Multiplicative ideal theory in the context of commutative monoids

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1 Introduction

General ideal theory of commutative rings has its origin in R. Dedekind's multiplicative theory of algebraic numbers from the 19th century. It became an autonomous theory by the work of W. Krull and E. Noether about 1930, and it proved to be a most powerful tool in algebraic and arithmetic geometry and complex analysis. Besides this mainstream movement towards algebraic geometry, there is a modern development of multiplicative ideal theory based on the works of W. Krull and H. Prüfer.

The main objective of multiplicative ideal theory is the investigation of the multiplicative structure of integral domains by means of ideals or certain systems of ideals of that domain. In doing so, Krull's concept of ideal systems proved to be fundamental. Its presentation in R. Gilmer's book [23], using the notion of star operations, influenced most of the research done in this area during the last 40 years, yielding a highly developed theory of integral domains characterized by ideal-theoretic or valuation-theoretic properties.

Fresh impetusses to the theory were given in the nineties by the concepts of spectral star operations and semistar operations. Spectral star operations were introduced by W. Fanggui and R.L. McCasland [11], [12] and shed new light on the connection between local and global behavior of integral domains. Semistar operations were introduce by A. Okabe and R. Matsuda [39] as a generalization of the concept of star operations. This new concept proved to be more flexible and made it possible to extend the theory obtained by star operations to a larger class of integral domains.

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Already in the early history of the theory, it was observed that a great deal of multiplicative ideal theory can be developed for commutative monoids disregarding the additive structure of integral domains. In an axiomatic way, this was first done by P. Lorenzen [34], and, in a more general setting, by K.E. Aubert [7]. A systematic presentation of this purely multiplicative theory was given in the volumes by P. Jaffard [32], J. Močkoř [37] and recently by the author [25].

The present article is based on the monograph [25]. Its main purpose is to outline the development of multiplicative ideal theory during the last 20 years (especially the concepts of spectral star operations and semistar operations) in the context of commutative monoids. In doing so, instead of being encyclopedically, we focus on the main results to outline the method, and we often only sketch proofs instead of giving them in full detail.

2 Notations and Preliminaries

By a *monoid* we always mean (deviating from the usual terminology) a commutative multiplicative semigroup K containing a unit element $1 \in K$ and a zero element $0 \in K$ (satisfying $0x = 0$ for all $x \in K$) such that every non-zero element $a \in K$ is cancellative (that is, $ab = ac$ implies $b = c$ for all $b,\, c \in K$).

For any set X, we denote by $X^\bullet$ the set of non-zero elements of X, by $\mathbb{P}_\mathsf{f}(X)$ the set of all finite subsets of X, and we set $\mathbb{P}_\mathsf{f}^\bullet(X) = \{E \in \mathbb{P}_\mathsf{f}(X) \mid E^\bullet \neq \emptyset\}$. A family $(X_\lambda)_{\lambda \in \Lambda}$ of subsets of X is called *directed* if, for any $\alpha,\, \beta \in \Lambda$ there exists some $\lambda \in \Lambda$ such that $X_\alpha \cup X_\beta \subset X_\lambda$.

For a monoid K, we denote by $K^\times$ the group of invertible elements of K. For subsets $X,\, Y \subset K$, we define $XY = \{xy \mid x \in X,\ y \in Y\}$ and $(X\!:\!Y) = (X\!:_K\! Y) = \{z \in K \mid zY \subset X\}$, and for $c \in K$ we set $Xc = X\{c\}$ and $(X\!:\!c) = (X\!:\!\{c\})$.

A submonoid $D \subset K$ is always assumed to contain 1 and 0, and a monoid homomorphism is assumed to respect 0 and 1.

In the sequel, let K be a monoid and $D \subset K$ a submonoid.

A subset $M \subset K$ is called a *D-module* if $DM = M$, and it is called an *ideal* of D if it is a D-submodule of D. A subset $T \subset K$ is called *multiplicatively closed* if $1 \in T$, $0 \notin T$ and $TT = T$. For a multiplicatively closed subset $T \subset K^\times$ and $X \subset K$, we define

$$T^{-1}X = \{t^{-1}x \mid t \in T,\ \ x \in X\,\} = \bigcup_{t \in T} t^{-1}X\,.$$

If $TX = X$, then the family $(t^{-1}X)_{t \in T}$ is directed. If $T \subset D$ is multiplicatively closed and X is a D-module, then $T^{-1}D \subset K$ is a submonoid, and

$T^{-1}X = (T^{-1}D)X$ is a $T^{-1}D$-module. We call $T^{-1}D$ the *quotient monoid* of D with respect to D.

We say that K a *quotient* of D and write $K = \mathsf{q}(D)$ if $D^\bullet \subset K^\times$ and $K = D^{\bullet\,-1}D$ (then $K^\bullet = K^\times$ is a quotient group of $D^\bullet$). Every monoid possesses a quotient which is unique up to canonical isomorphisms. If $K = \mathsf{q}(D)$, then a subset $X \subset K$ is called *D-fractional* if $cX \subset D$ for some $c \in D^\bullet$.

An ideal $P \subset D$ is called a *prime ideal* of D if $D \setminus P$ is multiplicatively closed. If $D \setminus P \subset K^\times$ and $X \subset K$, then we set $X_P = (D \setminus P)^{-1}X$.

In the following Lemma 2.1 we collect the elementary properties of quotient monoids. Proofs are easy and left to the reader.

Lemma 2.1. *Let $T \subset D \cap K^\times$ be a multiplicatively closed subset.*

1. *If $J \subset D$ is an ideal of D, then $T^{-1}J = (T^{-1}D)J \subset T^{-1}D$ is an ideal of $T^{-1}D$, $J \subset T^{-1}J \cap D$, and $T^{-1}J = T^{-1}D$ if and only if $J \cap T \neq \emptyset$.*
2. *If $\overline{J} \subset T^{-1}D$ is an ideal of $T^{-1}D$, then $\overline{J} = T^{-1}(\overline{J} \cap D)$.*
3. *The assignment $P \mapsto T^{-1}P$ defines a bijective map from the set of all prime ideals $P \subset D$ with $P \cap T = \emptyset$ onto the set of all prime ideals of $T^{-1}P$.*
4. *If $P \subset D$ is a prime ideal and $T \cap P = \emptyset$, then $P = T^{-1}P \cap D$, and if $T = D \setminus P$, then $T^{-1}P = PD_P = D_P \setminus D_P^\times$ is the greatest ideal of D_P.*
5. *If $X, Y \subset K$, then $T^{-1}(X:Y) \subset (T^{-1}X : T^{-1}Y) = (T^{-1}X : Y)$, and equality holds, if Y is finite.*

3 Definition and first properties of weak module systems

Let K be a monoid and $D \subset K$ a submonoid.

Definition 3.1. A *weak module system* on K is a map $r \colon \mathbb{P}(K) \to \mathbb{P}(K)$ such that, for all $c \in K$ and $X, Y \in \mathbb{P}(K)$ the following conditions are fulfilled:

M1. $X \cup \{0\} \subset X_r$.
M2. If $X \subset Y_r$, then $X_r \subset Y_r$.
M3. $cX_r \subset (cX)_r$.

A *module system* on K is a weak module system r on K such that, for all $X \subset K$ and $c \in K$,

M3′. $cX_r = (cX)_r$.

Let r be a weak module system on K. A subset $A \subset K$ is called an *r-module* if $A_r = A$, and D is called an *r-monoid* if it is an r-module. We denote by $\mathcal{M}_r(K)$ the set of all r-modules in K. An r-module $A \subset K$ is called *r-finite* or *r-finitely generated* if $A = E_r$ for some $E \in \mathbb{P}_\mathsf{f}(K)$. We denote by $\mathcal{M}_{r,\mathsf{f}}(K)$ the set of all r-finite r-modules.

A (weak) module system r on K is called a (*weak*) D*-module system* if every r-module is a D-module, and it is called a (*weak*) *ideal system* on K if it is a (weak) K-module system. If r is a (weak) ideal system on K, then the r-modules are called r*-ideals*, and in his case we shall often write $\mathcal{I}_r(K) = \mathcal{M}_r(K)$ (to be concordant with [25]).

The concept of a weak module system is a final step in a series of generalizations of the concepts of star and semistar operations on integral domains and that of Lorenzen's r-systems and Aubert's x-systems on commutative monoids. This concept also applies for not necessarily cancellative monoids, and in this setting it was presented in [27] (where a purely multiplicative analog of the Marot property for commutative rings was established). In this paper however, we shall restrict to cancellative monoids.

Examples will be discussed and presented later on in 5.6. In the meantime, the interested reader is invited to consult [25, Sections 2.2 and 11.4] to see examples of (weak) ideal systems and [30] to see examples of module systems.

In the following Proposition 3.2 we gather the elementary properties of weak module systems. We shall use them freely throughout this article. Their proofs are literally identical with those for weak ideal systems as presented in [25, Propositions 2.1, 2.3 and 2.4], and thus they will be omitted.

Proposition 3.2. *Let r be a weak module system on K and $X, Y \subset K$.*

1. *$\emptyset_r = \{0\}_r$ and if r is a module system, then $\{0\}_r = \{0\}$.*
2. *$(X_r)_r = X_r$, and if $X \subset Y$, then $X_r \subset Y_r$. In particular, X_r is the smallest r-module containing X.*
3. *The intersection of any family of r-modules is again an r-module.*
4. *For every family $(X_\lambda)_{\lambda \in \Lambda}$ in $\mathbb{P}(K)$ we have*
$$\bigcup_{\lambda \in \Lambda} (X_\lambda)_r \;\subset\; \Bigl(\bigcup_{\lambda \in \Lambda} X_\lambda\Bigr)_r = \Bigl(\bigcup_{\lambda \in \Lambda} (X_\lambda)_r\Bigr)_r .$$
5. *$(XY)_r = (X_rY)_r = (XY_r)_r = (X_rY_r)_r$, and for every family $(X_\lambda)_{\lambda \in \Lambda}$ in $\mathbb{P}(K)$ we have*
$$\Bigl(\bigcup_{\lambda \in \Lambda} X_\lambda Y\Bigr)_r = \Bigl(\bigcup_{\lambda \in \Lambda} (X_\lambda)_r Y\Bigr)_r = \Bigl(\bigcup_{\lambda \in \Lambda} (X_\lambda Y)_r\Bigr)_r .$$
   Equipped with the r-multiplication, defined by $(X, Y) \mapsto (XY)_r$, $\mathcal{M}_r(K)$ is a commutative semigroup with unit element $\{1\}_r$ and zero element $\emptyset_r$, and $\mathcal{M}_{r,\mathsf{f}}(K) \subset \mathcal{M}_r(K)$ is a subsemigroup.
6. *$(X:Y)_r \subset (X_r:Y) = (X_r:Y_r) = (X_r:Y)_r$, and equality holds, if Y is finite. In particular, if X is an r-module, then $(X:Y)$ is also an r-module.*

Proposition 3.3. *Let r be a weak module system on K.*

1. *D_r is an r-monoid, and if $A \subset K$ is a D-module, then A_r is a D_r-module. In particular, $\{1\}_r$ is the smallest r-monoid contained in K, r is a weak $\{1\}_r$-module system, and if $D \subset \{1\}_r$, then $\{1\}_r = D_r$.*

2. *Let r be a weak D-module system. Then $\{1\}_r = D_r$, and if $X \subset K$, then $X_r = D_r X_r = (DX)_r$.*
3. *r is a weak D-module system if and only if $cD \subset \{c\}_r$ for all $c \in K$, and if r is a D-module system, then $\{c\}_r = cD_r$ for all $c \in K$.*

Proof. 1. We have $D_r D_r \subset (DD)_r = D_r \subset D_r D_r$, and thus $D_r = D_r D_r \subset K$ is a submonoid. If $A \subset K$ is a D-module, then $D_r A_r \subset (DA)_r = A_r \subset D_r A_r$. Hence $A_r = D_r A_r$ is a D_r-module.

2. $\{1\}_r$ is a D-module containing 1, hence $D \subset \{1\}_r \subset D_r$ and thus $\{1\}_r = D_r$. If $X \subset K$, then $X_r \subset D_r X_r \subset (DX)_r = (DX_r)_r = (X_r)_r = X_r$, and thus equality holds.

3. If r is a weak D-module system and $c \in K$, then $\{c\}_r$ is a D-module containing c, which implies $cD \subset \{c\}_r$. If r is a D-module system, then $\{c\}_r = c\{1\}_r = cD_r$. Assume now that $cD \subset \{c\}_r$ for all $c \in K$, and let $A \in \mathcal{M}_r(K)$. Then $A \subset DA$, and if $c \in A$, then $Dc \subset \{c\}_r \subset A_r = A$, hence $DA = A$, and thus r is a weak D-module system. □

Definition 3.4. A weak module system r on K is called *finitary* or *of finite type* if

$$X_r = \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(X)} E_r \quad \text{for all} \quad X \subset K\,.$$

Theorem 3.5. *Let r be a weak module system on K. Then the following assertions are equivalent*:

(a) *r is finitary.*
(b) *For all $X \subset K$ and $a \in X_r$ there exists a finite subset $E \subset X$ such that $a \in E_r$.*
(c) *For every directed family $(X_\lambda)_{\lambda \in \Lambda}$ in $\mathbb{P}(K)$ we have*

$$\Big(\bigcup_{\lambda \in \Lambda} X_\lambda\Big)_r = \bigcup_{\lambda \in \Lambda} (X_\lambda)_r\,.$$

(d) *The union of every directed family of r-modules is again an r-module.*
(e) *If $X \subset K$, $A \in \mathcal{M}_{r,\mathsf{f}}(K)$ and $A \subset X_r$, then there is a finite subset $E \subset X$ satisfying $A \subset E_r$.*

In particular, if r is finitary, $X \subset K$ and $X_r \in \mathcal{M}_{r,\mathsf{f}}(K)$, then there exists a finite subset $E \subset X$ such that $E_r = X_r$.

Proof. The equivalence of (a) and (b) is obvious, and the equivalence of (a), (c) and (d) is proved as the corresponding statements for weak ideal systems in [25, Proposition 3.1].

(b) $\Rightarrow$ (e) Suppose that $X \subset K$ and $A = F_r \subset X_r$, where $F \in \mathbb{P}_{\mathsf{f}}(K)$. For every $c \in F$, there is some $E(c) \in \mathbb{P}_{\mathsf{f}}(X)$ such that $c \in E(c)_r$. Then

$$E = \bigcup_{c \in E} E(c) \in \mathbb{P}_{\mathsf{f}}(X)\,, \quad F \subset \bigcup_{c \in E} E(c)_r \subset E_r \quad \text{and thus} \quad A = F_r \subset E_r\,.$$

(e) $\Rightarrow$ (b) If $X \subset K$ and $a \in X_r$, then $\{a\}_r \in \mathcal{M}_{r,\mathsf{f}}(K)$ and $\{a\}_r \subset X_r$. Hence there exists a finite subset $E \subset X$ such that $a \in \{a\}_r \subset E_r$.

The final statement follows from (e) with $A = X_r$. $\square$

Theorem 3.6.

1. *Let* $r\colon \mathbb{P}_{\mathsf{f}}(K) \to \mathbb{P}(K)$ *be a map satisfying the conditions* **M1**, **M2** *and* **M3** *in* Definition 3.1 *for all* $X, Y \in \mathbb{P}_{\mathsf{f}}(K)$ *and* $c \in K$. *Then*

$$\overline{r}\colon \mathbb{P}(K) \to \mathbb{P}(K)\,, \quad \textit{defined by} \quad X_{\overline{r}} \ = \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(X)} E_r \quad \textit{for all} \quad X \subset K\,,$$

is the unique finitary weak module system on K *satisfying* $\overline{r}\,|\,\mathbb{P}_{\mathsf{f}}(K) = r$. *Moreover, if* r *has also the property* **M3**$'$ *for all* $X \in \mathbb{P}_{\mathsf{f}}(K)$ *and* $c \in K$, *then* $\overline{r}$ *is a module system, and if* $cD \subset \{c\}_r$ *for all* $c \in K$, *then* $\overline{r}$ *is a weak* D*-module system.*

2. *Let* r *be a* (*weak*) *module system on* K. *Then there exists a unique finitary* (*weak*) *module system* r_{f} *on* K *such that* $E_r = E_{r_{\mathsf{f}}}$ *for all finite subsets of* K. *It is given by*

$$X_{r_{\mathsf{f}}} = \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(X)} E_r \quad \textit{for all} \quad X \subset K\,,$$

it satisfies $(r_{\mathsf{f}})_{\mathsf{f}} = r_{\mathsf{f}}$, $X_{r_{\mathsf{f}}} \subset X_r$ *for all* $X \in \mathbb{P}(K)$, $\mathcal{M}_{r_{\mathsf{f}},\mathsf{f}}(K) = \mathcal{M}_{r,\mathsf{f}}(K)$, *and if* r *is a* (*weak*) D*-module system, then so is* r_{f}.

Proof. 1. It is easily checked that $\overline{r}$ satisfies the conditions **M1**, **M2** and **M3** resp. **M3**$'$ of Definition 3.1. Hence $\overline{r}$ is a weak module system resp. a module system, and obviously $E_{\overline{r}} = E_r$ for all finite subsets $E \subset K$. Hence $X_{\overline{r}} = \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(X)} E_{\overline{r}}$ for all $X \subset K$, and therefore $\overline{r}$ is finitary. If $\widetilde{r}$ is any finitary weak module system on K with $\widetilde{r}\,|\,\mathbb{P}_{\mathsf{f}}(K) = r$, then

$$X_{\widetilde{r}} = \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(X)} E_{\widetilde{r}} = \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(X)} E_r = X_{\overline{r}}\,, \quad \text{which implies} \quad \widetilde{r} = \overline{r}\,.$$

If $cD \subset \{c\}_r = \{c\}_{\overline{r}}$ for all $c \in K$, then $\overline{r}$ is a weak D-module system by Proposition 3.3.3.

2. By 1., applied for $r\,|\,\mathbb{P}_{\mathsf{f}}(X)$, there exists a unique (weak) module system r_{f} on K such that $E_{r_{\mathsf{f}}} = E_r$ for all $E \in \mathbb{P}_{\mathsf{f}}(X)$. If $X \subset K$, then $X_{r_{\mathsf{f}}}$ is given as asserted, and if r is a (weak) D-module system, then so is r_{f}. By definition, we have $\mathcal{M}_{r_{\mathsf{f}},\mathsf{f}}(K) = \mathcal{M}_{r,\mathsf{f}}(K)$, and by the uniqueness of r_{f} it follows that $r_{\mathsf{f}} = r$ if and only if r is finitary, and, in particular, $(r_{\mathsf{f}})_{\mathsf{f}} = r_{\mathsf{f}}$. $\square$

Definition 3.7. 1. Let $r\colon \mathbb{P}_{\mathsf{f}}(K) \to \mathbb{P}(K)$ be a map having the properties **M1**, **M2** and **M3** in Definition 3.1 for all $X, Y \in \mathbb{P}_{\mathsf{f}}(K)$ and $c \in K$. Then the unique weak module system on K which coincides with r on

$\mathbb{P}_{\mathsf{f}}(K)$ (see Theorem 3.5.1) is called the *total system* associated with r an is again denoted by r (instead of $\overline{r}$).

2. Let r be a (weak) module system on K. Then the unique finitary (weak) module system r_{f} on K defined in Theorem 3.5.2 is called the *finitary* (*weak*) *module system associated with* r.

4 Comparison and mappings of weak module systems

Let K be a monoid.

Definition 4.1. Let r and q be weak module systems on K. We call q *finer* than r and r *coarser* than q and write $r \le q$ if $X_r \subset X_q$ for all subsets $X \subset K$.

Proposition 4.2. *Let r and q be weak module systems on K. Then $r_{\mathsf{f}} \le r$, and the following assertions are equivalent*:

(a) $r \le q$.
(b) $X_q = (X_r)_q$ *for all subsets* $X \subset K$.
(c) $\mathcal{M}_q(K) \subset \mathcal{M}_r(K)$.

If r is finitary, then there are also equivalent:

(d) $E_q \subset E_r$ *for all finite subsets* $E \subset K$.
(e) $\mathcal{M}_{q_{\mathsf{f}}}(K) \subset \mathcal{M}_r(K)$.
(f) $\mathcal{M}_{q,\mathsf{f}}(K) \subset \mathcal{M}_r(K)$.
(g) $r \le q_{\mathsf{f}}$.

In particular, if r and q are both finitary, then $r = q$ if and only if $E_r = E_q$ for all finite subsets $E \subset K$.

Proof. Straightforward (see also [25, Proposition 5.1]). □

Definition 4.3. Let $\varphi\colon K \to L$ a monoid homomorphism, r a weak module system on K and q a weak module system on L.

φ is called an *(r,q)-homomorphism* if $\varphi(X_r) \subset \varphi(X)_q$ for all subsets $X \subset K$. We denote by $\mathrm{Hom}_{(r,q)}(K,L)$ the set of all (r,q)-homomorphisms $\varphi\colon K \to L$.

Proposition 4.4. *Let $\varphi\colon K \to L$ a monoid homomorphism, r a weak module system on K and q a weak module system on L.*

1. *φ is an (r,q)-homomorphism if and only if $\varphi^{-1}(A) \in \mathcal{M}_r(K)$ for all $A \in \mathcal{M}_q(L)$.*
2. *Let r be finitary and $\varphi(E_r) \subset \varphi(E)_q$ for all $E \in \mathbb{P}_{\mathsf{f}}(K)$. Then φ is an (r,q)-homomorphism.*

Proof. 1. If φ is an (r,q)-homomorphism and $A \in \mathcal{M}_q(L)$, then it follows that $\varphi\big(\varphi^{-1}(A)_r\big) \subset \varphi\big(\varphi^{-1}(A)\big)_q = \varphi\big(\varphi^{-1}(A)\big) \subset A$. Hence $\varphi^{-1}(A)_r \subset \varphi^{-1}(A)$, and thus $\varphi^{-1}(A) = \varphi^{-1}(A)_r \in \mathcal{M}_r(K)$.

Thus assume that $\varphi^{-1}(A) \in \mathcal{M}_r(K)$ for all $A \in \mathcal{M}_q(L)$, and let $X \subset K$. Then $\varphi^{-1}\big(\varphi(X)_q\big) \in \mathcal{M}_r(K)$, and as $X \subset \varphi^{-1}\big(\varphi(X)\big) \subset \varphi^{-1}\big(\varphi(X)_q\big)$, it follows that $X_r \subset \varphi^{-1}\big(\varphi(X)_q\big)$ and therefore $\varphi(X_r) \subset \varphi(X)_q$.

2. If $X \subset K$ and $a \in X_r$, then there is some $E \in \mathbb{P}_{\mathfrak{f}}(X)$ such that $a \in E_r$, and thus we obtain $\varphi(a) \in \varphi(E_r) \subset \varphi(E)_q \subset \varphi(X)_q$. $\square$

5 Extension and restriction of weak module systems

Let K be a monoid and $D \subset K$ a submonoid.

Definition 5.1. Let r be a weak module system on K. Then we define

$$r[D]\colon \mathbb{P}(K) \to \mathbb{P}(K) \quad \text{by} \quad X_{r[D]} = (XD)_r \quad \text{for all } X \subset K, \qquad \text{and}$$

$$r_D\colon \mathbb{P}(D) \to \mathbb{P}(D) \quad \text{by} \quad X_{r_D} = X_{r[D]} \cap D = (XD)_r \cap D \quad \text{for all} \quad X \subset D\,.$$

We call $r[D]$ the *extension* of r by D and r_D the *weak ideal system induced by r on D* (see Proposition 5.2.4).

Proposition 5.2. *Let r be a (weak) module system on K.*

1. *$r[D]$ is a (weak) D-module system on K, $\mathcal{M}_{r[D]}(K)$ consists of all r-modules which are equally D-modules, $r \le r[D]$, and $r = r[D]$ if and only if r is a (weak) D-module system.*
2. *$r_{\mathfrak{f}}[D]$ is finitary, $r_{\mathfrak{f}}[D] \le r[D]_{\mathfrak{f}}$, and if r is finitary, then $r[D]$ is also finitary.*
3. *$r_D = r[D]_D$ is a weak ideal system on D, and if r is finitary, then r_D is also finitary.*
4. *Suppose that r ist a weak D-module system and D is an r-monoid. Then $r_D = r\,|\,\mathbb{P}(D)$, and if r is a module system, then r_D is an ideal system on D.*
5. *If $A \in \mathcal{M}_r(K)$ is a D-module, then $A \cap D$ is an r_D-ideal of D.*
6. *If q is another weak module system on K and $r \le q$, then $r[D] \le q[D]$ and $r_D \le q_D$.*
7. *If $T \subset K$ is another submonoid, then $r[D][T] = r[TD]$.*

Proof. 1. It is easily checked that $r[D]$ satisfies the conditions of Definition 3.1, and thus it is a (weak) module system on K. If $A \in \mathcal{M}_{r[D]}(K)$, then $A = A_{r[D]} = (AD)_r$ is a D_r-module (hence a D-module) by Proposition 3.3.1. Conversely, if $A \in \mathcal{M}_r(K)$ is a D-module, then $A_{r[D]} = (AD)_r = A_r = A$ and thus $A \in \mathcal{M}_{r[D]}(K)$. Hence $\mathcal{M}_{r[D]}(K) \subset \mathcal{M}_r(K)$ and thus $r \le r[D]$.

Moreover, $r = r[D]$ holds if and only if every r-module is a D-module, that is, if and only if r is a weak D-module system.

2. If $X \subset K$ and $E \in \mathbb{P}_{\mathsf{f}}(XD)$, then there exists some $F \in \mathbb{P}_{\mathsf{f}}(X)$ such that $E \subset FD$. Hence

$$X_{r_{\mathsf{f}}[D]} = (XD)_{r_{\mathsf{f}}} = \bigcup_{E\in\mathbb{P}_{\mathsf{f}}(XD)} E_r \subset \bigcup_{F\in\mathbb{P}_{\mathsf{f}}(X)} (FD)_r = \bigcup_{F\in\mathbb{P}_{\mathsf{f}}(X)} F_{r[D]} = X_{r[D]_{\mathsf{f}}}\,,$$

and thus $r_{\mathsf{f}}[D] \le r[D]_{\mathsf{f}}$. Applying this reasoning for r_{f} instead of r, we obtain $r_{\mathsf{f}}[D] = (r_{\mathsf{f}})_{\mathsf{f}}[D] \le r_{\mathsf{f}}[D]_{\mathsf{f}} \le r_{\mathsf{f}}[D]$, and therefore $r_{\mathsf{f}}[D] = r_{\mathsf{f}}[D]_{\mathsf{f}}$ is finitary.

3. It is easily checked that $r_D = r[D]_D$ satisfies the conditions of Definition 3.1, and thus it is a (weak) module system on D.

If $c \in D$, then $cD \subset \{c\}_r D \cap D = \{c\}_{r_D}$, and thus r_D is a weak ideal system on D by Proposition 3.3.3. If r is finitary, $X \subset D$ and $a \in X_{r_D} = (XD)_r \cap D$, then there exists a finite subset $E \subset XD$ such that $a \in E_r \cap D$. In particular, there exists a finite subset $E \subset X$ such that $a \in (ED)_r \cap D = E_{r_D}$, and thus r_D is finitary.

4. If $X \subset D$, then $X_r \subset D$, and $X_{r_D} = (XD)_r \cap D = X_r \cap D = X_r$ by Proposition 3.3.2. If r is a module system, then $r_D = r \,|\, \mathbb{P}(D)$ is an ideal system on D.

5. If $A \in \mathcal{M}_r(K)$ is a D-module, then $A = A_r = AD \in \mathcal{M}_r(K)$, and therefore $A \cap D \subset (A \cap D)_{r_D} = [(A \cap D)D]_r \cap D \subset (AD)_r \cap D = A \cap D$.

6. and 7. are obvious by the definitions. $\square$

Proposition 5.3. *Let $T \subset D \cap K^\times$ be multiplicatively closed, r a finitary D-module system on K and $X \subset K$. Then $T^{-1}X_r = (T^{-1}X)_r = X_{r[T^{-1}D]}$, and if $X \subset T^{-1}D$, then $X_{r_{T^{-1}D}} = T^{-1}X_{r_D}$.*

Proof. Since $TDX = DX$ and r is finitary, it follows that

$$(T^{-1}DX)_r = \Big(\bigcup_{t\in T} t^{-1}DX\Big)_r = \bigcup_{t\in T}(t^{-1}DX)_r = \bigcup_{t\in T} t^{-1}(DX)_r = T^{-1}(DX)_r\,,$$

hence $T^{-1}X_r = T^{-1}(DX)_r = (T^{-1}DX)_r = (T^{-1}X)_r$ (by Proposition 3.3.2), and by definition we have $(T^{-1}DX)_r = X_{r[T^{-1}D]}$. If $X \subset T^{-1}D$, then $X_{r_{T^{-1}D}} = (XT^{-1}D)_r \cap T^{-1}D = T^{-1}X_r \cap T^{-1}D = T^{-1}X_{r_D}$. $\square$

Proposition 5.4. *Assume that $K = \mathsf{q}(D)$, and let $r\colon \mathbb{P}(D) \to \mathbb{P}(D)$ be a module system on D.*

1. *There exists a unique module system r_∞ on K such that $X_{r_\infty} = K$ if $X \subset K$ is not D-fractional, and $X_{r_\infty} = c^{-1}(cX)_r$ if $X \subset K$ and $c \in D^\bullet$ are such that $cX \subset D$.*
   In particular, $r_\infty \,|\, \mathbb{P}(D) = r$ and $D_{r_\infty} = D$. Moreover, r_∞ is a D-module system if and only if r is an ideal system on D, and then $(r_\infty)_D = r$.

2. *The module system* $(r_\infty)_{\mathsf f}$ *is the unique finitary module system on* K *satisfying* $(r_\infty)_{\mathsf f}\,|\,\mathbb{P}(D) = r_{\mathsf f}$. *Moreover,* $(r_\infty)_{\mathsf f}$ *is a* D*-module system on* K *if and only if* $r_{\mathsf f}$ *is an ideal system on* D*, and then* $((r_\infty)_{\mathsf f})_D = r_{\mathsf f}$.

Proof. 1. Uniqueness is obvious. To prove existence, we define r_∞ as in the assertion, making sure that for D-fractional subsets $X \subset K$ the definition of X_{r_∞} does not depend on the element $c \in D^\bullet$ with $cX \subset D$. Then it is easily checked that r_∞ has the properties of Definition 3.1.

We obviously have $r_\infty\,|\,\mathbb{P}(D) = r$. Hence, if r_∞ is a D-module system on K, then r is an ideal system on D. Conversely, let r be an ideal system on D. If $X \subset K$ is not D-fractional, then $X_{r_\infty} = K$ is a D-module. If $X \subset K$ is D-fractional and $c \in D^\bullet$ is such that $cX \subset D$, then $(cX)_r D = (cX)_r$, and $X_{r_\infty} D = c^{-1}(cX)_r D = c^{-1}(cX)_r = X_{r_\infty}$. Hence r_∞ is a D-module system, and $(r_\infty)_D = r$ by definition.

2. $(r_\infty)_{\mathsf f}$ is a finitary module system on K. If $X \subset D$, then

$$X_{(r_\infty)_{\mathsf f}} = \bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} E_{r_\infty} = \bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} E_r = X_{r_{\mathsf f}}\,, \quad \text{hence} \quad (r_\infty)_{\mathsf f}\,|\,\mathbb{P}(D) = r_{\mathsf f}\,.$$

Consequently, if $(r_\infty)_{\mathsf f}$ is a D-module system on K, then $r_{\mathsf f}$ is an ideal system on D. Conversely, let $r_{\mathsf f}$ be an ideal system on D and $X \subset K$. If $E \in \mathbb{P}_{\mathsf f}(X)$ and $c \in D^\bullet$ is such that $cE \subset D$, then

$$E_{r_\infty} D = c^{-1}(cE)_r D = c^{-1}(cE)_{r_{\mathsf f}} D = c^{-1}(cE)_{r_{\mathsf f}} = c^{-1}(cE)_r = E_{r_\infty}\,.$$

Hence $X_{(r_\infty)_{\mathsf f}} D = \bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} E_{r_\infty} D = \bigcup_{E \in \mathbb{P}_{\mathsf f}(X)} E_{r_\infty} = X_{(r_\infty)_{\mathsf f}}$, thus $(r_\infty)_{\mathsf f}$ is a D-module system, and $((r_\infty)_{\mathsf f})_D = r_{\mathsf f}$ by definition.

It remains to prove uniqueness. Let x be any finitary module system on K satisfying $x\,|\,\mathbb{P}(D) = r_{\mathsf f}$. If $E \in \mathbb{P}_{\mathsf f}(K)$ and $c \in D^\bullet$ is such that $cE \subset D$, then $E_x = [c^{-1}(cX)]_x = c^{-1}(cE)_x = c^{-1}(cE)_{r_{\mathsf f}} = E_{r_\infty} = E_{(r_\infty)_{\mathsf f}}$, and thus $x = (r_\infty)_{\mathsf f}$ by Proposition 4.2. $\square$

Definition 5.5. Assume that $K = \mathsf{q}(D)$, and let r be a module system on D. Then the module system r_∞ on K constructed in Proposition 5.4 is called the *trivial extension* of r to a module system on K.

If r is a finitary module system on D, then $(r_\infty)_{\mathsf f}$ is called the *natural extension* of r to a module system on K. In this case, we say that $(r_\infty)_{\mathsf f}$ is *induced* by r, and (as there will be no risk of confusion) we write again r instead of $(r_\infty)_{\mathsf f}$.

With this identification, every finitary module system r on D is a finitary module system on K, and r is even a finitary ideal system on D if and only if r is a finitary D-module system on K satisfying $D_r = \{1\}_r = D$.

Examples 5.6 (Examples of ideal systems and module systems)

1. The *semigroup system* $s(D)\colon \mathbb{P}(D) \to \mathbb{P}(D)$ is defined by $X_{s(D)} = DX$ for all $X \subset D$. It is a finitary ideal system on D, and $\mathcal{M}_{s(D)}(D)$ is the set

of ordinary semigroup ideals of D. For every ideal system r on D, we have $s(D) \le r$.

The *identical system* $s\colon \mathbb{P}(K) \to \mathbb{P}(K)$ is defined by $X_s = X \cup \{0\}$ for all $X \subset K$. It is a finitary module system on K, for every subset $X \subset K$ we have $X_{s[D]} = DX$ (the D-module generated by X), and $s_D = s(D)$.

2. Assume that $K = \mathsf{q}(D)$. Then $s(D) = s[D]$ is the finitary module system on K induced by the semigroup system $s(D)$ (according to Definition 5.5).

The module system $v(D)$ on K is defined by $X_{v(D)} = (D:(D:X))$ for all subsets $X \subset K$. If $X \subset K$ is not D-fractional, then $X_{v(D)} = K$, and thus $v(D)$ is the trivial extension of the classical "Vielfachensystem" v_D on D (compare [25, Section 11.4] and [23, §34]). Note that v_D (and thus also $v(D)$) is usually not finitary. If $X \subset K$ is D-fractional, then

$$X_{v(D)} = \bigcap_{\substack{b \in K \\ X \subset bD}} Db\,,$$

and for every ideal system r on D we have $r \le v_D$.

The associated finitary ideal system on D (which is identified with its natural extension to a finitary module system on K) is the classical "t-system" denoted by $t(D) = v(D)_{\mathsf{f}}$. If r is any finitary ideal system on D, then $r \le t(D)$. But note that for an overmonoid $T \supset D$ in general $t(D)[T] \ne t(T)$.

3. Let D be a ring. The *Dedekind system* $d(D)\colon \mathbb{P}(D) \to \mathbb{P}(D)$ is defined by $X_{d(R)} = {}_R\langle X\rangle$ (the usual ring ideal generated by X).

4. Let D be an integral domain and $K = \mathsf{q}(D)$. The *additive system* $d\colon \mathbb{P}(K) \to \mathbb{P}(K)$ is given by $X_d = {}_{\mathbb{Z}}\langle X\rangle$ (the additive group generated by X) for all $X \subset K$. It is a finitary module system on K, and $d[D] = d(D)$ ($X_{d[D]}$ is the D-submodule of K generated by X for every subset $X \subset K$).

Recall that a *semistar operation* $*$ on D is a map

$$\mathcal{M}_{d[D]}(K) \to \mathcal{M}_{d[D]}(K)\,, \quad X \mapsto X^*$$

having the following properties for all $X, Y \subset K$ and $c \in K$:

($*$1) $(cX)^* = cX^*$; **($*$2)** $X \subset X^* = X^{**}$ **($*$3)** $X \subset Y \implies X^* \subset Y^*$.

A (*semi*)*star operation* on D is a semistar operation satisfying $D^* = D$ (then $*\,|\,\mathcal{F}(D) \cap \mathcal{M}_{d[D]}(K)$ is a star operation in the classical sense, see [23, §32]).

If $*$ is a semistar operation on D, then the map $r^*\colon \mathbb{P}(K) \to \mathbb{P}(K)$, defined by $X_{r^*} = (X_{d[D]})^*$, is a D-module system on K such that $d[D] \le r^*$ and $r^*\,|\,\mathcal{M}_{d[D]}(K) = *$. Moreover, $*$ is a (semi)star operation if and only if D is an r^*-monoid (then $*\,|\,\mathcal{F}(D)$ is a star operation and $r^*\,|\,\mathbb{P}(D)$ is an ideal system on D). r^* is called the *module system induced by* $*$.

Conversely, let r be a module system on K such that $d[D] \le r$. Then $*_r = r\,|\,\mathcal{M}_{d[D]}(K)$ is a semistar operation on D, and $r = r^{*_r}$ is the module system induced by $*_r$.

6 Prime and maximal ideals, spectral module systems

Let K be a monoid and $D \subset K$ a submonoid.

Proposition 6.1. *Let $(r_\lambda)_{\lambda \in \Lambda}$ be a family of (weak) D-module systems on K, and let $r\colon \mathbb{P}(K) \to \mathbb{P}(K)$ be defined by*

$$X_r = \bigcap_{\lambda \in \Lambda} X_{r_\lambda} \quad \text{for all} \quad X \subset K$$

(if $\Lambda = \emptyset$, then r is the trivial weak module system on K, defined by $X_r = K$ for all $X \subset K$).

Then r is a (weak) D-module system on K, and $r = \inf\{r_\lambda \mid \lambda \in \Lambda\}$ is the Infimum of the family $(r_\lambda)_{\lambda \in \Lambda}$ in the partially ordered set of all weak D-module systems on K [that is, for every weak module system x on K we have $x \le r$ if and only if $x \le r_\lambda$ for all $\lambda \in \Lambda$].

Proof. Obvious. $\square$

Definition 6.2. Let r be a weak ideal system on D. We denote by r-spec(D) the set of all prime r-ideals of D and by r-max(D) the set of all maximal elements in $\mathcal{I}_r(D) \setminus \{D\}$ (called r-maximal r-ideals). We say that r *has enough primes* if for every $J \in \mathcal{I}_r(D) \setminus \{D\}$ there is some $P \in r$-spec(D) such that $J \subset P$.

Proposition 6.3. *Let r be a finitary weak ideal system on D. Then r has enough primes. More precisely, the following assertions hold:*

1. *If $J \in \mathcal{I}_r(D)$ and $T \subset D^\bullet$ is a multiplicatively closed subset such that $J \cap T = \emptyset$, then the set $\Omega = \{P \in \mathcal{I}_r(D) \mid J \subset P \text{ and } P \cap T = \emptyset\}$ has maximal elements, and every maximal element in Ω is prime.*
2. *Every r-ideal $J \in \mathcal{I}_r(D) \setminus \{D\}$, is contained in an r-maximal r-ideal of D, and r-max$(D) \subset r$-spec(D)*

Proof. [25, Theorems 6.3 and 6.4]. $\square$

Proposition 6.4. *Assume that $K = \mathsf{q}(D)$, let r be a finitary module system on K and $A \in \mathcal{M}_r(K)$ a D-module. Then*

$$A = \bigcap_{P \in r_D\text{-max}(D)} A_P\,. \quad \textit{If } D \textit{ is an } r\textit{-monoid, then} \quad D = \bigcap_{P \in r_D\text{-max}(D)} D_P\,.$$

Proof. Obviously, $A \subset A_P$ for all $P \in r_D$-max(D). Thus assume that $z \in A_P^\bullet$ for all $P \in r_D$-max(D). Then $I = z^{-1}A \cap D$ is an r_D-ideal of D. For each $P \in r_D$-max(D), there exists some $s \in D \setminus P$ such that $sz \in A$, hence $s \in I$ and $I \not\subset P$. Therefore we obtain $1 \in I$ and $z \in A$ by Proposition 6.3. $\square$

In the sequel we investigate two closely connected special classes of module systems, spectral and stable ones (see Definition 6.10 for a formal definition). In the case of semistar operations, they were introduced in [13] where its deep connection with localizing systems was established. For the connection with localizing systems in a purely multiplicative context we refer to [30]. In the case of integral domains, spectral module systems describe the ideal theory of generalized Nagata rings (see [19], [20]).

Theorem 6.5. *Assume that* $K = \mathsf{q}(D)$*, let* q *be a finitary* D*-module system on* K*,* $\varDelta \subset q_D\text{-spec}(D)$ *and* $q_\varDelta = \inf\{q[D_P] \mid P \in \varDelta\}$ *(see* Proposition 6.1*).*

1. $q_\varDelta$ *is a* D*-module system on* K *satisfying* $q \le q_\varDelta$*. If* $X \subset K$*, then*

$$D_P X_q = D_P X_{q_\varDelta} \quad \text{for all } P \in \varDelta\,, \quad \text{and} \quad X_{q_\varDelta} = \bigcap_{P \in \varDelta} D_P X_q\,.$$

2. *For all* $A,\, B \in \mathcal{M}_q(K)$ *we have* $(A \cap B)_{q_\varDelta} = A_{q_\varDelta} \cap B_{q_\varDelta}$*.*
3. *For all* $P \in \varDelta$ *we have* $P_{q_\varDelta} \cap D = P$ *(and thus* $\varDelta \subset (q_\varDelta)_D\text{-spec}(D)$ *).*
4. *If* $J \subset D$ *is an ideal such that* $1 \notin J_{q_\varDelta}$*, then there exists some* $P \in \varDelta$ *such that* $J \subset P$*. In particular,* $(q_\varDelta)_D$ *has enough primes.*

Proof. 1. By Proposition 6.1, $q_\varDelta = \inf\{q[D_P] \mid P \in \varDelta\}$ is a D-module system on K. Since $q \le q[D_P]$ for all $P \in \varDelta$, it follows that $q \le q_\varDelta$. If $X \subset K$, then $X_{q[D_P]} = D_P X_q$ by Proposition 5.3, and thus

$$X_{q_\varDelta} = \bigcap_{P \in \varDelta} X_{q[D_P]} = \bigcap_{P \in \varDelta} D_P X_q\,.$$

Now $X_q \subset X_{q_\varDelta} \subset D_P X_q$ implies $D_P X_q \subset D_P X_{q_\varDelta} \subset D_P D_P X_q = D_P X_q$ and thus $D_P X_q = D_P X_{q_\varDelta}$.

2. If $A,\, B \in \mathcal{M}_q(K)$, then $A \cap B \in \mathcal{M}_q(K)$, and

$$(A \cap B)_{q_\varDelta} = \bigcap_{P \in \varDelta} D_P(A \cap B) = \bigcap_{P \in \varDelta} D_P A \cap \bigcap_{P \in \varDelta} D_P B = A_{q_\varDelta} \cap B_{q_\varDelta}\,.$$

3. Let $P,\, Q \in \varDelta$. If $P \not\subset Q$, then $D \subset D_Q = P D_Q \subset P_q D_Q$, and if $P \subset Q$, then $P_q D_Q \supset P_q D_P$. Hence we obtain

$$P_{q_\varDelta} \cap D = \bigcap_{Q \in \varDelta} P_q D_Q \cap D = P_q D_P \cap D \supset P\,,$$

and it remains to prove that $P_q D_P \cap D \subset P$. If $z \in P_q D_P \cap D$, then there is some $s \in D \setminus P$ such that $sz \in P_q \cap D = P$ and therefore $z \in P$.

4. If $J \subset D$ is an ideal and $1 \notin J_{q_\varDelta}$, then $1 \notin J_q D_P$ and thus $1 \notin J_q D_P \cap D_P$ for some $P \in \varDelta$. Since $J_q D_P \cap D_P \subset D_P$ is an ideal and $P D_P = D_P \setminus D_P^\times$, we obtain $J_q D_P \cap D_P \subset P D_P$ and $J \subset J_q D_P \cap D \subset P D_P \cap D = P$. $\square$

Theorem 6.6. *Let q be a finitary (weak) D-module system on K, r a weak module system on K, and define $r[q]\colon \mathbb{P}(K) \to \mathbb{P}(K)$ by*

$$X_{r[q]} = \bigcup_{\substack{B \subset D \\ 1 \in B_r}} (X_q : B) \quad \textit{for all} \quad X \subset K\,.$$

1. *$r[q]$ is a finitary (weak) D-module system on K satisfying $q \le r[q]$, and*

$$X_{r[q]} = \{x \in K \mid 1 \in [(X_q : x) \cap D]_r\,\} \quad \textit{for all} \ \ X \subset K.$$

2. *For all $X, Y \in \mathcal{M}_q(K)$ we have $(X \cap Y)_{r[q]} = X_{r[q]} \cap Y_{r[q]}$.*
3. *If $B \subset D$ and $1 \in B_r$, then $1 \in B_{r[q]}$.*
4. *If $q \le r$, then $r[q] \le r$ and $(r[q])[q] = r[q]$. In particular, $q[q] = q$.*
5. *If $q \le r$, then r_D-$\max(D) \subset r[q]_D$-$\max(D)$, and equality holds if r_D has enough primes.*

Proof. 1. Let $X \subset K$. Then $(X_q : B)$ is a D-module for every $B \subset D$, and thus $X_{r[q]}$ is a D-module. If B', $B'' \subset D$ are such that $1 \in B'_r$ and $1 \in B''_r$, then $1 \in B'_r B''_r \subset (B'B'')_r$ and $(X_q : B') \cup (X_q : B'') \subset (X_q : B'B'')$ (since X_q is a D-module). Hence $\{(X_q : B) \mid B \subset D,\ 1 \in B_r\}$ is directed, and since q is finitary, it follows that

$$(X_{r[q]})_q = \bigcup_{\substack{B \subset D \\ 1 \in B_r}} (X_q : B)_q = \bigcup_{\substack{B \subset D \\ 1 \in B_r}} (X_q : B) = X_{r[q]}\,, \quad \text{and}$$

$$X_{r[q]} = \bigcup_{\substack{B \subset D \\ 1 \in B_r}} \Big(\Big[\bigcup_{E \in \mathbb{P}_{\mathrm{f}}(X)} E_q \Big] : B \Big) = \bigcup_{\substack{B \subset D \\ 1 \in B_r}} \bigcup_{E \in \mathbb{P}_{\mathrm{f}}(X)} (E_q : B) = \bigcup_{E \in \mathbb{P}_{\mathrm{f}}(X)} E_{r[q]}\,.$$

We show now that $r[q]$ satisfies the conditions of Definition 3.1. Once this is done, then by the above considerations $r[q]$ is a finitary D-module system satisfying $q \le r[q]$. Thus let $X, Y \subset K$ and $c \in K$.

M1. If $B \subset D$, then $XB \subset XD \subset X_q$ and thus $X \subset (X_q : B) \subset X_{r[q]}$.

M2. If $X \subset Y_{r[q]}$ and $z \in X_{r[q]}$, then there is some $B \subset D$ such that $zB \subset X_q \subset (Y_{r[q]})_q = Y_{r[q]}$ and thus $z \in (Y_{r[q]} : B) \subset Y_{r[q]}$, since $Y_{r[q]}$ is a D-module.

M3. and **M3′.** If $B \subset D$, then $((cX)_q : B) \supseteq (cX_q : B) = c(X_q : B)$, and thus we obtain $(cX)_{r[q]} \supseteq cX_{r[q]}$.

It remains to prove that $X_{r[q]} = \{x \in K \mid 1 \in [(X_q : x) \cap D]_r\,\}$.

If $x \in X_{r[q]}$, then there is some $B \subset D$ such that $1 \in B_r$ and $xB \subset X_q$, whence $B \subset (X_q : x) \cap D$ and $1 \in B_r \subset [(X_q : x) \cap D]_r$. Conversely, if $x \in K$ and $1 \in [(X_q : x) \cap D]_r$, then $B = (X_q : x) \cap D \subset D$, $1 \in B_r$ and $x \in (X_q : B) \subset X_{r[q]}$.

2. If $X, Y \in \mathcal{M}_q(K)$, then obviously $(X \cap Y)_{r[q]} \subset X_{r[q]} \cap Y_{r[q]}$. To prove the reverse inclusion, let $z \in X_{r[q]} \cap Y_{r[q]}$ and B', $B'' \subset D$ such that $1 \in B'_r$, $1 \in B''_r$, $zB' \subset X_q = X$ and $zB'' \subset Y_q = Y$. Then $1 \in B'_r B''_r \subset (B'B'')_r$,

and since X and Y are D-modules, it follows that $zB'B'' \subset X \cap Y$, whence $z \in (X \cap Y : B'B'') \subset (X \cap Y)_{r[q]}$.

3. If $B \subset D$ and $1 \in B_r$, then $1 \in (B_q : B) \subset B_{r[q]}$.

4. Assume that $q \le r$, and let $X \subset K$. If $x \in X_{r[q]}$, then it follows that $1 \in [(X_q : x) \cap D]_r \subset (X_r : x)_r = (X_r : x)$, which implies $x \in X_r$. Hence we obtain $X_{r[q]} \subset X_q$ and thus $r[q] \le r$. Applied with $r[q]$ instead of r, this argument shows that $(r[q])[q] \le r[q]$. To prove $r[q] \le (r[q])[q]$, let $X \subset K$ and $x \in X_{r[q]}$. Then $1 \in [(X_q : x) \cap D]_r \subset [(X_{r[q]} : x) \cap D]_r$, hence $1 \in [(X_{r[q]} : x) \cap D]_{r[q]}$ by 3. and thus $x \in X_{(r[q])[q]}$.

5. Assume that $q \le r$, and let $P \in r_D\text{-max}(D)$. Then $r[q] \le r$ by 4., hence $r[q]_D \le r_D$ and thus $P \in \mathcal{I}_{r[q]_D}(D)$. Since $r[q]$ (and thus also $r[q]_D$) is finitary, there exists some $P' \in r[q]_D\text{-max}(D)$ such that $P \subset P'$. If $P \subsetneq P'$, then $1 \in P'_{r_D} \subset P'_r$, and thus $1 \in P'_{r[q]} \cap D = P'_{r[q]_D}$, a contradiction. Hence it follows that $P = P' \in r[q]_D\text{-max}(D)$.

Assume now that r_D has enough primes, and let $P \in r[q]_D\text{-max}(D)$. Then $1 \notin P = P_{r[q]} \cap D$ and thus $1 \notin P_r \cap D = P_{r_D}$. Therefore there exists some $P' \in r_D\text{-spec}(D) \subset \mathcal{I}_{r[q]_D}(D)$ such that $P_{r_D} \subset P'$. Hence $P \subset P'$ and thus $P = P' \in r_D\text{-spec}(D)$. If $P' \in r_D\text{-max}(D)$, we are done. Otherwise, there exists some $P'' \in \mathcal{I}_{r_D}(D) \subset \mathcal{I}_{r[q]_D}(D)$ such that $P' \subsetneq P''$, and then $P \subsetneq P''$ yields a contradiction. □

Definition 6.7. Let q be a finitary (weak) D-module system and r a weak module system on K. The finitary (weak) D-module system $r[q]$ defined in Theorem 6.6 is called the *q-stabilizer* of r on D or the *spectral extension* of q by r on D.

Theorem 6.8. *Assume that $K = \mathsf{q}(D)$, let q be a finitary D-module system and r a module system on K.*

1. $r[q][D_P] = q[D_P]$ *for all* $P \in r_D\text{-spec}(D)$, *and if* $q \le r$, *this holds for all* $P \in r[q]_D\text{-spec}(D)$.
2. $r[q] \le \inf\{\, q[D_P] \mid P \in r_D\text{-spec}(D)\}$ (*see* Proposition 6.1). *Equality holds if* r_D *has enough primes, and* $r[q] = \inf\{q[D_P] \mid P \in r_D\text{-max}(D)\}$ *if* r *is finitary.*
3. *If* $q \le r$, *then* $r[q] = \inf\{\, q[D_P] \mid P \in q_D\text{-spec}(D)\,,\ 1 \notin P_r\}$.

Proof. 1. Let $P \in r_D\text{-spec}(D)$. Then $q \le r[q]$ implies $q[D_P] \le r[q][D_P]$. To prove the reverse inequality, we must show that $X_{r[q][D_P]} \subset X_{q[D_P]}$ for all $X \subset K$. If $X \subset K$ and $z \in X_{r[q][D_P]} = X_{r[q]}D_P$, let $s \in D \setminus P$ be such that $sz \in X_{r[q]}$. Then there is some $B \subset D$ such that $1 \in B_r$ and $szB \subset X_q$. Since $1 \in B_r$, it follows that $B \not\subset P = P_r \cap D$, and if $t \in B \setminus P$, then $stz \in X_q$ and $z \in X_q D_P = X_{q[D_P]}$.

Assume now that $q \le r$. Then $(r[q])[q] = r[q]$, and we apply what we have just proved for $r[q]$ instead of r and obtain $r[q][D_P] = (r[q])[q][D_P] = q[D_P]$ for all $P \in r[q]_D\text{-spec}(D)$.

2. We must prove that $X_{r[q]} \subset X_{q[D_P]} = X_q D_P$ for all $P \in r_D$-spec(D) and $X \subset K$. Thus let $P \in r_D$-spec(D), $X \subset K$, $x \in X_{r[q]}$ and $B \subset D$ such that $1 \in B_r$ and $xB \subset X_q$. Then it follows that $B \not\subset P = P_r \cap D$, and if $s \in B \setminus P$, then $xs \in X_q$, which implies $x \in X_q D_P$.

Assume now that r_D has enough primes and $x \in X_{q[D_P]} = X_q D_P$ for for all $P \in r_D$-spec(D). For each $P \in r_D$-spec(D), let $s_P \in D \setminus P$ be such that $s_P z \in X_q$. Then $B = \{s_P \mid P \in r_D\text{-spec}(D)\} \subset D$ and $B \not\subset P$ for all $P \in r_D$-spec(D). Hence $B_{r_D} = B_r \cap D = D$, whence $1 \in B_r$ and $z \in (X_q \colon B) \subset X_{r[q]}$.

If r is finitary, then so is r_D. In particular, r_D has enough primes, and for every $P \in r_D$-spec(D) there exists some $M \in r_D$-max(D) such that $P \subset M$, hence $D_M \subset D_P$, and it follows that

$$\bigcap_{P \in r_D\text{-spec}(D)} X_{q[D_P]} = \bigcap_{P \in r_D\text{-max}(D)} X_{q[D_P]} \quad \text{for all } X \subset K$$

and consequently $r[q] = \inf\{\, q[D_P] \mid P \in r_D\text{-max}(D)\}$.

3. If $q \le r$, then $q_D \le r_D$, r_D-spec$(D) \subset \{P \in q_D\text{-spec}(D) \mid 1 \notin P_r\}$ and thus $\inf\{\, q[D_P] \mid P \in q_D\text{-spec}(D)\,,\ 1 \notin P_r\} \le r[q]$. To prove the reverse inequality, it suffices to show that $r[q] \le q[D_P]$ for all $P \in q_D$-spec(D) such that $1 \notin P_r$. Thus let $P \in q_D$-spec(D), $1 \notin P_r$, $X \subset K$, $x \in X_{r[q]}$ and $B \subset D$ such that $1 \in B_r$ and $xB \subset X_q$. Then we have $B \not\subset P$, and if $x \in B \setminus P$, then $xs \in X_q$, whence $x \in X_q D_P = X_{q[D_P]}$. □

Remark 6.9. Let D be an integral domain with quotient field K, $*$ a semistar operation on D and $r = r^*$ the D-module system on K induced by $*$ (see Example 5.6.4). If $\widetilde{*}$ is the spectral semistar operation associated with $*$ (see [13]), then Theorem 6.6 implies $r[d] = r^{\widetilde{*}}$, and in the case of star operations we also obtain $r[d] = r^{*_w}$ (where $*_w$ is the star operation introduced in [3]) and $t[d] = r^w = r^{\widetilde{v}}$ (where $w = \widetilde{v}$ is the star operation introduced in [11]).

Definition 6.10. Let q be a finitary D-module system and r a module system on K such that $q \le r$. Then r is called

- *q-stable* if $X_r \cap Y_r = (X \cap Y)_r$ for all $X, Y \in \mathcal{M}_q(K)$.
- *q-spectral* if $r = q_\Delta$ for some subset $\Delta \subset q_D$-spec(D) (see Theorem 6.5).

Theorem 6.11. *Assume that $K = \mathsf{q}(D)$, let q be a finitary D-module system on K such that $D = D_q$ and r a module system on K such that $q \le r$.*

1. *The following assertions are equivalent*:
   (a) $r = r[q]$.
   (b) *r is q-stable.*
   (c) $[(X \colon E) \cap D]_r = (X_r \colon E) \cap D_r$ *for all $E \in \mathbb{P}_{\mathsf{f}}(K)$ and $X \in \mathcal{M}_q(K)$.*
2. *r is q-spectral if and only if r is q-stable and r_D has enough primes.*

Proof. 1. (a) $\Rightarrow$ (b) By Theorem 6.6.2.

(b) $\Rightarrow$ (c) Let $E \in \mathbb{P}_{\mathsf{f}}(K)$ and $X \in \mathcal{M}_q(K)$. Then, as $D = D_q$,

$$[(X:E) \cap D]_r = \Big(\bigcap_{x \in E^\bullet} x^{-1}X \cap D \Big)_r = \bigcap_{x \in E^\bullet} x^{-1}X_r \cap D_r = (X_r:E) \cap D_r\,.$$

(c) $\Rightarrow$ (a) By Theorem 6.6 we have $q \le r[q] \le r$, and thus it suffices to prove that $X_r \subset X_{r[q]}$ for all $X \in \mathcal{M}_q(K)$. Thus let $X \in \mathcal{M}_q(K)$ and $x \in X_r$. Then $1 \in (X_r:x) \cap D_r = [(X:x) \cap D]_r$ and therefore $x \in X_{r[q]}$.

2. If r is q-spectral, then r is q-stable and r_D has enough primes by Theorem 6.5. If r is q-stable, then $r = r[q]$ by 1., and if r_D has enough primes, then $r[q]$ is q-stable by Theorem 6.8.3. ☐

7 A survey on valuation monoids and GCD-monoids.

Let K be a monoid and $D \subset K$ a submonoid such that $K = \mathsf{q}(D)$.

In this section we gather several facts concerning GCD-monoids, valuation monoids and their homomorphisms. For a more concise presentation of this topic we refer to [25, Chapers 10, 15 and 18].

Definition 7.1.

1. Let $X \subset D$. An element $d \in D$ is called a *greatest common divisor* of X if dD is the smallest principal ideal containing X [equivalently, $d \,|\, x$ for all $x \in E$, and if $e \in D$ and $e \mid x$ for all $x \in E$, then $e \mid d$ (where the notion of divisibility in D is used in the common way)]. If $\mathrm{GCD}(X) = \mathrm{GCD}_D(X)$ denotes the set of all greatest common divisors of X, then $\mathrm{GCD}(X) = dD^\times$ for every $d \in X$. If D is reduced, then X has at most one greatest common divisor, and we write $d = \gcd(X)$ instead of $\mathrm{GCD}(X) = \{d\}$. If $X = \{a_1, \ldots, a_n\}$, we set $\mathrm{GCD}(a_1, \ldots, a_n) = \mathrm{GCD}(X)$ resp. $\gcd(a_1, \ldots, a_n) = \gcd(X)$.
2. D is called a GCD-*monoid* if $\mathrm{GCD}(E) \ne \emptyset$ for all $E \in \mathbb{P}_{\mathsf{f}}(D)$ [equivalently, $\mathrm{GCD}(a,b) \ne \emptyset$ for all $a,\, b \in D^\bullet$].
3. D is called a *valuation monoid* if, for all $a,\, b \in D$, either $a \,|\, b$ or $b \,|\, a$. If r is a module system on K, then D is called an *r-valuation monoid* (of K) if D is a valuation monoid satisfying $D_r = D$.
4. A homomorphism $\varphi\colon G_1 \to G_2$ of GCD-monoids is called a GCD-*homomorphism* if $\varphi(\mathrm{GCD}(E)) \subset \mathrm{GCD}(\varphi(E))$ for every $E \in \mathbb{P}_{\mathsf{f}}(G_1)$. We denote by $\mathrm{Hom}_{\mathrm{GCD}}(G_1, G_2)$ the set of all GCD-homomorphisms $\varphi\colon G_1 \to G_2$.

By definition, D is a valuation monoid if and only if for every $z \in K^\times$ either $z \in D$ or $z^{-1} \in D$. If D is a valuation monoid, then every monoid T such that

$D \subset T \subset K$ is also a valuation monoid. Obviously, every valuation monoid is a GCD-monoid.

If D is a valuation monoid and $E \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)$, then (after a suitable numbering) $E = \{a_1, \ldots, a_n\}$ with $a_1 D \subset a_2 D \subset \ldots \subset a_n D$, hence $ED = a_n D$, and if $E \subset D$, then $\mathrm{GCD}(E) = a_n D^{\times}$. In particular, the s-system is the only finitary ideal system on D. We identify it with its natural extension to a D-module system on K, whence $s(D) = t(D)$ and $D = \{1\}_{t(D)}$ (see Example 5.6.2).

Lemma 7.2. *Let D be a* GCD*-monoid.*

1. *If $E, F \in \mathbb{P}_{\mathsf{f}}(D)$ and $b \in D$, then* $\mathrm{GCD}(EF) = \mathrm{GCD}(E)\,\mathrm{GCD}(F)$ *and* $\mathrm{GCD}(bE) = b\,\mathrm{GCD}(E)$.
2. *If $a, b, c \in D$, $\mathrm{GCD}(a,b) = D^{\times}$ and $a \,|\, bc$, then $a \,|\, c$.*
3. *Every $z \in K$ has a representation in the form $z = a^{-1}b$ with $a \in D^{\bullet}$ and $b \in D$ such that* $\mathrm{GCD}(a,b) = D^{\times}$*. In this representation $aD^{\times}$ and $bD^{\times}$ are uniquely determined by z.*
4. *If $v = v(D)$, $X \subset D$ and $d \in D$, then*
$$X_v = \bigcap_{\substack{a \in D \\ X \subset aD}} aD\,, \quad \textit{and} \quad X_v = dD \quad \textit{if and only if} \quad d \in \mathrm{GCD}(X)\,.$$
In particular, if $E \in \mathbb{P}_(D)$ and $d \in \mathrm{GCD}(E)$, then $E_{t(D)} = dD$.
5. *$\mathcal{M}_{t(D),\mathsf{f}}(K) = \{aD \mid a \in K\}$, and $\mathcal{M}_{t(D),\mathsf{f}}(K)^{\bullet} \cong K^{\times}/D^{\bullet}$ is cancellative.*

Proof. 1., 2. and 3. are easy exercises in elementary number theory (see [25, Ch. 10]).

4. If $a \in D$ and $X \subset aD$, then $X_v \subset aD$, which implies $\subset$. To prove the reverse inclusion, let $z \in D$ be such that $z \in aD$ for all $a \in D$ satisfying $X \subset aD$. We must prove that $z \in X_v = (D\!:\!(D\!:\!X))$, that is, $zx \in D$ for all $x \in (D\!:\!X)$. Thus let $x \in (D\!:\!X) \subset K$, say $x = c^{-1}b$, where $c, b \in D$ and $\mathrm{GCD}(b,c) = D^{\times}$. Then $c^{-1}bX \subset D$, hence $X \subset cb^{-1}D \cap D$, and we assert that $cb^{-1}D \cap D \subset cD$. Indeed, if $v \in D$ and $cb^{-1}v \in D$, then $b \,|\, cv$, hence $b \,|\, v$ and thus $cb^{-1}v \in cD$. Now $X \subset cD$ implies $z \in cX$ and $zx \in bD \subset D$.

Hence it follows that $X_v = dD$ if and only if dD is the smallest principal ideal containing X, which by definition is equivalent to $d \in \mathrm{GCD}(X)$.

5. If $E \in \mathbb{P}_{\mathsf{f}}(K)$, let $c \in D^{\bullet}$ be such that $cE \subset D$ and $d \in \mathrm{GCD}(cE)$. Then $cED = dD = (cE)_{t(D)}$ and thus $E_{t(D)} = c^{-1}dD$. Hence the map $\partial\colon K^{\times} \to \mathcal{M}_{t(D),\mathsf{f}}(K)$, defined by $\partial(a) = aD$, is a group epimorphism with kernel $D^{\times}$ and induces an isomorphism $\mathcal{M}_{t(D),\mathsf{f}}(K)^{\bullet} \overset{\sim}{\to} K^{\times}/D^{\bullet}$. $\square$

Lemma 7.3. *For $i \in \{1,2\}$, let G_i be a* GCD*-monoid, $K_i = \mathsf{q}(G_i)$ and $t_i = t(G_i)$. A monoid homomorphism $\varphi\colon K_1 \to K_2$ is a (t_1,t_2)-homomorphism if and only if $\varphi(G_1) \subset G_2$ and $\varphi \,|\, G_1\colon G_1 \to G_2$ is a* GCD*-homomorphism. In particular, there is a bijective map*
$$\mathrm{Hom}_{(t_1,t_2)}(K_1,K_2) \;\to\; \mathrm{Hom}_{\mathrm{GCD}}(G_1,G_2)\,, \quad \textit{defined by} \quad \varphi \;\mapsto\; \varphi \,|\, G_1\,.$$

Proof. Let first φ be a (t_1, t_2)-homomorphism. Then

$$\varphi(G_1) = \varphi(\{1\}_{t_1}) \subset \{\varphi(1)\}_{t_2} = \{1\}_{t_2} = G_2\,.$$

Let $E \subset G_1$ be finite, $d_1 \in \mathrm{GCD}(E)$ and $d_2 \in \mathrm{GCD}(\varphi(E))$. Then $E_{t_1} = d_1G_1$, and $\varphi(D)_{t_2} = d_2G_2$. Since $d_1 \,|\, x$ for all $x \in E$, it follows that $\varphi(d_1) \,|\, y$ for all $y \in \varphi(E)$, and thus $\varphi(d_1) \,|\, d_2$. But $\varphi(d_1) \in \varphi(E_{t_1}) \subset \varphi(E)_{t_2} = d_2G_2$ implies $d_2 \,|\, \varphi(d_1)$ and therefore $\varphi(d_1) \in d_2G_2^\times = \mathrm{GCD}(\varphi(E))$.

Assume now that $\varphi(G_1) \subset G_2$, and let $\varphi \mid G_1 \colon G_1 \to G_2$ be a GCD-homomorphism. It is obviously sufficient to prove $\varphi(E_{t_1}) \subset \varphi(E)_{t_2}$ for all $E \in \mathbb{P}_{\mathsf{f}}(G_1)$. If $E \in \mathbb{P}_{\mathsf{f}}(G_1)$ and $d \in \mathrm{GCD}(E)$, then $\varphi(d) \in \mathrm{GCD}(\varphi(E))$ and therefore $\varphi(E_{t_1}) = \varphi(dG_1) \subset \varphi(d)G_2 = \varphi(E)_{t_2}$. $\square$

Lemma 7.4. *Let r be a finitary module system on K and $V \subset K$ a valuation monoid. Then $V = V_r$ if and only if id_K is an $(r, t(V))$-homomorphism.*

Proof. If id_K is an $(r, t(V))$-homomorphism, the $V \subset V_r \subset V_{t(V)} = V$ and thus $V = V_r$. Conversely, assume that $V = V_r$. If $E \in \mathbb{P}_{\mathsf{f}}(K)$, then Lemma 7.2.5 implies that $E_{t(V)} = EV = aV$ for some $a \in E$, and therefore we obtain $E_r \subset (aV)_r = aV = E_{t(V)}$. Hence id_K is an $(r, t(V))$-homomorphism by Proposition 4.4. $\square$

Proposition 7.5. *Let G be a* GCD*-monoid, $K = \mathsf{q}(G)$, $V \subset K$ a submonoid and $t = t(G)$.*

1. *Let V be a valuation monoid. Then $V = V_t$ if and only if $G \subset V$ and $G \hookrightarrow V$ is a* GCD*-homomorphism.*
2. *V is a t-valuation monoid if and only if $V = G_P$ for some $P \in t$-*spec*(G). In particular, G is the intersection of all t-valuation monoids of K.*

Proof. 1. By Lemma 7.4 we have $V = V_t$ if and only if id_K is a $(t, t(V))$-homomorphism, and by Lemma 7.3 this holds if and only if $G \subset V$ and $G \hookrightarrow V$ is a GCD-homomorphism.

2. Let first V be a t-valuation monoid. By Lemma 7.4, $j = (G \hookrightarrow V)$ is a $(t, t(V))$-homomorphism, and since $t(V) = s(V)$, it follows by Proposition 4.4 that $P = G \setminus V^\times = j^{-1}(V \setminus V^\times) \in t\text{-spec}(G)$. Since $G \setminus P \subset V^\times$, we obtain $G_P \subset V$. To prove the reverse inclusion, let $z = a^{-1}b \in V$, where $a, b \in G$ and $\mathrm{GCD}_G(a, b) = G^\times$. By 1., $G \hookrightarrow V$ is a GCD-homomorphism, hence $\mathrm{GCD}_V(a, b) = V^\times$, and thus either $a \in V^\times$ or $b \in V^\times$. If $a \in V^\times$, then $a \notin P$ and thus $z \in G_P$. If $b \in V^\times$, then $z \in V$ implies $b \in aV$, hence $a \in V$ and again $z \in G_P$.

Assume now that $P \in t\text{-spec}(G)$ and $z = a^{-1}b \in K$, where $a, b \in D$ and $\mathrm{GCD}(a, b) = D^\times$. Then $\{a, b\}_t = D$, hence $\{a, b\}_t \not\subset P = P_t$ and thus either $a \notin P$ or $b \notin P$. If $a \notin P$, then $z \in G_P$, and if $b \notin P$, then $z^{-1} \in G_P$. Therefore G_P is a valuation monoid, and $(G_P)_t = (G_t)_P = G_P$.

By Proposition 6.4, this implies that G is the intersection of all t-valuation monoids of K. $\square$

8 Integral closures and cancellation properties

Let K be a monoid and $D \subset K$ a submonoid.

Proposition 8.1. *Let r be a weak module system on K and $A \in \mathcal{M}_{r,\mathsf{f}}(K)$.*

1. *The following assertions are equivalent:*
   (a) *A is cancellative in $\mathcal{M}_{r,\mathsf{f}}(K)$ (that is, for all finite subsets $X, Y \subset K$, if $(AX)_r = (AY)_r$, then $X_r = Y_r$).*
   (b) *For all finite subsets $X, Y \subset K$, if $(AX)_r \subset (AY)_r$, then $X_r \subset Y_r$.*
   (c) *For all finite subsets $X \subset K$ and $c \in K$, if $cA \subset (AX)_r$, then $c \in X_r$.*
   (d) *For all finite subsets $X \subset K$ we have $((AX)_r : A) \subset X_r$*

   In each of the above assertions, the statement "for all finite subsets" can be replaced by the statement "for all r-finite r-modules".
2. *$\mathcal{M}_{r,\mathsf{f}}(K)^\bullet$ is cancellative if and only if $((EF)_r : E) \subset F_r$ for all $E \in \mathbb{P}_{\mathsf{f}}^\bullet(K)$ and $F \in \mathbb{P}_{\mathsf{f}}(K)$.*

Proof. 1. (a) $\Rightarrow$ (b) If $(AX)_r \subset (AY)_r$, then

$$(AY)_r = [(AX)_r \cup (AY)_r]_r = (AX \cup AY)_r = [A(X \cup Y)]_r \,,$$

and therefore $X_r \subset (X \cup Y)_r = Y_r$.

(b) $\Rightarrow$ (c) If $cA \subset (AX)_r$, then $(A\{c\})_r = (cA)_r \subset (AX)_r$, and thus $c \in \{c\}_r \subset X_r$.

(c) $\Rightarrow$ (d) If $z \in ((AX)_r : A)$, then $zA \in (AX)_r$ and therefore $z \in X_r$.

(d) $\Rightarrow$ (a) If $(AX)_r = (AY)_r$, then $AX_r \subset (AY)_r$ and $AY_r \subset (AX)_r$, hence $X_r \subset ((AY)_r : A) \subset Y_r$ and $Y_r \subset ((AX)_r : A) \subset X_r$, whence $X_r = Y_r$.

If $X \subset K$, then $(AX)_r = (AX_r)_r$, and thus the statement "for all finite subsets" can always be replaced by the statement "for all r-finite r-modules".

2. By 1.(d), since $\mathcal{M}_{r,\mathsf{f}}(K)^\bullet$ is cancellative if and only if E_r is cancellative for all $E \in \mathbb{P}_{\mathsf{f}}^\bullet(K)$. $\square$

Theorem 8.2. *Let r be a finitary weak module system on K, and let*

$$r_{\mathsf{a}} \colon \mathbb{P}(K) \to \mathbb{P}(K) \quad \text{be defined by} \quad X_{r_{\mathsf{a}}} = \bigcup_{B \in \mathbb{P}_{\mathsf{f}}^\bullet(K)} ((XB)_r : B) \,.$$

1. *r_{a} is a finitary weak module system on K, $r \le r_{\mathsf{a}}$, and if r is a module system, then so is r_{a}.*
2. *$\mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)^\bullet$ is cancellative, and if q is any finitary weak module system on K such that $r \le q$ and $\mathcal{M}_{q,\mathsf{f}}(K)^\bullet$ is cancellative, then $r_{\mathsf{a}} \le q$. In particular, $(r_{\mathsf{a}})_{\mathsf{a}} = r_{\mathsf{a}}$, and $\mathcal{M}_{r,\mathsf{f}}(K)^\bullet$ is cancellative if and only if $r = r_{\mathsf{a}}$.*
3. *$r[D]_{\mathsf{a}} = r_{\mathsf{a}}[D]$, and if r is a weak D-module system, then so is r_{a}.*

4. *If* G *is a reduced* GCD*-monoid and* $L = \mathsf{q}(G)$*, then*

$$\mathrm{Hom}_{(r,t(G))}(K,L) = \mathrm{Hom}_{(r_{\mathsf{a}},t(G))}(K,L)\,.$$

In particular, every r*-valuation monoid of* K *is an* r_{a}*-valuation monoid of* K*.*

Proof. 1. If $X \subset K$ and $B \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)$, then $X_rB \subset (XB)_r$, hence $X_r \subset ((XB)_r\!:\!B) \subset X_{r_{\mathsf{a}}}$ and, since r is finitary,

$$X_{r_{\mathsf{a}}} = \bigcup_{B\in\mathbb{P}_{\mathsf{f}}^{\bullet}(K)} \Big(\big(\bigcup_{E\in\mathbb{P}_{\mathsf{f}}(X)} EB\big)_r\!:\!B\Big) = \bigcup_{B\in\mathbb{P}_{\mathsf{f}}^{\bullet}(K)}\ \bigcup_{E\in\mathbb{P}_{\mathsf{f}}(X)} ((EB)_r\!:\!B) = \bigcup_{E\in\mathbb{P}_{\mathsf{f}}(X)} E_{r_{\mathsf{a}}}\,.$$

Therefore it remains to prove that r_{a} is a (weak) module system, and by Theorem 3.6 we have to check the conditions of Definition 3.1 for all finite subsets $X, Y \subset K$ and $c \in K$. Thus let $X, Y \in \mathbb{P}_{\mathsf{f}}(K)$ and $c \in K$. The verification of **M1.**, **M3.** and **M3′.** is straightforward.

M2. Let $X \subset Y_{r_{\mathsf{a}}}$ and $z \in X_{r_{\mathsf{a}}}$. Then there exists some $F \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)$ such that $z \in ((XF)_r\!:\!F)$, and since $\{((YB)_r\!:\!B) \mid B \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)\}$ is directed, there exists some $B \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)$ such that $X \subset ((YB)_r\!:\!B)$. Then

$$zFB \subset (XF)_rB \subset (XBF)_r \subset [(YB)_rF]_r = (YFB)_r$$

and thus $z \in ((YFB)_r\!:\!FB) \subset Y_{r_{\mathsf{a}}}$, since $FB \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)$.

2. By Proposition 8.1 we must prove that $((EF)_{r_{\mathsf{a}}}:E) \subset F_{r_{\mathsf{a}}}$ holds for all $E \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)$ and $F \in \mathbb{P}_{\mathsf{f}}(K)$. Thus let $E \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)$, $F \in \mathbb{P}_{\mathsf{f}}(K)$ and $z \in ((EF)_{r_{\mathsf{a}}}\!:\!E)$. Then $zE \subset (EF)_{r_{\mathsf{a}}}$ implies $zE \subset ((EFB)_r\!:\!B)$ for some $B \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)$ (since $\{((EFB)_r\!:\!B) \mid B \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)\}$ is directed). Hence it follows that $zEB \subset (EFB)_r$ and $z \in ((EFB)_r\!:\!EB) \subset F_{r_{\mathsf{a}}}$, since $EB \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)$.

Let now q be any finitary weak module system on K such that $r \le q$ and $\mathcal{M}_{q,\mathsf{f}}(K)^{\bullet}$ is cancellative. For any $X \in \mathbb{P}_{\mathsf{f}}(K)$ and $B \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)$, Proposition 8.1 implies $((XB)_r\!:\!B) \subset ((XB)_q\!:\!B) \subset X_q$, and thus $r_{\mathsf{a}} \le q$ by Proposition 4.4.2.

3. For $X \subset K$, it is easily checked that $X_{r_{\mathsf{a}}[D]} = X_{r[D]_{\mathsf{a}}}$.

4. Since $r \le r_{\mathsf{a}}$, every (r_{a},t)-homomorphism is an (r,t)-homomorphism. If $\varphi\colon K \to L$ is an (r,t)-homomorphism, then by Proposition 4.4.2 we must prove that $\varphi(X_{r_{\mathsf{a}}}) \subset \varphi(X)_{t(G)}$ for all $X \in \mathbb{P}_{\mathsf{f}}(K)$. If $X \in \mathbb{P}_{\mathsf{f}}(K)$, $z \in X_{r_{\mathsf{a}}}$ and $B \in \mathbb{P}_{\mathsf{f}}^{\bullet}(K)$ are such that $zB \subset (XB)_r$, then

$$\varphi(z)\varphi(B) \subset \varphi((XB)_r) \subset \varphi(XB)_t = [\varphi(X)\varphi(B)]_t$$

and therefore $\varphi(z) \in \big([\varphi(X)\varphi(B)]_t : \varphi(B)\big) \subset \varphi(X)_t$ by Proposition 8.1 and Lemma 7.2.4.

If $V \subset K$ is a valuation monoid, then it follows by Lemma 7.4 that V is an r- (resp. r_{a}-)valuation monoid if and only if id_K is an $(r,t(V))$- (resp.

$(r_{\mathsf{a}}, t(V))$-homomorphism. Hence every r-valuation monoid is an r_{a}-valuation monoid. □

Definition 8.3. Let r be a finitary weak module system on K. The finitary weak module system r_{a} is called the *cancellative extension* of r. An element $a \in K$ is called r*-integral* over D if $a \in D_{r_{\mathsf{a}}}$. A subset $X \subset K$ is called r-integral over D if $X \subset D_{r_{\mathsf{a}}}$. The monoid $D_{r_{\mathsf{a}}}$ is called the r*-closure* of D, and D is called r*-closed* if $D = D_{r_{\mathsf{a}}}$.

Remark 8.4. The notion of r-integrality generalizes the concept of integral elements in commutative ring theory. If D is an integral domain and $d = d(D)$ is the module system induced by the Dedekind system on K, then $D_{d_{\mathsf{a}}}$ is the integral closure of D. Most results of the classical theory of integral elements (transitivity and localization properties) continue to hold for r-integrality (see [25, Ch.14] for details, [27] for a version for not necessarily cancellative monoids and [15, Example 2.1] for the history of the concept). In Krull's ancient terminology (which is still used in the theory of semistar operations, see [23, §32]) ideal systems x for which $\mathcal{M}_{x,\mathsf{f}}(K)^\bullet$ is cancellative, are called "e.a.b." (endlich arithmetisch brauchbar). In the case of ideal systems on monoids, the construction of r_{a} goes back to P. Lorenzen [34] who constructed a multiplicative substitute for the Kronecker function ring. A readable overview of the development of the concepts and results related to Kronecker function rings and semistar operations was given by M. Fontana and K.A. Loper [20].

Definition 8.5. Let r be a finitary module system on K. We denote by $\Lambda_r(K) = \mathsf{q}(\mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K))$ the quotient of the monoid $\mathcal{M}_{r_{\mathsf{a}}}(K)$ ($\mathcal{M}_{r_{\mathsf{a}}}(K)^\bullet$ is cancellative, see Theorem 8.2.2). The group $\Lambda_r(K)^\times$ is a quotient group of $\mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)^\bullet$ and is called the *Lorenzen r-group*. For $X \in \Lambda_r(K)^\bullet$, we denote by $X^{[-1]}$ its inverse in the group $\Lambda_r(K)^\times$. Then we obtain, by the very definition,

$$\Lambda_r(K) = \{C^{[-1]}A \mid A \in \mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)\,,\ C \in \mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)^\bullet\,,\}\,.$$

If $A, A' \in \mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)$ and $C, C' \in \mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)^\bullet$, then $C^{[-1]}A = C'^{[-1]}A'$ if and only if $(AC')_{r_{\mathsf{a}}} = (A'C)_{r_{\mathsf{a}}}$, and multiplication in $\Lambda_r(K)$ is given by the formula $(C^{[-1]}A)\cdot(C'^{[-1]}A') = (CC')_r{}^{[-1]}(AA')_r$. In particular, $D_{r_{\mathsf{a}}} = \{1\}_{r_{\mathsf{a}}}$ is the unit element of $\Lambda_r(K)$. The submonoid

$$\Lambda_r^+(K) = \{C^{[-1]}A \ \mid \ A \in \mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)\,,\ C \in \mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)^\bullet\,,\ A \subset C\} \ \subset \ \Lambda_r(K)$$

is called the *Lorenzen r-monoid*. It is easily checked that $\Lambda_r^+(K) \subset \Lambda_r(K)$ is really a submonoid, and $\mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K) \subset \Lambda_r(K)$. The *Lorenzen homomorphism* $\tau_r \colon K \to \Lambda_r(K)$ is defined by $\tau_r(a) = \{a\}_{r_{\mathsf{a}}} = aD_{r_{\mathsf{a}}} \in \mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K) \subset \Lambda_r(K)$ for all $a \in K$.

Theorem 8.6. *Let r be a finitary module system on K, $D \subset \{1\}_{r_{\mathsf{a}}}$ and $K = \mathsf{q}(D)$. Let $t = t(\Lambda_r^+(K))$ be the t-system on $\Lambda_r(K)$ induced from $\Lambda_r^+(K)$.*

1. *If* $A \in \mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)$ *and* $C \in \mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)^\bullet$, *then* $C^{[-1]}A \in \Lambda_r^+(K)$ *if and only if* $A \subset C$.
2. $\Lambda_r^+(K)$ *is a reduced* GCD*-monoid, and* $\Lambda_r(K)$ *is a quotient of* $\Lambda_r^+(K)$. *If* $X, Y \in \Lambda_r^+(K)$, *then there exist* $A, B \in \mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)$ *and* $C \in \mathcal{M}_{r_{\mathsf{a}},\mathsf{f}}(K)^\bullet$ *such that* $A \cup B \subset C$, $X = C^{[-1]}A$ *and* $Y = C^{[-1]}B$. *In this case, we have* $X \,|\, Y$ *if and only if* $B \subset A$, *and* $\gcd(X,Y) = C^{[-1]}(A \cup B)_{r_{\mathsf{a}}}$.
3. *For every* $X \in \Lambda_r^+(D)$ *there exist* $E \in \mathbb{P}_{\mathsf{f}}(D)$ *and* $E' \in \mathbb{P}_{\mathsf{f}}^\bullet(D)$ *such that* $E_{r_{\mathsf{a}}} \subset E'_{r_{\mathsf{a}}}$ *and* $X = {E'_{r_{\mathsf{a}}}}^{[-1]} E_{r_{\mathsf{a}}} = \gcd(\tau_r(E'))^{[-1]} \gcd(\tau_r(E)$.
4. *The Lorenzen homomorphism* $\tau_r \colon K \to \Lambda_r(K)$ *is an* (r_{a}, t)*-homomorphism and* $\tau_r \,|\, K^\times \colon K^\times \to \Lambda_r(K)^\times$ *is a group homomorphism satisfying* $\mathrm{Ker}(\tau_r \,|\, K^\times) = D_{r_{\mathsf{a}}}^\times$.
5. *For every* $Z \subset K$ *we have* $Z_{r_{\mathsf{a}}} = \tau_r^{-1}[\tau_r(Z)_t] = \{c \in K \mid \{c\}_{r_{\mathsf{a}}} \in \tau_r(Z)_t\}$, *and in particular* $\tau_r^{-1}(\Lambda_r^+(K)) = D_{r_{\mathsf{a}}}$.

Proof. The assertions 1. to 4. follow immediately from the definitions.

5. Let now first $Z \subset K$ be finite, say $Z = a^{-1}A$, where $a \in D^\bullet$ and $A = \{a_1, \ldots, a_n\} \subset D \subset \{1\}_{r_{\mathsf{a}}}$. Then

$$\begin{aligned} A_{r_{\mathsf{a}}} = (\{a_1\}_{r_{\mathsf{a}}} \cup \ldots \cup \{a_n\}_{r_{\mathsf{a}}})_{r_{\mathsf{a}}} &= \gcd(\{a_1\}_{r_{\mathsf{a}}}, \ldots, \{a_n\}_{r_{\mathsf{a}}}) \\ &= \gcd(\tau_r(a_1), \ldots, \tau_r(a_n)) = \gcd(\tau_r(A)) \end{aligned}$$

and therefore $\tau_r(A)_t = A_{r_{\mathsf{a}}} \Lambda_r^+(K)$ by Lemma 7.2.4. For $c \in K$, we have $c \in \tau_r^{-1}[\tau_r(Z)_t]$ if and only if

$$\tau_r(ac) = \tau_r(a)\tau_r(c) \in \tau_r(a)\tau_r(Z)_t = \tau_r(aZ)_t = \tau_r(A)_t = A_{r_{\mathsf{a}}}\Lambda_r^+(K)\,,$$

and therefore we obtain

$$\begin{aligned} \tau_r(A)_t = A_{r_{\mathsf{a}}}\Lambda_r^+(K) &\iff {A_{r_{\mathsf{a}}}}^{[-1]} ac_{r_{\mathsf{a}}} \in \Lambda_r^+(K) \iff \{ac\}_{r_{\mathsf{a}}} \subset A_{r_{\mathsf{a}}} \\ &\iff ac \in A_{r_{\mathsf{a}}} = aZ_{r_{\mathsf{a}}} \iff c \in Z_{r_{\mathsf{a}}}\,. \end{aligned}$$

Hence $Z_{r_{\mathsf{a}}} = \tau_r^{-1}(\tau_r(Z)_t)$ and $D_{r_{\mathsf{a}}} = \tau_r^{-1}(\tau_r(\{1\}_t) = \tau_r^{-1}(\Lambda_r^+(K))$. If finally $Z \subset K$ is arbitrary, then

$$Z_{r_{\mathsf{a}}} = \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(Z)} E_{r_{\mathsf{a}}} = \bigcup_{E \in \mathbb{P}_{\mathsf{f}}(Z)} \tau_r^{-1}[\tau_r(E)_t] = \tau_r^{-1}\Big(\bigcup_{F \in \mathbb{P}_{\mathsf{f}}(\tau_r(Z))} F_t\Big) = \tau_r^{-1}(\tau_r(Z)_t)\,.$$

In particular, it follows that $\tau_r(Z_{r_{\mathsf{a}}}) \subset \tau_r(Z)_t$, and thus τ_r is an (r_{a}, t)-homomorphism. □

Remark 8.7. Let D be an integral domain with quotient field K, $*$ a semistar operation on D and $r = r^*$ the module system on K induced by $*$. Then the Lorenzen r-monoid $\Lambda_r^+(K)$ is isomorphic to the monoid $(\mathrm{Kr}(D,*))$ of principal ideals of the semistar Kronecker function ring $\mathrm{Kr}(D,*)$ (see [19]). We recall the definition: $\mathrm{Kr}(D,*)$ consists of all rational functions f/g with $f, g \in D[X]$ such that $g \neq 0$ and there exists some $h \in D[X]^\bullet$ satisfying

$[\boldsymbol{c}(f)\boldsymbol{c}(h)]^* \subset [\boldsymbol{c}(g)\boldsymbol{c}(h)]^*$. An isomorphism $(\mathrm{Kr}(*, D)) \to \Lambda_r^+(K)$ is given by the assignment $(f/g) \mapsto \boldsymbol{c}(g)_{r_\mathsf{a}}^{[-1]}\boldsymbol{c}(f)_{r_\mathsf{a}}$.

Theorem 8.8 (Universal property of the Lorenzen monoid). *Let r be a finitary module system on K, $D \subset \{1\}_{r_\mathsf{a}}$, $K = \mathsf{q}(D)$ and $t = t(\Lambda_r^+(K))$ the t-system on $\Lambda_r(K)$ induced from $\Lambda_r^+(K)$. If G is a reduced* GCD*-monoid and $L = \mathsf{q}(G)$, then there is a bijective map*

$$\mathrm{Hom}_{(t,t(G))}(\Lambda_r(K), L) \;\to\; \mathrm{Hom}_{(r,t(G))}(K, L)\,, \quad \textit{defined by} \quad \phi \;\mapsto \phi \circ \tau_r\,.$$

Proof. If $\Phi\colon \Lambda_r(K) \to L$ is a $(t, t(G))$-homomorphism, then $\Phi \circ \tau_r \colon K \to L$ is an $(r, t(G))$-homomorphism, since τ_r is an (r_a, t)-homomorphism and thus also an (r, t)-homomorphism. We prove that for every $\varphi \in \mathrm{Hom}_{(r,t(G))}(K, L)$ there is a unique $\Phi \in \mathrm{Hom}_{(t,t(G))}(\Lambda_r(K), L)$ such that $\Phi \circ \tau_r = \varphi$. Thus let $\varphi \in \mathrm{Hom}_{(r,t(G))}(K, L)$.

By Lemma 7.3, the map $\mathrm{Hom}_{(t,t(G))}(\Lambda_r(K), L) \to \mathrm{Hom}_{\mathrm{GCD}}(\Lambda_r^+(K), G)$, defined by $\Phi \mapsto \Phi\,|\,\Lambda_r^+(K)$, is bijective, and for $\Phi \in \mathrm{Hom}_{(t,t(G))}(\Lambda_r(K), L)$ we have $\Phi \circ \tau_r = \varphi$ if and only if $[\Phi\,|\,\Lambda_r^+(K)] \circ (\tau_r\,|\,D) = \varphi\,|\,D$ (since $K = \mathsf{q}(D)$). Hence it suffices to prove that there exists a unique $\psi \in \mathrm{Hom}_{\mathrm{GCD}}(\Lambda_r^+(K), G)$ such that $\psi \circ \tau_r(a) = \varphi(a)$ for all $a \in D^\bullet$.

Uniqueness: If $\psi \in \mathrm{Hom}_{\mathrm{GCD}}(\Lambda_r^+(K), G)$ be such that $\psi \circ \tau_r(a) = \varphi(a)$ for all $a \in D^\bullet$ and $X = \gcd(\tau_r(E'))^{[-1]} \gcd(\tau_r(E)) \in \Lambda_r^+(K)$ (where $E \in \mathbb{P}_\mathsf{f}(D)$, $E' \in \mathbb{P}_\mathsf{f}^\bullet(D)$ and $E_{r_\mathsf{a}} \subset E'_{r_\mathsf{a}}$), then

$$\psi(X) = \gcd[\psi(\tau_r(E'))]^{-1} \gcd[\psi(\tau_r(E))] = \gcd[\varphi(E')]^{-1} \gcd[\varphi(E)]\,,$$

and thus ψ is uniquely determined by φ.

Existence: Define ψ provisionally by $\psi(X) = \gcd(\varphi(E'))^{-1} \gcd(\varphi(E))$ if $X = \gcd(\tau_r(E'))^{[-1]} \gcd(\tau_r(E))$ with $E \in \mathbb{P}_\mathsf{f}(D)$, $E' \in \mathbb{P}_\mathsf{f}^\bullet(D)$ and $E_{r_\mathsf{a}} \subset E'_{r_\mathsf{a}}$. We must prove the following assertions: **1)** $\psi(X) \subset G$; **2)** the definition is independent of the choice of E and E'; **3)** ψ is a GCD-homomorphism. The proofs are lengthy but straightforward and are left to the reader. □

Theorem 8.9. *Let r be a finitary module system on K, $D \subset \{1\}_{r_\mathsf{a}}$ and $K = \mathsf{q}(D)$. Let $t = t(\Lambda_r^+(K))$ the t-system on $\Lambda_r(K)$ induced from $\Lambda_r^+(K)$. Let $\mathcal{V}$ be the set of all r-valuation monoids in K and $\mathcal{W}$ the set of all t-valuation monoids in $\Lambda_r(K)$. Then $\mathcal{V} = \{\tau_r^{-1}(W) \mid W \in \mathcal{W}\}$.*

Proof. If $W \in \mathcal{W}$ and $x \in K \setminus \tau_r^{-1}(W)$, then $\tau_r(x)^{-1} = \tau_r(x^{-1}) \in W$ and therefore $x^{-1} \in \tau_r^{-1}(W)$. Hence $\tau_r^{-1}(W)$ is a valuation monoid, and since τ_r is an (r, t)-homomorphism, it is even an r-valuation monoid and lies in $\mathcal{V}$.

Let now $V \in \mathcal{V}$ and $\pi\colon K \to K/V^\times$ the canonical epimorphism. Then $V/V^\times$ is a reduced valuation monoid, $\mathsf{q}(V/V^\times) = K/V^\times$, and we denote by $t^* = t(V/V^\times) = s(V/V^\times)$ the module system on $K/V^\times$ which is induced by the t-system on $V/V^\times$. Since $r \le r_V = s(V)$, it follows that π is an (r, t^*)-homomorphism. By Theorem 8.8, the assignment $\Phi \mapsto \Phi \circ \tau_r$ defines a bijective map $\mathrm{Hom}_{(t,t^*)}(\Lambda_r(K), K/V^\times) \to \mathrm{Hom}_{(r,t^*)}(K, K/V^\times)$. Hence there is a unique (t, t^*)-homomorphism $\Phi\colon \Lambda_r(K) \to K/V^\times$ such that $\Phi \circ \tau_r = \pi$, and Φ is surjective, since π is surjective. Now $W = \Phi^{-1}(V/V^\times) \subset \Lambda_r(K)$ is a valuation monoid, and $\tau_r^{-1}(W) = (\Phi\circ\tau_r)^{-1}(V/V^\times) = \pi^{-1}(V/V^\times) = V$. Thus it remains to prove that $W_t = W$. Since Φ is a (t, t^*)-homomorphism, it follows that $\Phi(W_t) \subset \Phi(W)_{t^*} = (V/V^\times)_{t^*} = V/V^\times$ and $W_t \subset \Phi^{-1}(V/V^\times) = W$, whence $W_t = W$. □

Theorem 8.10. *Let r be a finitary module system on K, $D \subset \{1\}_{r_{\mathsf{a}}}$ and $K = \mathsf{q}(D)$. If $\mathcal{V}_r(D)$ denotes the set of all r-valuation monoids of K containing D, then $\mathcal{V}_r(D) = \mathcal{V}_{r_{\mathsf{a}}}(D_{r_{\mathsf{a}}})$ and*

$$D_{r_{\mathsf{a}}} = \{1\}_{r_{\mathsf{a}}} = \bigcap_{V\in\mathcal{V}_r(D)} V\,.$$

Proof. By Theorem 8.2.4, a monoid $V \subset K$ is an r-valuation monoid if and only if it is an r_{a}-valuation monoid. Hence $\mathcal{V}_r(D) = \mathcal{V}_{r_{\mathsf{a}}}(D) \supset \mathcal{V}_{r_{\mathsf{a}}}(D_{r_{\mathsf{a}}})$, and if $V \in \mathcal{V}_r(D)$, then $\{1\}_{r_{\mathsf{a}}} = D_{r_{\mathsf{a}}} \subset V_{r_{\mathsf{a}}} = V$ and thus $V \in \mathcal{V}_{r_{\mathsf{a}}}(D_{r_{\mathsf{a}}})$.

Let $\tau_r\colon K \to \Lambda_r(K)$ be the Lorenzen homomorphism, $t = t(\Lambda_r^+(K))$ and $\mathcal{W}$ the set of all t-valuation monoids in $\Lambda_r(K)$. By Theorem 8.9 we have $\mathcal{V}_r(D) = \{\tau_r^{-1}(W) \mid W \in \mathcal{W}\}$ and, applying Proposition 7.5.2 and Theorem 8.6.3, we obtain

$$D_{r_{\mathsf{a}}} = \tau_r^{-1}(\Lambda_r^+(K)) = \tau_r^{-1}\Big(\bigcap_{W\in\mathcal{W}} W\Big) = \bigcap_{W\in\mathcal{W}} \tau_r^{-1}(W) = \bigcap_{V\in\mathcal{V}_r(D)} V\,. \qquad \square$$

Corollary 8.11. *Let $K = \mathsf{q}(D)$ and r a finitary ideal system on D. Then $D_{r_{\mathsf{a}}}$ is the intersection of all r-valuation monoids in K.*

Remark 8.12. In the case of integral domains, Theorem 8.10 generalizes the connection between semistar Kronecker function rings and valuation overrings as developed in [18]. In particular, Corollary 8.11 contains the classical fact that the integral closure of an integral domain is the intersection of its valuation overrings (see [23, (19.8)]).

9 Invertible modules and Prüfer-like conditions

Let K be a monoid and $D \subset K$ a submonoid such that $K = \mathsf{q}(D)$.

This final section contains the basics of a purely multiplicative theory of semistar invertibility and semistar Prüfer domains as it was developed only recently by M. Fontana with several co-authors (see [9], [17], [18], [21], [22], [15], [16], [6]). In particular, we refer to the examples presented in these papers which show the semistar approach covers really new classes of integral domains.

Definition 9.1. Let r be a module system on K. A D-module $A \subset K$ is called (r*-finitely*) r*-invertible* (relative D) if there exists a (finite) subset $B \subset (D\!:\!A)$ such that $(AB)_r = D_r$ [equivalently, $1 \in (AB)_r$].

By definition, A is r-invertible if and only if A is $r[D]$-invertible. If A is r-invertible, then A is q-invertible for every module system q on K satisfying $r \le q$, and every D-module A' with $A \subset A' \subset A_r$ is also r-invertible.

Lemma 9.2. *Let $A \subset K$ be a D-module and $B \subset K$ such that $D = AB$. Then $A = aD$ for some $a \in K$.*

Proof. Let $P = D \setminus D^\times$. Then $PA \subset A$, and we assert that $PA \ne A$. Indeed, if $PA = A$, then $P = PD = PAB = AB = D$, a contradiction. If $a \in A \setminus AP$, then $aD \subset A$, hence $aBD \subset AB = D$. If $aBD \ne D$, then $aBD \subset P$, since aBD is an ideal of D, and then $a \in aD = aABD \subset AP$, a contradiction. Hence $aBD = D$, and consequently $A = aABD = aD$. $\square$

Proposition 9.3. *Let r be a module system on K, $c \in K^\times$, and let $A \subset K$ be a D-module.*

1. *A is r-invertible if and only if $[A(D\!:\!A)]_r = D_r$, and then $(D\!:\!A)$ and $A_{v(D)}$ are also r-invertible.*
2. *If A is r-invertible, then cA is also r-invertible, and A_r is cancellative in $\mathcal{M}_r(K)$.*
3. *A is r-invertible (relative D) if and only if A_r is r-invertible (relative D_r) and $(D_r\!:\!A) = (D\!:\!A)_r$.*
4. *If $A_1, A_2 \subset K$ are D-modules, then A_1A_2 is r-invertible if and only if A_1 and A_2 are both r-invertible.*

Proof. 1. If $[A(D\!:\!A)]_r = D_r$, then A is r-invertible. If A is r-invertible, then there is some $B \subset (D\!:\!A)$ such that $(AB)_r = D_r$, and since $[A(D\!:\!A)]_r \subset D_r$, it follows that $[A(D : A)]_r = D_r$. Hence $(D : A)$ is r-invertible, and (by an iteration of the argument) $A_v = (D\!:\!(D\!:\!A))$ is also r-invertible.

2. Let A be r-invertible and $B \subset (D : A)$ such that $(AB)_r = D_r$. Since $c^{-1}B \subset (D : cA)$ and $((cA)(c^{-1}B))_r = D_r$, it follows that cA is also r-invertible. If $X, Y \in \mathcal{M}_r(D)$ and $(A_rX)_r = (A_rY)_r$, then it follows that $X = [(BA)_rX]_r = [B(A_rX)_r]_r = [B(A_rY)_r]_r = [(BA)_rY]_r = Y$, and thus A_r is cancellative.

3. By Proposition 3.3.3, A_r is a D_r-module. If A is r-invertible, then $D_r = [A(D\!:\!A)]_r \subset [A_r(D\!:\!A)_r]_r \subset [A_r(D_r\!:\!A)]_r = [A_r(D_r\!:\!A_r)]_r \subset D_r$, hence equality holds, A_r is r-invertible (relative D_r), and since A_r is cancellative in $\mathcal{M}_r(D)$, it follows that $(D\!:\!A)_r = (D_r\!:\!A)$. To prove the converse, let A_r be r-invertible (relative D_r) and $(D : A)_r = (D_r : A)$. Then it follows that $[A(D\!:\!A)]_r = [A_r(D\!:\!A)_r]_r = [A_r(D_r\!:\!A)_r]_r = [A_r(D_r\!:\!A_r)]_r = D_r$, and thus A is r-invertible relative D.

4. If A_1A_2 is r-invertible, then there is some $B \subset (D : A_1A_2)$ such that $(A_1A_2B)_r = D_r$. Since $A_1B \subset (D\colon A_2)$ and $A_2B \subset (D\colon A_1)$, it follows that A_1 and A_2 are both r-invertible. If A_1 and A_2 are r-invertible, then there exist $B_1 \subset (D\colon A_1)$ and $B_2 \subset (D\colon A_2)$ such that $(A_1B_1)_r = (A_2B_2)_r = D_r$. Now $(A_1A_2B_1B_2)_r = [(A_1B_1)_r(A_2B_2)_r]_r = D_r$ and $B_1B_2 \subset (D : A_1A_2)$ implies that A_1A_2 is r-invertible. $\square$

Proposition 9.4. *Let r be a finitary module system on K and $A \subset K$ a D-module.*

1. *The following assertions are equivalent*:
   (a) *A is r-invertible (relative D).*
   (b) *There exists a finite subset $F \subset (D\colon A)$ such that $1 \in (AF)_r$.*
   (c) *For all $P \in r_D\text{-}\max(D)$ we have $A(D\colon A) \not\subset P$.*
2. *If A is r-invertible, then A_r is $r[D]$-finite and A is r-finitely r-invertible.*
3. *If $T \subset D$ is multiplicatively closed and A is r-invertible, then $T^{-1}A$ is r-invertible (relative $T^{-1}D$).*

Proof. 1. (a) $\Rightarrow$ (b) If $B \subset (D\colon A)$ is such that $1 \in (AB)_r$, then (since r is finitary) there exists a finite subset $F \subset B$ such that $1 \in (AF)_r$.

(b) $\Rightarrow$ (c) Assume that $A(D : A) \subset P$ for some $P \in r_D\text{-}\max(D)$, and let $F \subset (D : A)$ be finite such that $1 \in (AF)_r$. Then it follows that $1 \in (AF)_r \cap D \subset [A(D\colon A)]_r \cap D \subset P_r \cap D = P$, a contradiction.

(c) $\Rightarrow$ (a) Since $A(D\colon A) \subset [A(D\colon A)]_r \cap D$, it follows that the r_D-ideal $[A(D : A)]_r \cap D$ is contained in no $P \in r_D\text{-}\max(D)$. Hence it follows that $[A(D\colon A)]_r \cap D = D \subset [A(D\colon A)]_r$ and therefore $[A(D\colon A)]_r = D_r$.

2. Let $B \subset (D\colon A)$ be such that $1 \in (AB)_r$, and let $E \subset A$ and $F \subset B$ be finite subsets satisfying $1 \in (EF)_r$. Then $D_r \subset (DEF)_r \subset (AF)_r \subset D_r$, which implies $D_r = (AF)_r$, and thus A is r-finitely r-invertible relative D. Moreover, it follows that $A_r = D_rA_r = (DEFA)_r = (DE)_r = E_{r[D]}$, and therefore A_r is $r[D]$-finite.

3. If $B \subset (D\colon A)$ is such that $(AB)_r = D_r$, then $B \subset (T^{-1}D\colon T^{-1}A)$ and $(T^{-1}AB)_r = (T^{-1}D)_r$. Hence $T^{-1}A$ is r-invertible (relative $T^{-1}D$). $\square$

Theorem 9.5. *Let r be a finitary module system on K and $A \subset K$ a D-module.*

1. *If A is r-invertible and $P \in r_D\text{-spec}(D)$, then $A_P = aD_P$ for some $a \in K^\times$.*
2. *Suppose that for every $P \in r_D\text{-}\max(D)$ there is some $a_P \in K^\times$ such that $A_P = a_PD_P$. If y is a finitary module system on K such that $D_y = D$ and A is y-finite, then A is r-invertible.*

Proof. 1. Let A be r-invertible, $B \subset (D : A)$ such that $(AB)_r = D_r$ and $P \in r_D\text{-spec}(D)$. Then $AB \not\subset P$, and since $AB \subset D$ is an ideal, we obtain $D_P = (AB)_P = A_PB_P$. Now the assertion follows by Lemma 9.2.

2. Suppose that $A = E_y$ for some $E \in \mathbb{P}_{\mathsf{f}}(K)$ and that A is not r-invertible. By Proposition 9.4, there is some $P \in r_D\text{-spec}(D)$ such that $A(D\!:\!A) \subset P$ and thus $a_P(D\!:\!A)_P \subset PD_P$. Since $D = D_y$, it follows that

$$(D\!:\!A)_P = (D\!:\!E)_P = (D_P\!:\!E) = (D_P\!:\!A_P) = a_P^{-1}D_P$$

and thus $PD_P \supset a_P(D\!:\!A)_P = D_P$, a contradiction. □

Definition 9.6. Let r and y be finitary module systems on K such that $y \le r$ and $D_y = D$. Then D is called a *y-basic r-Prüfer monoid* if every $A \in \mathcal{M}_{y,\mathsf{f}}(K)$ is r-invertible.

Remark 9.7. Let D be an integral domain, $*$ a semistar operation on D and $r = r^*$ the D-module system on K induced by $*$. Then D is a P$*$MD (as defined in [15]) if and only if D is a basic $d(D)$-Prüfer monoid.

Theorem 9.8. *Let r, q and y be finitary module systems on K such that q is a D-module system, $y \le q \le r$ and $D_y = D$.*

1. *If D is an y-basic r-Prüfer monoid, then D_P is a valuation monoid for every $P \in r_D\text{-spec}(D)$.*
2. *The the following assertions are equivalent:*
   - (a) *D is a y-basic r-Prüfer monoid.*
   - (b) *D is a y-basic $r[q]$-Prüfer monoid.*
   - (c) *D_P is a valuation monoid for every $P \in r_D\text{-max}(D)$.*

Proof. 1. Let $P \in r_D\text{-spec}(D)$. Since $D_P = D_P^\times D$, it suffices to prove that for all $a,\, b \in D^\bullet$ we have either $a \in bD_P$ or $b \in aD_P$. If $a,\, b \in D^\bullet$, then $\{a,b\}_y$ is r-invertible by the assumption: Let $B \subset (D\!:\!\{a,b\}_y) = (D\!:\!\{a,b\})$ be such that $1 \in (\{a,b\}_yB)_r$. We assert that even $1 \in \{a,b\}BD_P$. Indeed, if not, then $\{a,b\}BD_P \subset PD_P$, which implies $\{a,b\}B \subset PD_P \cap D = P$ and $1 \in (\{a,b\}_yB)_r \cap D = (\{a,b\}B)_r \cap D \subset P_r \cap D = P$, a contradiction.

Now it follows that $D_P = (\{a,b\}D_P)B$ and thus $\{a,b\}D_P = cD_P$ for some $c \in D_P$ by Lemma 9.2. Hence there exist $u,\, v \in D_P$ such that $a = cu$, $b = cv$, and $\{u,v\}D_P = D_P$. Therefore we have either $u \in D_P^\times$ or $v \in D_P^\times$ and thus either $b \in aD_P$ or $a \in bD_P$.

2. (a) $\Rightarrow$ (c) By 1.

(c) $\Rightarrow$ (a) Let $A = E_y \in \mathcal{M}_{y,\mathsf{f}}(K)$, where $E \in \mathbb{P}_{\mathsf{f}}(K)$, and assume that A is not r-invertible. By Proposition 9.4 there exists some $P \in r_D\text{-max}(D)$ such that $A(D\!:\!A) \subset P$. Since D_P is a valuation monoid, we obtain $ED_P = aD_P$ for some $a \in E$, and thus also $A_P = E_yD_P = (ED_P)_y = aD_P$. Since $[A(D:A)]_P = A_P(D:E)_P = A_P(D_P:ED_P) = aD_P(D_P:aD_P) = D_P$, we obtain $PD_P \supset [A(D\!:\!A)]_P = D_P$, a contradiction.

(a) $\Leftrightarrow$ (b) By Theorem 6.6.5 we have $r_D\text{-max}(D) = r[q]_D\text{-max}(D)$. We apply the equivalence of (a) and (c) with $r[q]$ instead of r and obtain the equivalence of (a) and (b). □

Corollary 9.9. *Let r and y be finitary module systems on K such that $y \leq r$ and $D_y = D$. If D is an y-basic r-Prüfer monoid, then every y-monoid T satisfying $D \subset T \subset K$ is also an y-basic r-Prüfer monoid.*

Proof. By Theorem 9.8 it suffices to prove that T_P is a valuation monoid if $P \in r_T\text{-max}(T)$. If $P \in r_T\text{-max}(T)$, then $P \cap D = P_r \cap T \cap D = P_r \cap D$. Hence $P \cap D \in r_D\text{-spec}(D)$, $D_{P \cap D}$ is a valuation monoid, and since $D_{P\cap D} \subset T_P$, it follows that T_P is also a valuation monoid.

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