_\Lambda) = (\mathfrak{D}_\Lambda \!:\! \mathfrak{D}_\Lambda) = \big((\Lambda \!:\! \Lambda') \!:\! (\Lambda \!:\! \Lambda')\big) = (\Lambda \!:\! \mathfrak{a})$. Hence $\Lambda' = \mathfrak{a}\Lambda'$, and by Lemma 3.1 it follows that $1 \in \mathfrak{a}$, whence $\mathfrak{a} = \Lambda$ and Λ' is Λ-invertible.

(b) $\Rightarrow$ (d) Since Λ' is Λ-invertible, we have $\mathfrak{D}_\Lambda \Lambda' = \Lambda$ and thus, by Lemma 5.1.3, $\mathfrak{f}_\Lambda\,\mathfrak{D}_{\overline{R}} = \mathfrak{D}_\Lambda \Lambda' \mathfrak{f}_\Lambda\,\mathfrak{D}_{\overline{R}} = \mathfrak{D}_\Lambda \overline{R}'\,\mathfrak{D}_{\overline{R}} = \mathfrak{D}_\Lambda \overline{R}$.

(d) $\Rightarrow$ (a) From $\overline{R}\,\mathfrak{D}_\Lambda = \mathfrak{f}_\Lambda \mathfrak{D}_{\overline{R}}$ we obtain $\overline{R}'\mathfrak{D}_\Lambda = \mathfrak{f}_\Lambda \mathfrak{D}_{\overline{R}} \overline{R}' = \mathfrak{f}_\Lambda$ and, using Lemma 5.1.3, $\Lambda'\mathfrak{f}_\Lambda = \overline{R}' = (\Lambda'\mathfrak{D}_\Lambda)\overline{R}'$. Since $\Lambda'\mathfrak{D}_\Lambda$ is an ideal of Λ, we obtain $(\Lambda'\mathfrak{D}_\Lambda)^2 \subset \Lambda'\mathfrak{D}_\Lambda$, and by Lemma 3.1 it follows that $\Lambda'\mathfrak{D}_\Lambda = \Lambda$. Let now $\mathfrak{a} \in \mathcal{F}(\Lambda)$ with $\Lambda = \mathsf{R}(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{a})$. Then $\Lambda' = \mathfrak{a}\mathfrak{a}'$, hence $\mathfrak{a}\mathfrak{a}'\mathfrak{D}_\Lambda = \Lambda$, and therefore $\mathfrak{a}$ is Λ-invertible. □

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