CLIFFORD SEMIGROUPS OF IDEALS IN MONOIDS AND DOMAINS

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Abstract. We investigate the ideal semigroup and the ideal class semigroup built by the fractional ideals of an ideal system on a monoid or on a domain. We provide criteria for these semigroups to be Clifford semigroups or Boolean semigroups. In particular, we consider the case of valuation monoids (domains) and of Prüfer-like monoids (domains). By the way, we prove that a monoid (domain) is of Krull type if every locally principal ideal is finite.

1. Introduction

One of the main aims of multiplicative ideal theory is the description of an integral domain by means of the multiplicative semigroup of fractional ideals. In this context the ideal class group (built by the isomorphism classes of invertible fractional ideals) has been one of the major objects of investigations. Starting with the ideal class group of the ring of integers of algebraic number fields, this notion has obtained several important generalizations in commutative algebra. Among them, the class groups associated with star operations and ideal systems are the most general and fruitful ones (see [1], [9] and [17, Ch.12]). In particular, the divisor class groups of Krull domains and Krull monoids are special cases of these concepts (for their arithmetical relevance the interested reader is invited to consult [15]).

Only recently the class semigroup (built by the isomorphism classes of all non-zero fractional ideals) has been introduced and investigated by several authors. E.C. Dade, O. Taussky and H. Zassenhaus [11] investigated the structure of the class semigroup of a non-principal order in an algebraic number field. More generally, this was done in [18] for the semigroup of lattices over Dedekind domains. S. Bazzoni and L. Salce [7] proved that the ideal class semigroup of a valuation domain is a Clifford semigroup, and almost contemporaneously P. Zanardo and U. Zannier [23] did the same for orders in quadratic number fields (reproving results from [11]). They also observed that an integrally closed domain with Clifford class semigroup must be a Prüfer domain. A systematic study of integral domains with Clifford class semigroup was made in a series of papers by S. Bazzoni [2], [3], [4], [5]. Among others, she proved that a Prüfer domain has Clifford semigroup if and only if it has finite character, and she disclosed the connections with the theory of stable domains. S. Kabbaj and A. Mimouni in [19] and [20] continued the work of S. Bazzoni. They investigated not only the question whether a class semigroup is a Clifford semigroup but also the question whether it is a Boolean semigroup, they generalized several results for noetherian domains to Mori domains and, above all, they generalized Bazzoni's result for Prüfer domains by characterizing Prüfer v-multiplication domains (pseudo-Prüfer domains in the sense of Bourbaki [8, Ch.VII, Exercise 19]) with Clifford t-class semigroup.

Many of the results concerning integral domains with Clifford or Boolean class semigroup turn out to be purely multiplicative in nature and thus they can be formulated and proved in the language of ideal systems on cancellative commutative monoids. Also, it turns out, that the results concerning the structure of the ideal class semigroup are in fact results concerning the

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multiplicative semigroup of fractional ideals itself. In this paper, we shall take this point of view in a systematic way and investigate the semigroup of fractional ideals of a monoid defined by a finitary ideal system. Under this hypothesis, we shall generalize and unify several results of the literature and equip them with simpler proofs.

After reviewing the basic facts from the theory of semigroups in Section 2 and the theory of ideal systems in Section 3, we present the general concepts concerning N-regularity and stability of ideals relative to an ideal system in a purely multiplicative setting in Section 4. In Section 5 we sketch the results for valuation monoids (which are are almost identical with those for valuation domains). Finally, Section 6 deals with r-Prüfer monoids and contains the main results of the paper. We prove that the semigroup of fractional r-ideals of an r-Prüfer monoid D is a Clifford semigroup if and only if D is of Krull type, and that this is equivalent with the local invertibility property (as conjectured by S. Bazzoni [5, Question 2]). Finally, we strengthen the main result of Mimouni and Kabbaj on Prüfer-v-multiplication domains ([20, Theorem 3.2]) and prove that the semigroup of fractional ideals of a v-domain D (that is, of a regularly integrally closed domain in the sense of Bourbaki [8, Ch.VII, Exercise 30]) is a Clifford semigroup if and only if D is a domain of Krull type.

2. Commutative semigroups

By a *semigroup* S we always mean a multiplicative commutative semigroup containing a unit element 1 (satisfying $1x = x$ for all $x \in S$) and a zero element 0 (satisfying $0x = 0$ for all $x \in S$). An element $x \in S$ is called

- *invertible* if there is some (unique) $x' \in S$ such that $xx' = 1$.
- *cancellative* if $xy = xz$ implies $y = z$ for all $y, z \in D$.
- *von Neumann regular* (*N-regular* for short), if $x^2y = x$ for some $y \in S$.

Note that $0 \in S$ is idempotent, and every idempotent element is N-regular. Obviously, an element $x \in S$ is invertible if and only if it is cancellative and N-regular. We denote by $S^\times$ the group of invertible elements of S, and for $x \in S^\times$, we denote by x^{-1} its inverse. For any set X, we set $X^\bullet = X \setminus \{0\}$.

By a *monoid* we mean (deviating from the usual terminology) a semigroup D for which every $x \in D^\bullet$ is cancellative. If D is a monoid, then a monoid K is called a *quotient monoid* of D if $K^\bullet$ is a quotient group of $D^\bullet$ (and then $K^\bullet = K^\times$). Every monoid D possesses a quotient monoid which is unique up to canonical isomorphisms and is denoted by $\mathsf{q}(D)$. By an *overmonoid* of a monoid D we mean a monoid E such that $D \subset E \subset \mathsf{q}(D)$. By a *multiplicatively closed subset* of a monoid D we mean a subset $T \subset D^\bullet$ with $1 \in T$ and $TT = T$. By definition, the sets $\{1\}$, $D^\times$ and $D^\bullet$ are multiplicatively closed subsets of D. A subsemigroup or a submonoid is always assumed to contain 1, and a semigroup homomorphism is always assumed to respect 1.

Let S be a multiplicative commutative semigroup and $\mathrm{Id(S)}$ the subsemigroup of idempotent elements of S. If $S = \mathrm{Id(S)}$, then S is called a *Boolean semigroup.* For $e \in \mathrm{Id(S)}$, let S_e denote the set of all $x \in S$ such that $xe = x$ and $xy = e$ for some $y \in S$. The semigroup S is called a *Clifford semigroup* if

$$S = \bigcup_{e \in \mathrm{Id(S)}} S_e\,,$$

and for $e \in \mathrm{Id(S)}$ we call S_e the *constituent group* of e (see Lemma 2.1). By definition, we have $S_0 = \{0\}$.

If $\varphi\colon S \to S'$ is a semigroup homomorphism and $x \in S$ is idempotent [N-regular], then $\varphi(x)$ is idempotent [N-regular]. In particular, the homomorphic image of a Boolean semigroup [Clifford semigroup] is again a Boolean semigroup [Clifford semigroup].

Lemma 2.1. *Let S be a semigroup.*

1. *Let $e, f \in \mathrm{Id(S)}$. Then S_e is a group with unit element e, and if $e \ne f$, then $S_e \cap S_f = \emptyset$.*
2. *For $x \in S$, the following assertions are equivalent:*
   (a) *x is N-regular.*
   (b) *There is a (unique) idempotent element $e \in \mathrm{Id}(S)$ such that $x \in S_e$.*
   (c) *x is contained in some group $G \subset S$.*

   In particular, S is a Clifford semigroup if and only if every element of S is N-regular.

Proof. 1. By definition, we have $e \in S_e$, and if $x, y \in S_e$, then also $xy \in S_e$. If $x \in S_e$ and $y \in S$ are such that $xy = e$, then $x(ye) = e$ and $(ye)e = ye$. Hence ye is the inverse of x in S_e, and thus S_e is a group.

Assume now that $e, f \in \mathrm{Id}(S)$ and $x \in S_e \cap S_f$. Then $xe = x = xf$, and there exist $y, z \in S$ such that $xy = e$ and $xz = f$. But then $f = xz = xez = x^2yz = (xy)(xz) = ef$, and similarly $e = ef$, which implies $e = f$.

2. (a) $\Rightarrow$ (b) The uniqueness of e follows from 1. If $x^2y = x$ for some $y \in S$, then $e = xy \in \mathrm{Id}(S)$ and $ex = x$, whence $x \in S_e$.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) If $G \subset S$ is a group with unit element e and $x \in G$, then $xy = e$ for some $y \in G$, and $x^2y = xe = x$. □

Let S be a semigroup and $G \subset S^\times$ a subgroup. Two elements $x, y \in S$ are called *congruent modulo* G, $x \equiv y \bmod G$ if $x = uy$ for some $u \in G$. Congruence modulo G is a congruence relation on S and we denote by $S/G = \{aG \mid a \in S\}$ the quotient semigroup of S under this congruence relation.

Lemma 2.2. *Let S be a semigroup, $G \subset S^\times$ a subgroup and $\rho\colon S \to S/G$ the residue class epimorphism.*

1. *$\rho(S^\times) = (S/G)^\times$. In particular, $S^\bullet$ is a group if and only if $(S/G)^\bullet$ is a group.*
2. *An element $a \in S$ is N-regular if and only if $\rho(a) \in S/G$ is N-regular.*
3. *$\mathrm{Id}(S/G) = \rho(\mathrm{Id}(S)) = \{aG \mid a \in \mathrm{Id}(S)\}$, and if $e \in \mathrm{Id}(S)$, then $(S/G)_{\rho(e)} = \rho(S_e)$ and $\mathrm{Ker}(\rho \,|\, S_e) = Ge \subset S_e$. In particular, S is a Clifford semigroup if and only if S/G is a Clifford semigroup, and if S is a Boolean semigroup, then so is S/G.*

Proof. Since ρ is a homomorphism, it follows that $\rho(S^\times) \subset (S/G)^\times$, $\rho(\mathrm{Id}(S)) \subset \mathrm{Id}(S/G)$ and $\rho(S_e) \subset (S/G)_{\rho(e)}$ for every $e \in \mathrm{Id}(S)$. Also, if $a \in S$ is N-regular, then so is $\rho(a)$. Let now $a \in S$.

If $aG \in (S/G)^\times$, then $abG = G$ for some $b \in S$, hence $ab \in G \subset S^\times$ and thus also $a \in S^\times$. If $aG \in S/G$ is N-regular, then there exists some $x \in S$ such that $a^2xG = aG$, say $a^2x = au$ for some $u \in G$. Then $a^2(xu^{-1}) = a$, and thus a is N-regular.

If $aG \in \mathrm{Id}(S/G)$, then $a^2G = aG$, hence $a^2 = au$ for some $u \in G$, and therefore $(au^{-1})^2 = au^{-1}$. This implies $au^{-1} \in \mathrm{Id}(S)$ and $aG = au^{-1}G \in \rho(\mathrm{Id}(S))$.

Assume finally that $e \in \mathrm{Id}(S)$ and $aG \in (S/G)_{eG}$. Then there exists some $b \in G$ such that $aeG = aG$ and $abG = eG$, and thus there exist $u, v \in G$ satisfying $aeu = a$ and $abv = e$. Hence it follows that $a=a(abv)u=a^2buv$, and thus $abuv=ue\in \mathrm{Id}(S)$. From $u^2e=u^2e^2=ue$ we deduce $ue=e$, $ae=a$ and $a(bv)=e$, whence $a\in S_e$. Therefore we obtain $(S/G)_{\rho(e)}=\rho(S_e)$, and by the very definition it follows that $\mathrm{Ker}(\rho\,|\,S_e)=Ge$. $\square$

3. Review on ideal systems

Throughout this section, let D be a monoid and $K=\mathsf{q}(D)$.

The most important example we have in mind is when D is the multiplicative monoid of an integral domain and K is its quotient field (it is this case why we admit a zero element in D). The main reference for the theory of ideal systems is [17]. All undefined notions are used as there, but for the convenience of the reader we repeat the most central notions.

For any subsets $X,\,Y\subset K$, we set $XY=\{xy\mid x\in X\,,\ y\in Y\}$, $(X:Y)=\{z\in K\mid zY\subset X\}$ and (if there is no doubt concerning D) $X^{-1}=(D\,{:}\,X)$. A subset $X\subset K$ is called *D-fractional* if $cX\subset D$ for some $c\in D^\bullet$. We denote by $\boldsymbol{F}(D)$ the set of all D-fractional subsets of K. A subset $P\subset D$ is called a *prime ideal* if $\emptyset\ne P\subsetneq D$, $DP=P$ and $D\setminus P$ is a submonoid of D. It is easily checked that the union and the intersection of any chain of prime ideals is again a prime ideal.

By an *ideal system* on D we mean a map $r\colon \boldsymbol{F}(D)\to\boldsymbol{F}(D)$ with the following properties for all subsets $X,\,Y\in\boldsymbol{F}(D)$ and all $c\in K$:

- $X\cup\{0\}\subset X_r$.
- $X\subset Y_r$ implies $X_r\subset Y_r$.
- $(cX)_r=cX_r$.
- $D_r=D$.

A D-fractional subset $J\subset K$ is called a *fractional r-ideal* if $J_r=J$. A fractional r-ideal J is called an *r-ideal* if $J\subset D$. A fractional r-ideal J is called *r-finite* if $J=F_r$ for some finite set F. If $a\in K$, then aD is a fractional r-ideal (called a *fractional principal ideal*). We denote by $\mathcal{P}(D)$ the set of all fractional principal ideals, by $\mathcal{F}_r(D)$ the set of all fractional r-ideals, by $\mathcal{I}_r(D)$ the set of all r-ideals, by $\mathcal{F}_{r,\mathsf{f}}(D)$ the set of all r-finite fractional r-ideals and by $\mathcal{I}_{r,\mathsf{f}}(D)$ the set of r-finite r-ideals of D. For any set $\mathcal{J}$ of ideals, we denote by $\mathcal{J}^\bullet$ the set of non-zero ideals in $\mathcal{J}$. For $I,\,J\in\mathcal{F}_r(D)$, we define their *r-product* by $I\cdot_r J=(IJ)_r$. Equipped with the r-multiplication, $\mathcal{F}_r(D)$ is a semigroup (with unit element D and zero element $\{0\}$), and $\mathcal{I}_r(D)$, $\mathcal{F}_{r,\mathsf{f}}(D)$ and $\mathcal{I}_{r,\mathsf{f}}(D)$ are subsemigroups of $\mathcal{F}_r(D)$. For all $X,\,Y\subset\boldsymbol{F}(D)$ we have $(XY)_r=(X_rY)_r=(X_rY_r)_r$, and if $Y\cap K^\bullet\ne\emptyset$, then $(X_r{:}Y)=(X_r{:}Y_r)=(X_r{:}Y)_r\in\mathcal{F}_r(D)$.

We denote by r-$\max(D)$ the set of all r-maximal r-ideals (that is, of all maximal elements of $\mathcal{I}_r(D)\setminus\{D\}$), and by r-$\mathrm{spec}(D)$ the set of all prime r-ideals of D. For a subset $X\subset D$, we set r-$\max(D,X)=\{M\in r\text{-}\max(D)\mid X\subset M\}$.

For any ideal system r on D, the *associated finitary ideal system* r_s is defined by

$$X_{r_s}=\bigcup_{\substack{F\subset X\\ F\text{ finite}}}F_r\quad\text{for}\quad X\in\boldsymbol{F}(D)\,.$$

For every finite subset $Y\subset K$ we have $Y_r=Y_{r_s}$. The ideal system r is called *finitary* if $r_s=r$. If r is finitary, then every r-ideal $J\ne D$ is contained in an r-maximal r-ideal.

The intersection of any family of r-ideals is an r-ideal, and if r is finitary, then the union of any chain of r-ideals is also an r-ideal.

The most common ideal systems on an arbitrary monoid are the systems $s = s(D)$, $v = v(D)$ and $t = t(D) = v_s$, defined by $X_s = DX$ and $X_v = (X^{-1})^{-1}$ for all $X \in \boldsymbol{F}(D)$. $s(D)$ and $t(D)$ are finitary ideal systems, $v(D)$ is not finitary. If r and q are ideal systems, we set $r \le q$, if $\mathcal{F}_q(D) \subset \mathcal{F}_r(D)$ (equivalently, $X_r \subset X_q$ for all $X \in \boldsymbol{F}(D)$). If r is any ideal system on D, then $s(D) \le r_s \le r \le v(D)$ and $r_s \le t(D)$.

If D is an integral domain, then the ideal system $d = d(D)$ is defined by $X_d = {}_D(X)$ (the ordinary ring ideal generated by X). It is a finitary ideal system. If r is any ideal system on D satisfying $r \ge d$, then the map $\mathcal{F}_d(D)^\bullet \to \mathcal{F}_d(D)$, defined by $J \mapsto J_r$, is a star operation in the sense of [16, §32].

For a multiplicatively closed subset $T \subset D$, let $T^{-1}D = \{t^{-1}c \mid c \in D\,,\ t \in T\}$ be the quotient monoid with respect to T. For a finitary ideal system r on D, the quotient system $T^{-1}r$ is the unique finitary ideal system on $T^{-1}D$ satisfying $\mathcal{F}_{T^{-1}r}(T^{-1}D) = \{T^{-1}J \mid J \in \mathcal{F}_r(D)\}$. For all $X \in \boldsymbol{F}(D)$ we have $X_{T^{-1}r} = (T^{-1}X)_{T^{-1}r} = T^{-1}X_r$. If $I \in \mathcal{I}_{T^{-1}r}(T^{-1}D)$, then $I \cap D \in \mathcal{I}_r(D)$ and $T^{-1}(I \cap D) = I$. If $I,\, J \in \mathcal{F}_r(D)$, then $(T^{-1}I : J) = (T^{-1}I : T^{-1}J) \supset T^{-1}(I : J)$, with equality if $J \in \mathcal{I}_{r,\mathsf{f}}(D)$. We obviously have $T^{-1}s(D) = s(T^{-1}D)$, and if D is a domain, then $T^{-1}d(D) = d(T^{-1}D)$. If $P \subset D$ is a prime ideal, we write $X_P = (D \setminus P)^{-1}X$ for every $X \in \boldsymbol{F}(D)$ and $r_P = (D \setminus P)^{-1}r$ for every finitary ideal system r on D.

Let r be a finitary ideal system on D. If $E \in \mathcal{F}_r(D)$ is an overmonoid of D, then E is called an *r-overmonoid*, and we define the (finitary) ideal system $r[E]$ on E by $X_{r[E]} = (XE)_r$ for $X \in \boldsymbol{F}(E) = \boldsymbol{F}(D)$. For every $I \in \mathcal{F}_r(D)^\bullet$, the monoid $(I : I)$ is an r-overmonoid of D.

Let $J \in \mathcal{F}_r(D)$. For every $P \in r\text{-spec}(D)$, we have $J_P = J_{r_P}$,

$$J = \bigcap_{M \in r\text{-}\max(D)} J_M = \bigcap_{M \in r\text{-}\max(D)} J_{r_M}\,, \quad \text{and in particular} \quad D = \bigcap_{M \in r\text{-}\max(D)} D_M\,.$$

J is called *r-locally principal* if for every $M \in r\text{-}\max(D)$ there exists some $a_M \in K^\times$ such that $J_M = aD_M$. If J is r-locally principal, then J is a cancellative element of $\mathcal{F}_r(D)$.

A fractional r-ideal $I \in \mathcal{F}_r(D)^\times$ is called *r-invertible*. If I is r-invertible, then I^{-1} is its inverse in $\mathcal{F}_r(D)$, that is, $(II^{-1})_r = D$. The group $\mathcal{F}_r(D)^\times$ is a subgroup of $\mathcal{F}_v(D)^\times$, and if r is finitary, then $\mathcal{F}_r(D)^\times = \mathcal{F}_{r,\mathsf{f}}(D)^\times$ is a subgroup of $\mathcal{F}_{t,\mathsf{f}}(D)^\times$.

4. Ideal (class) semigroups

Throughout this section, let D be a monoid, $K = \mathsf{q}(D)$ and r a finitary ideal system on D.

Two fractional r-ideals $I,\, J \in \mathcal{F}_r(D)$ are called *equivalent*, $I \sim J$, if $J = cI$ for some $c \in K^\times$. Obviously, this is an equivalence relation on $\mathcal{F}_r(D)$, we call $\mathcal{S}_r(D) = \mathcal{F}_r(D)/\sim$ the *r-ideal class semigroup* of D, and for $I \in \mathcal{F}_r(D)$ we denote by $[I]_r \in \mathcal{S}_r(D)$ the class of I. By definition, we have $\mathcal{S}_r(D) = \{[I]_r \mid I \in \mathcal{I}_r(D)\}$. By Lemma 2.2, $\mathcal{S}_r(D)^\times = \{[I]_r \mid I \in \mathcal{F}_r(D)^\times\}$ is the r-class group of D. In particular, $\mathcal{S}_r(D)^\bullet$ is a group if and only if every non-zero (fractional) r-ideal of D is r-invertible.

A fractional r-ideal $J \in \mathcal{F}_r(D)$ is called *r-regular* if it is an N-regular element of $\mathcal{F}_r(D)$ [equivalently, there exists some $Z \in \boldsymbol{F}(D)$ such that $(J^2Z)_r = J$]. By Lemma 2.2, a fractional r-ideal $J \in \mathcal{F}_r(D)$ is r-regular if and only if $[J]_r$ is an N-regular element of $\mathcal{S}_r(D)$. In particular, $\mathcal{F}_r(D)$ is a Clifford semigroup if and only if $\mathcal{S}_r(D)$ is a Clifford semigroup. In this case (following the terminology of [20] and [5]), we call D *Clifford r-regular*. Note that D is Clifford r-regular if and only if every $I \in \mathcal{I}_r(D)$ is N-regular in $\mathcal{F}_r(D)$.

If $J \in \mathcal{F}_r(D)$, then Lemma 2.2 implies that $[J]_r \in \mathcal{S}_r(D)$ is an idempotent if and only if there exists some $a \in K^\times$ such that aJ is an idempotent in $\mathcal{F}_r(D)$. The monoid D is called *Boolean r-regular* if $\mathcal{S}_r(D)$ is a Boolean semigroup.

Lemma 4.1. *Let $I \in \mathcal{F}_r(D)$ be r-regular [an idempotent of $\mathcal{F}_r(D)$].*

1. *If $E \supset D$ is an r-overmonoid of D, then $I_{r[E]} = (IE)_r$ is $r[E]$-regular [an idempotent element of $\mathcal{F}_{r[E]}(E)$].*
2. *If q is an ideal system on D with $q \geq r$, then I_q is q-regular [an idempotent element of $\mathcal{F}_q(D)$]. Moreover, if D is Clifford r-regular [Boolean r-regular], then D is also Clifford q-regular [Boolean q-regular].*
3. *Let $T \subset D$ be a multiplicatively closed subset. Then $T^{-1}I$ is $T^{-1}r$-regular [an idempotent element of $\mathcal{F}_{T^{-1}r}(T^{-1}D)$]. Moreover, if D is Clifford r-regular [Boolean r-regular], then $T^{-1}D$ is Clifford $T^{-1}r$-regular [Boolean $T^{-1}r$-regular].*

Proof. The maps $\theta\colon \mathcal{F}_r(D) \to \mathcal{F}_{r[T]}(T)$, $\eta\colon \mathcal{F}_r(D) \to \mathcal{F}_q(D)$ and $\vartheta\colon \mathcal{F}_r(D) \to \mathcal{F}_{T^{-1}r}(T^{-1}D)$, defined by $\theta(I) = I_{r[T]} = (IT)_r$, $\eta(I) = I_q$ and $\vartheta(I) = T^{-1}I$, are homomorphisms which act trivially on fractional principal ideals. Moreover, η and ϑ are surjective. From these observations the assertions follow. □

Proposition 4.2. *Let $I \in \mathcal{F}_r(D)^\bullet$, $T = (I(I\!:\!I^2))_r$ and $E = (I\!:\!I)$. Then $T = (I(E\!:\!I))_r$, and the following assertions are equivalent:*

(a) *I is r-regular, that is, $I = (I^2X)_r$ for some $X \in \mathcal{F}_r(D)$.*

(b) *$I = (I^2(I\!:\!I^2))_r$.*

(c) *$I = (IT)_r$.*

If $X \in \mathcal{F}_r(D)$ is such that $I = (I^2X)_r$, then $T = (IX)_r \in \mathcal{I}_{r[E]}(E)$, $(T^2)_r = (T^2)_{r[E]} = T$, and $E = (T\!:\!T) = (E\!:\!T)$.

If I is r-finite and r-regular, then there exists some $Y \in \mathcal{F}_{r,\mathsf{f}}(D)$ such that $I = (I^2Y)_r$, and T is r-finite. In particular, if I is r-finite and r-regular, then I is N-regular in $\mathcal{F}_{r,\mathsf{f}}(D)$.

Proof. 1. Obviously, $(I\!:\!I^2) = ((I\!:\!I)\!:\!I) = (E\!:\!I)$ and $T = (I(E\!:\!I))_r \subset E_r = E$. Hence it follows that $(IT)_r = (I^2(E\!:\!I))_r = (I^2(I\!:\!I^2))_r$. Therefore (b) and (c) are equivalent, and (b) implies (a) by the very definition.

Suppose that $I = (I^2X)_r$ for some $X \in \mathcal{F}_r(D)$. Then $I^2X \subset I$ implies $X \subset (I\!:\!I^2)$, and

$$(IX)_r \subset (I(I\!:\!I^2))_r = T = ((I^2X)_r(I\!:\!I^2))_r = (I^2X(I\!:\!I^2))_r \subset (IX)_r\,.$$

Hence $(IX)_r = T$, and $(T^2)_r = (I^2X^2)_r = ((I^2X)_rX)_r = (IX)_r = T$. Since $1 \in E$, we obtain $T \subset ET \subset ((I\!:\!I)I(E\!:\!I))_r \subset (I(E\!:\!I))_r = T$, hence $ET = T$ and thus $T \in \mathcal{I}_{r[E]}(E)$. Moreover, we have $E = (I\!:\!I) = (I\!:\!(IT)_r) = (I\!:\!IT) = ((I\!:\!I)\!:\!T) = (E\!:\!T) \supset (T\!:\!T) \supset E$, and therefore $E = (T\!:\!T) = (E\!:\!T)$.

Assume now that I is r-finite and r-regular, and let $X \in \mathcal{F}_r(D)$ be such that $I = (I^2X)_r$. Let $F \subset I$ be finite with $I = F_r$. Then there exists a finite subset $Z \subset X$ such that $F \subset (I^2Z)_r$, and if $Y = Z_r$, then $Y \subset X$ and $I = F_r \subset (I^2Z)_r \subset (I^2Y)_r \subset (I^2X)_r = I$. Hence $I = (I^2Y)_r$, and by the above we obtain $T = (IY)_r$, whence T is also r-finite. □

Lemma 4.3. *Let $I \in \mathcal{F}_{r,\mathsf{f}}(D)$. If I_M is r_M-regular for all $M \in r\text{-}\max(D)$, then I is r-regular.*

Proof. Since I is r-finite, we obtain

$$(I^2(I{:}I^2))_r = \bigcap_{M\in r\text{-}\max(D)} (I^2(I{:}I^2))_{r_M} = \bigcap_{M\in r\text{-}\max(D)} (I_M^2(I_M{:}I_M^2))_{r_M} = \bigcap_{M\in r\text{-}\max(D)} I_M = I\,. \quad \square$$

Lemma 4.4. *For $J \in \mathcal{F}_r(D)$, the following conditions are equivalent:*

(a) *J is r-invertible.*

(b) *J is r-finite and r-locally principal.*

(c) *J is r-regular and r-locally principal.*

In particular, if D is Clifford r-regular, then every r-locally principal r-ideal is r-finite.

Proof. The equivalence of (a) and (b) is well known (see [17, Theorem 12.3]). For the equivalence with (c) note that J is invertible in $\mathcal{F}_r(D)$ if and only if J is N-regular and cancellative in $\mathcal{F}_r(D)$, and that every r-locally principal ideal is cancellative in $\mathcal{F}_r(D)$. $\square$

The connection between Clifford regular and stable domains outlined by S. Bazzoni [4] has its counterpart in the theory of ideal systems (although here the connection is not so fruitful since there is no rich purely multiplicative theory of stable ideals). We merely mention the main notions and results.

Definition 4.5. Let $I \in \mathcal{F}_r(D)^\bullet$ and $E = (I:I)$. Then I is called *r-stable* if I is $r[E]$-invertible [that is, $(I(E{:}I))_r = E$, or, equivalently, $(IC)_r = E$ for some $C \in \mathcal{F}_r(D)$ with $EC = C$].
D is called (*finitely*) *r-stable* if every non-zero (r-finite) fractional r-ideal is r-stable.

Lemma 4.6. *Let $I \in \mathcal{F}_r(D)^\bullet$. If I is r-invertible, then I is r-stable, and if I is r-stable, then I is r-regular. In particular, if D is r-stable, then D is Clifford r-regular.*

Proof. Let $E = (I:I)$. If I is r-invertible and $X \in \mathcal{F}_r(D)$ is such that $(IX)_r = D$, then $(I(XE))_r = E$. Hence I is r-stable. Conversely, if I is r-stable, then $T = (I(E:I))_r = E$, hence $I = IT = (IT)_r$, and thus I is r-regular by Lemma 4.2. $\square$

In [4, Proposition 2.3] it is proved that a Clifford regular domain is finitely stable. In the case of ideal systems we obtain only the following weaker result (because in general there is no analog to Nakayama's Lemma).

Lemma 4.7. *Let $I \in \mathcal{F}_{r,\mathsf{f}}(D)^\bullet$ be r-regular, $E = (I{:}I)$, and assume that $\mathcal{I}_{r[E],\mathsf{f}}(E)$ has no non-trivial idempotents. Then $I_{r[E]}$ is $r[E]$-invertible.*

Proof. By Proposition 4.2 it follows that $T = (I(E{:}I))_r \in \mathcal{I}_{r[E]}(E)$ is an r-finite, and therefore it is also $r[E]$-finite. Since $(T^2)_{r[E]} = T$, we obtain $T = E$, and I is $r[E]$-invertible. $\square$

Our next result shows that (for suitably closed monoids) r-stability may be used to characterize r-Dedekind and r-Prüfer monoids (for these concepts see [17]).

Proposition 4.8.

1. *If D is completely integrally closed, then D is r-stable if and only if D is an r-Dedekind monoid.*
2. *If D is r-closed, then D is finitely r-stable if and only if D is an r-Prüfer monoid.*

Proof. Let $I \in \mathcal{F}_r(D)^\bullet$. If either D is completely integrally closed or D is r-closed and I is r-finite, then $(I:I) = D$. Hence, by the very definition, I is r-stable if and only if I is r-invertible. □

5. Valuation monoids

For the sake of completeness we rephrase here the results of [7] in the language of monoids and give full proofs (although they are not really different from those in the case of integral domains). After that, we characterize Boolean regular valuation monoids by means of their value group (Theorem 5.7). Basic facts on valuation monoids my be found in [17, Ch. 15 and Ch. 16].

Throughout this section, let D be a valuation monoid.

We denote by $\mathsf{M}_D = D \setminus D^\times$ the maximal ideal of D. The s-system is the only finitary ideal system on D. We only consider s-ideals, and thus we suppress the specification s whenever we denote sets of ideals of D (hence $\mathcal{F}(D)$, $\mathcal{I}(D)$, $\operatorname{spec}(D)$ etc. have the obvious meaning, and $\mathcal{F}(D)^\times = \mathcal{P}(D)^\bullet$, the set of all non-zero fractional principal ideals). Note that for any subsets $U, V, W \subset K$ we have $DU \subset DV$ or $DV \subset DU$ and thus $(DU \cap DV)W = DUW \cap DVW$.

Lemma 5.1.

1. *If $V \supset D$ is an overmonoid, then V is a valuation monoid, $\mathsf{M}_V \subset D$ is a prime ideal, and $V = D_{\mathsf{M}_V}$. In particular, if $V \neq K$, then $V \in \mathcal{F}(D)$.*
2. *If $L \in \operatorname{spec}(D)$, then $L = \mathsf{M}_{D_L}$ is the maximal ideal of D_L.*

Proof. 1. We may assume that $V \neq K$. If $x \in K \setminus V$, then $x \notin D$ and thus $x^{-1} \in D \subset V$. Hence V is a valuation monoid. If $x \in \mathsf{M}_V^\bullet$, then $x^{-1} \notin V$, hence $x^{-1} \notin D$ and thus $x \in D$. This proves $\mathsf{M}_V \subset D$. Clearly, M_V is a prime ideal of D and $D_{\mathsf{M}_V} \subset V$. If $x \in V \setminus D$, then $x^{-1} \in D \subset V$, hence $x^{-1} \in V^\times \cap D = D \setminus \mathsf{M}_V$ and $x = (x^{-1})^{-1} \in D_{\mathsf{M}_V}$.

2. If $L \in \operatorname{spec}(D)$, then $\mathsf{M}_{D_L} = LD_L \subset D$ and therefore $L = LD_L \cap D = \mathsf{M}_{D_L}$. □

For a non-zero fractional ideal $I \in \mathcal{F}(D)^\bullet$, we set $I^\# = \mathsf{M}_{(I:I)}$. Then Lemma 5.1 implies $I^\# \in \operatorname{spec}(D)$, and by the very definition we have

$$II^\# \subset I\,, \quad I^\# = \{x \in D \mid xI \subsetneq I\} \quad \text{and} \quad D \setminus I^\# = \{x \in D \mid I = xI\} = \{x \in I \mid x^{-1}I \subset I\}\,.$$

Lemma 5.2. *Let $I \in \mathcal{F}(D)^\bullet$.*

1. *If I is principal, then $(I:I) = D$ and $I^\# = \mathsf{M}_D$.*
2. *If I is not principal, then $I^\# = II^{-1}$.*

Proof. 1. Obvious.

2. Since $II^{-1}(I:I) \subset II^{-1} \subset D \subset (I:I)$, it follows that II^{-1} is an ideal of $(I:I)$. Since I is not principal, it is not invertible and thus $1 \notin II^{-1}$. Hence $II^{-1} \subset I^\#$. Let now $z \in I^\#$ and $a \in I \setminus zI$. Then $zI \subset aD$, hence $a^{-1}zI \subset D$ and therefore $a^{-1}z \in I^{-1}$. It follows that $z = a(a^{-1}z) \in II^{-1}$. □

Lemma 5.3. *Let $I \in \mathcal{F}(D)^\bullet$. Then $ID_{I^\#} = I$, and I is a principal ideal of $D_{I^\#}$ if and only if $II^\# \subsetneq I$.*

Proof. If $a \in I$ and $s \in D \setminus I^\#$, then $s^{-1}a \in s^{-1}I \subset I$. Hence $ID_{I^\#} \subset I$, and the other inclusion is obvious.

Assume now that $I = aD_{I^\#}$ for some $a \in I$. We will prove that $a \notin II^\#$. Assume to the contrary that $a = br$ for some $b \in I$ and $r \in I^\#$. Then $aD_{I^\#} = rbD_{I^\#} \subset rID_{I^\#} = rI \subsetneq I$, a contradiction.

For the converse, let $a \in I \setminus II^\#$. Then $aD_{I^\#} \subset I$, and we shall prove that equality holds. Let $b \in I$. If $b \in aD$, there is nothing to do. If $b \notin aD$, then $a \in bD$, say $a = bt$ for some $t \in D$. Since $a \notin II^\#$, we have $t \notin I^\#$ and $b = t^{-1}a \in aD_{I^\#}$. □

The subsequent Theorem 5.4 is essentially [7, Theorem 3]. Recall that every idempotent ideal of D is a prime ideal (see [17, Proposition 16.1]).

Theorem 5.4.

1. *D is Clifford regular (that is, every $I \in \mathcal{F}(D)$ is N-regular).*
2. *$L \in \mathcal{F}(D)$ is an idempotent element of $\mathcal{F}(D)$ if and only if either $L \supset D$ is an overmonoid or $L \in \operatorname{spec}(D)$ is an idempotent prime ideal.*

Proof. 1. It suffices to prove that every $I \in \mathcal{I}(D)^\bullet$ is N-regular. Thus let $I \in \mathcal{I}(D)^\bullet$. If I is not principal in $D_{I^\#}$, then I is not a principal ideal of D and (using the Lemmas 5.2 and 5.3) we obtain $I = II^\# = I^2I^{-1}$. If $I = aD_{I^\#}$ for some $a \in I$, then $I^2 = aID_{I^\#} = aI$ and $I = I^2(a^{-1}D)$.

2. Obviously, overmonoids and idempotent prime ideals are idempotent elements of $\mathcal{F}(D)$. Conversely, if $L \in \mathcal{F}(D)$ and $L^2 = L$, then either $L \supset D$ (and then L is an overmonoid) or $L \subset D$ (and then L is an idempotent ideal). □

We close this section with the determination of the constituent groups of $\mathcal{F}(D)$ and a criterion for D to be Boolean regular. Following [14], a fractional ideal $I \in \mathcal{F}(D)$ is called *archimedean* if $I \neq \{0\}$ and $I^\# = \mathsf{M}_D$. Let $\mathcal{G}(D)$ denote the set of all archimedean fractional ideals. Then $\mathcal{P}(D)^\bullet \subset \mathcal{G}(D)$, and $\mathcal{P}(D)^\bullet \cong K^\times/D^\times$. Following [6], we let $\mathsf{G}_D = \mathcal{G}(D) \setminus \mathcal{P}(D)$ denote the set of all non-principal archimedean fractional ideals.

Lemma 5.5. *$\mathcal{G}(D) \subset \mathcal{F}(D)$ is a subsemigroup, and if M_D is principal, then $\mathcal{G}(D) = \mathcal{P}(D)^\bullet$.*

If M_D is not principal, then $\mathsf{M}_D^2 = \mathsf{M}_D$, and G_D consists of all $I \in \mathcal{F}(D)^\bullet$ such that $I\mathsf{M}_D = I$ and $IJ = \mathsf{M}_D$ for some $J \in \mathcal{F}(D)$. In particular, G_D is the constituent group of the idempotent M_D in $\mathcal{F}(D)$, and $\mathsf{M}_D\mathcal{P}(D)^\bullet \subset \mathsf{G}_D$ is a subgroup.

Proof. If $I \in \mathcal{G}(D)$, then $D_{I^\#} = D$, and (by Lemma 5.3) I is principal if and only if $I\mathsf{M}_D \neq I$. Thus, if $I \in \mathsf{G}_D$, then I is not invertible, and $II^{-1} = \mathsf{M}_D$ by Lemma 5.2. Consequently, if $\mathsf{G}_D \neq \emptyset$, then M_D is not invertible, hence not principal, and $\mathsf{M}_D^2 = \mathsf{M}_D$.

Assume now that M_D is not principal. If $I \in \mathsf{G}_D$, then we have already seen that $I\mathsf{M}_D = I$ and $II^{-1} = \mathsf{M}_D$. As to the converse, assume that $I\mathsf{M}_D = I$ and $IJ = \mathsf{M}_D$ for some $J \in \mathcal{F}(D)$. Then I is not principal and $J \subset I^{-1}$. Thus it follows that $\mathsf{M}_D \supset II^{-1} \supset IJ = \mathsf{M}_D$. Hence $I^\# = II^{-1} = \mathsf{M}_D$ and $I \in \mathcal{G}(D)$.

It remains to prove that $\mathcal{G}(D) \subset \mathcal{F}(D)$ is a subsemigroup. We must show that the product of two archimedean fractional ideals is again archimedean. But this follows immediately from the characterization just given. □

Theorem 5.6. *Let $L \in \mathcal{F}(D)$ be an idempotent, $\mathcal{F}(D)_L$ its constituent group in $\mathcal{F}(D)$ and $\mathcal{S}(D)_L$ the constituent group of $[L]$ in the ideal class semigroup $\mathcal{S}(D)$.*

1. *If* $L \supset D$ *is an overmonoid, then* $\mathcal{F}(D)_L = \mathcal{P}(L)^\bullet \cong K^\times/L^\times$*, and* $\mathcal{S}(D)_L$ *is trivial.*
2. *If* $L \in \mathrm{spec}(D)^\bullet$ *is an idempotent non-zero prime ideal, then* $\mathcal{F}(D)_L = \mathsf{G}_{D_L}$*, and* $\mathcal{S}(D)_L = \mathsf{G}_{D_L}/L\mathcal{P}(D_L)^\bullet$*.*

Proof. By Lemma 2.2 it suffices to prove the assertions concerning $\mathcal{F}(D)_L$. Let $I \in \mathcal{F}(D)$. By definition, we have $I \in \mathcal{F}(D)_L$ if and only if $IL = I$ and $IJ = L$ for some $J \in \mathcal{F}(D)$.

1. The conditions for $I \in \mathcal{F}(D)_L$ are fulfilled if and only if I is an invertible fractional ideal of L, that is, if $I \in \mathcal{P}(L)^\bullet$.

2. By Lemma 5.5, the conditions for $I \in \mathcal{F}(D)_L$ are fulfilled if and only if I is a non-principal archimedean fractional ideal of D_L (observe that $IL = I$ implies $ID_L = ILD_L = IL = I$). □

We finally give a description of G_D using the completion of the value group associated with D. Let Γ_D be the additively written group $K^\times/D^\times$. For $a,\, b \in K^\times$, we define $aD^\times \le bD^\times$ if $bD \subset aD$. With this definition, Γ_D becomes a totally ordered abelian group, and we denote by $\widehat{\Gamma_D}$ its completion in the order topology (details concerning the construction of $\widehat{\Gamma_D}$ may be found in [10] or [13, Kap. V.15]). If M_D is principal, then Γ_D has a smallest positive element, hence Γ_D is discrete and $\widehat{\Gamma_D} = \Gamma_D$. We define $v_D \colon K^\times \to \Gamma_D$ by $v_D(a) = aD^\times$, we call Γ_D the *value group* and v_D the *valuation* associated with D. If $P \in \mathrm{spec}(D)$, then there is natural epimorphism $\Gamma_D \to \Gamma_{D_P}$ with kernel $\Delta_P \cong D_P^\times/D^\times$, and $P \mapsto \Delta_P$ is a bijective map from $\mathrm{spec}(D)$ onto the set of all convex subgroups of Γ_D. Following [21], Γ_D is called *algebraically complete* if $\widehat{\Gamma_{D_P}} = \Gamma_{D_P}$ for all $P \in \mathrm{spec}(D)$. If P is not idempotent, then PD_P is principal. Therefore Γ_D is algebraically complete if and only if $\widehat{\Gamma_{D_P}} = \Gamma_{D_P}$ for all idempotent non-zero prime ideals $P \in \mathrm{spec}(D)$.

Theorem 5.7.

1. *Let* $L \in \mathrm{spec}(D)^\bullet$ *be an idempotent non-zero prime ideal. Then there is an isomorphism*
$$\theta_L \colon \mathsf{G}_{D_L} \to \widehat{\Gamma_{D_L}} \quad \textit{satisfying} \quad \theta_L(L\mathcal{P}(D_L)^\bullet) = \Gamma_{D_L}\,.$$
In particular, we have an isomorphism $\mathcal{S}(D)_L \cong \widehat{\Gamma_{D_L}}/\Gamma_{D_L}$*.*
2. *D is Boolean regular if and only if Γ_D is algebraically complete.*

Proof. 1. By Theorem 5.6.2 we may assume that $L = \mathsf{M}_D$ is idempotent, and then we must establish an isomorphism $\theta \colon \mathsf{G}_D \to \widehat{\Gamma_D}$ satisfying $\theta(\mathsf{M}_D\mathcal{P}(D)^\bullet) = \Gamma_D$. For this, we have to recall the construction of $\widehat{\Gamma_D}$.

For $\alpha \in \Gamma_D$, we set $\Gamma_D(\alpha) = \{\gamma \in \Gamma_D \mid \gamma \ge \alpha\}$. A non-empty proper subset $U \subsetneq \Gamma_D$ is called a *filter* if $\Gamma_D(\alpha) \subset U$ for all $\alpha \in U$. A filter $U \subset \Gamma_D$ is called a *Cauchy filter* if for every positive $\varepsilon \in \Gamma_D$ there is some $\alpha \in U$ such that $U \subset \Gamma_D(\alpha - \varepsilon)$, and U is called a *principal filter* if $U = \Gamma_D(\alpha)$ for some $\alpha \in \Gamma_D$. By construction, $\widehat{\Gamma_D} = \{\inf(U) \mid U \subset \Gamma_D \text{ is a Cauchy filter}\}$.

For $I \in \mathcal{F}(D)^\bullet$, we define $v_D^*(I) = \{v_D(a) \mid a \in I^\bullet\} \subset \Gamma_D$. Then v_D^* is a bijection from $\mathcal{F}(D)^\bullet$ onto the set of all filters of Γ_D. If $I \in \mathcal{F}(D)^\bullet$, then $v_D^*(I)$ is a Cauchy filter if and only if I is archimedean, and $v_D^*(I)$ is a principal filter if and only if I is principal. We define $\theta \colon \mathsf{G}_D \to \widehat{\Gamma_D}$ by $\theta(I) = \inf v_D^*(I)$. By definition, θ is a surjective homomorphism, and since $\mathsf{G}_D \cap \mathcal{P}(D) = \emptyset$, we obtain $\mathrm{Ker}(\theta) = \{\mathsf{M}_D\}$ (the unit element of G_D). Hence θ is an isomorphism. If $a \in K^\times$, then $\theta(a\mathsf{M}_D) = v_D(a) \in \Gamma_D$. Conversely, if $I \in \mathsf{G}_D$ and $\theta(I) = v_D(a) \in \Gamma_D$ for some $a \in K^\times$, then $\theta(a^{-1}I) = 0$, whence $a^{-1}I = \mathsf{M}_D$ and $I \in \mathsf{M}_D\mathcal{P}(D)^\bullet$.

2. Obvious by 1. and Theorem 5.6. □

Note that the characterization of Boolean regular valuation domains given in [19, Lemma 3.5] is not correct (valuation domains with value group $\mathbb{R}$ are the simplest counterexample; this was also observed in [22]).

6. Prüfer monoids

For the definition and the elementary properties of r-Prüfer monoids we refer to [17, Ch. 17]. The most important examples we have in mind are multiplicative monoids of Prüfer domains and of Prüfer v-multiplication domains. However, our theory is purely multiplicative, and thus we formulate it in the language of monoids.

Throughout this section, let D be an r-Prüfer monoid for some finitary ideal system r on D.

Recall that D is r-closed (that is, $(J:J) = D$ for all $J \in \mathcal{F}_{r,\mathsf{f}}(D)^\bullet$), $r = t(D)$, every r-finite non-zero r-ideal is r-invertible, and for every $P \in r\text{-spec}(D)$ the localization D_P is a valuation monoid. We start with some additional facts concerning the ideal theory of r-Prüfer monoids.

Lemma 6.1.

1. *Let $P \in r\text{-spec}(D)$ and $\Omega = \{Q \in s\text{-spec}(D) \mid Q \subset P\}$ the set of prime s-ideals of D contained in P. Then $\Omega \subset r\text{-spec}(D)$, and Ω is a chain.*
2. *Suppose that $P, Q \in r\text{-spec}(D)$ and $P \subsetneq Q$. Then there exist $P_1, Q_1 \in r\text{-spec}(D)$ such that $P \subset P_1 \subsetneq Q_1 \subset Q$, and there is no prime r-ideal lying strictly between P_1 and Q_1.*
3. *For any $P, Q \in r\text{-spec}(D)$, the set $\Omega = \{N \in r\text{-spec}(D) \mid N \subset P \cap Q\}$ has a greatest element.*

Proof. 1. Observe that $s_P = r_P$ (since D_P is a valuation monoid). Hence, if $Q \in \Omega$, then $Q = Q_P \cap D \in r\text{-spec}(D)$. The map $\Omega \to \text{spec}(D_P)$, defined by $Q \mapsto Q_P$, is an inclusion-preserving bijection, and therefore Ω is a chain.

2. By 1., the set $\Omega = \big\{R \in r\text{-spec}(D) \mid P \subset R \subset Q\big\}$ is a chain. Let $a \in Q \setminus P$,

$$P_1 = \bigcup_{\substack{R \in \Omega \\ a \notin R}} R \quad \text{and} \quad Q_1 = \bigcap_{\substack{R \in \Omega \\ a \in R}} R\,.$$

Then $P_1, Q_1 \in r\text{-spec}(D)$, $P \subset P_1 \subsetneq Q_1 \subset Q$, and there is no prime r-ideal strictly between P_1 and Q_1 (recall that the union and the intersection of any chain of prime r-ideals is again a prime r-ideal).

3. By 1., Ω is a chain, and therefore

$$R = \bigcup_{N \in \Omega} N \in r\text{-spec}(D) \qquad \text{is the greatest element of } \Omega. \qquad \square$$

Lemma 6.2. *Let $I \in \mathcal{I}_r(D)$, and suppose that for some $a \in I$ the set $r\text{-}\max(D, \{a\})$ is finite.*

1. *There exists some $J \in \mathcal{I}_{r,\mathsf{f}}(D)$ such that $J \subset I$ and $r\text{-}\max(D, J) = r\text{-}\max(D, I)$.*
2. *Let*
$$C = \bigcap_{\substack{M \in r\text{-}\max(D) \\ I \not\subset M}} D_M$$
and $N \in r\text{-}\max(D)$. Then $C_N = D_P$ for some $P \in r\text{-spec}(D)$ with $I \not\subset P$.

Proof. 1. Let $\{M_1,\ldots,M_n\} = r\text{-}\max(D,\{a\}) \setminus r\text{-}\max(D,I)$ (where $n \in \mathbb{N}_0$). For $i \in [1,n]$, let $y_i \in I \setminus M_i$. Then $J = \{a, y_1, \ldots, y_n\}_r$ has the required property.

2. By definition, $C_N \supset D_N$ is an overmonoid, and as D_N is a valuation monoid, it follows that $C_N = (D_N)_{\overline{P}}$ for some $\overline{P} \in s\text{-spec}(D_N)$. Then there is some $P \in r\text{-spec}(D)$ satisfying $\overline{P} = P_N$, $P \subset N$ and $C_N = D_P$. It remains to prove that $I \not\subset P$.

Let $J \in \mathcal{I}_{r,\mathsf{f}}(D)$ be such that $J \subset I$ and $r\text{-max}(D,J) = r\text{-max}(D,I)$. If $M \in r\text{-max}(D)$ and $I \not\subset M$, then $J \not\subset M$ and thus $J^{-1} \subset D_M$ [indeed, if $x \in J^{-1}$ and $y \in J \setminus M$, then $xy \in D$ and $x = y^{-1}(xy) \in D_M$]. Hence it follows that $J^{-1} \subset C \subset D_P$. Since $J \in \mathcal{I}_{r,\mathsf{f}}(D)$, we obtain $J_P = aD_P$ for some $a \in D$ and $(J^{-1})_P = (D_P\!:\!J_P) = a^{-1}D_P \subset D_P$. Hence $J_P = D_P$, $J \not\subset P$, and thus also $I \not\subset P$. □

Recall that D is a monoid of Krull type if and only if for every $a \in D^\bullet$ the set $r\text{-max}(D,\{a\})$ is finite [17, Theorem 22.4].

Theorem 6.3. *If D is a monoid of Krull type, then D is Clifford r-regular.*

Proof. Let $I \in \mathcal{I}_r(D)$ and $r\text{-max}(D,I) = \{M_1,\ldots,M_n\}$. We shall prove that I is N-regular. We may assume that $I \ne D$, hence $n \ge 1$, and we must prove that $[(I^2(I\!:\!I^2))_r]_M = I_M$ for all $M \in r\text{-max}(D)$. If $M \in r\text{-max}(D)$, then D_M is a valuation monoid, hence $r_M = s_M$ and $[(I^2(I\!:\!I^2))_r]_M = I^2(I\!:\!I^2)D_M$. If $M \notin \{M_1,\ldots,M_n\}$, then $I^2(I\!:\!I^2) \not\subset M$ and therefore $I^2(I\!:\!I^2)D_M = I_M = D_M$. Thus let $M \in \{M_1,\ldots,M_n\}$, say $M = M_1$. Then

$$I^2(I\!:\!I^2)D_{M_1} = I^2\Big(\bigcap_{M\in r\text{-}\max(D)} I_M : I^2\Big)D_{M_1} = I^2D_{M_1}\Big[(I_{M_1}\!:\!I^2) \cap \bigcap_{i=2}^{n}(I_{M_i}\!:\!I^2)D_{M_1} \cap CD_{M_1}\Big],$$

where

$$C = \bigcap_{\substack{M\in r\text{-}\max(D)\\ I\not\subset M}} D_M\,.$$

Since D_{M_1} is a valuation monoid, the product distributes over the intersections, and therefore

$$I^2(I\!:\!I^2)D_{M_1} = I^2D_{M_1}(I_{M_1}\!:\!I^2) \cap \bigcap_{i=2}^{n} I^2D_{M_1}(I_{M_i}\!:\!I^2) \cap I^2CD_{M_1}\,.$$

Being valuation monoids, the monoids D_{M_i} are Clifford regular, and thus we obtain

$$I^2D_{M_1}(I_{M_1}\!:\!I^2) = I_{M_1}^2(I_{M_1}\!:\!I_{M_1}^2) = I_{M_1} \quad \text{and, for } i \in [2,n],$$

$$I^2D_{M_1}(I_{M_i}\!:\!I^2) = I^2D_{M_1}D_{M_i}(I_{M_i}\!:\!I^2) = D_{M_1}I_{M_i}^2(I_{M_i}\!:\!I_{M_i}^2) = D_{M_1}I_{M_i} \supset D_{M_1}I = I_{M_1}\,.$$

Lemma 6.2 implies that $CD_{M_1} = C_{M_1} = D_P$ for some $P \in r\text{-spec}(D)$ with $I \not\subset P$, hence $I^2 \not\subset P$ and therefore $I^2CD_{M_1} = I^2D_P = D_P \supset I_{M_1}$. Putting all together, it follows that $I^2(I\!:\!I^2)D_{M_1} = I_{M_1}$. □

Lemma 6.4. *Let $P \in r\text{-spec}(D)^\bullet$.*

1. *If $P \notin r\text{-}\max(D)$, then P is not r-invertible, and*

$$P^{-1} = (P\!:\!P) = D_P \cap \bigcap_{\substack{M\in r\text{-}\max(D)\\ P\not\subset M}} D_M\,.$$

2. *If $P \in r\text{-}\max(D)$ and P is not r-invertible, then $P^{-1} = (P\!:\!P) = D$.*

3. *If* $Q \in r\text{-spec}(D)$, $Q \subset P$, $z \in P \setminus Q$ *and* $J = (Q \cup \{z\})_r$, *then* J *is* r*-locally principal, and* $Q \subsetneq J \subset P$.

Proof. 1. If $P \notin r\text{-}\max(D)$, then P is not r-invertible by [12, Lemma 4.7]. Hence $P^{-1} = (P\colon P)$ by [12, Proposition 4.8.2], and by [12, Theorem 4.6] P^{-1} is an intersection of localizations as asserted.

2. By [12, Proposition 4.8.2].

3. It suffices to prove that J is r-locally principal. Let $M \in r\text{-max}(D)$. If $J \not\subset M$, then $J_M = D_M$. If $J \subset M$, then $Q_M \cap D = Q$ implies $z \notin Q_M$, hence $Q_M \subset zD_M$ and $J_M = zD_M$. □

Lemma 6.5. *Let* $E \supset D$ *be an* r*-overmonoid,* $q = r[E]$ *and* $\Omega = \{P \in r\text{-spec}(D) \mid P_q \neq E\}$.

1. E *is a* q*-Prüfer monoid.*
2. *If* $\overline{P} \in q\text{-spec}(E)$ *and* $P = \overline{P} \cap D$, *then* $P \in \Omega$, $\overline{P} = P_q$ *and* $E_{\overline{P}} = D_P$.
3. $\Omega = \{P \in r\text{-spec}(D) \mid E \subset D_P\}$, *and if* $P \in \Omega$, *then* $P_q = PD_P \cap E \in q\text{-spec}(E)$.

In particular, the map $\theta\colon q\text{-spec}(E) \to \Omega$, *defined by* $\theta(\overline{P}) = \overline{P} \cap D$, *is bijective.*

Proof. By [17, Theorem 27.2 and Supplement]. □

Lemma 6.6. *Let* $P \in r\text{-spec}(D)$,

$$a \in \bigcap_{\substack{N \in r\text{-max}(D)\\ P \not\subset N}} D_N \setminus D_P \quad \text{and} \quad I = a^{-1}D \cap D\,.$$

Then we have $I \in \mathcal{I}_{r,\mathfrak{f}}(D)$, $I \subset P$ *and* $r\text{-max}(D, I) = r\text{-max}(D, P)$.

Proof. By [17, Theorem 17.6] we have $I \in \mathcal{F}_r(D)^\times$ and thus I is r-finite (note that an r-Prüfer monoid is an r-GCD-monoid). Since $a \notin D_P$, it follows that $a^{-1} \in PD_P$ and $I \subset PD_P \cap D = P$, whence $r\text{-max}(D, P) \subset r\text{-max}(D, I)$. If $N \in r\text{-max}(D)$ and $P \not\subset N$, then $a = s^{-1}c$ for some $c \in D$ and $s \in D \setminus N$, and consequently we obtain $s = a^{-1}c \in I \setminus N$, whence $N \notin r\text{-max}(D, I)$. □

A prime ideal $P \in r\text{-spec}(D)$ is called *branched* if there is some prime ideal $P_0 \subsetneq P$ such that there is no prime ideal lying strictly between P_0 and P. Note that P is branched if and only if for every family $(P_\lambda)_{\lambda \in \Lambda}$ of prime ideals $P_\lambda \subsetneq P$ we have

$$\bigcup_{\lambda \in \Lambda} P_\lambda \subsetneq P\,.$$

For any P, $Q \in r\text{-spec}(D)$, we denote by $R \wedge_r Q$ the greatest prime r-ideal contained in $P \cap Q$ (see Lemma 6.1.3).

Lemma 6.7. *Suppose that every* r*-locally principal* r*-ideal of* D *is* r*-finite. Let* $P \in r\text{-spec}(D)$ *be branched, and let* $P_0 \in r\text{-spec}(D)$ *be such that* $P_0 \subsetneq P$ *and there is no prime* r*-ideal strictly between* P *and* P_0. *Then*

$$(*) \qquad \bigcap_{\substack{N \in r\text{-max}(D)\\ P \not\subset N}} D_N \not\subset D_P\,,$$

and there exists some $I \in \mathcal{I}_{r,\mathfrak{f}}(D)$ *such that* $P_0 \subsetneq I \subset P$ *and* $r\text{-max}(D, I) = r\text{-max}(D, P)$.

Proof. Assume that $(*)$ holds, let

$$a \in \bigcap_{\substack{N \in r\text{-max}(D) \\ P \not\subset N}} D_N \setminus D_P \quad \text{and} \quad I_0 = a^{-1}D \cap D\,.$$

By Lemma 6.6 we have $I_0 \in \mathcal{I}_{r,\mathsf{f}}(D)$, $I_0 \subset P$ and $r\text{-max}(D, I_0) = r\text{-max}(D, P)$. If $z \in P \setminus P_0$, then $J = (P_0 \cup \{z\})_r$ is r-locally principal by Lemma 6.4.3, hence r-finite by assumption, and therefore $I = (I_0 \cup J)_r \in \mathcal{I}_{r,\mathsf{f}}(D)$ fulfills our requirements.

It remains to prove $(*)$.

Suppose first that $P \in r\text{-max}(D)$. Since P is branched, we obtain

$$P \supsetneq \bigcup_{\substack{N \in r\text{-max}(D) \\ N \neq P}} P \wedge_r N\,, \quad \text{and if } a \in P \setminus \bigcup_{\substack{N \in r\text{-max}(D) \\ N \neq P}} P \wedge_r N\,, \quad \text{then } a^{-1} \in \bigcap_{\substack{N \in r\text{-max}(D) \\ P \not\subset N}} D_N \setminus D_P\,.$$

Thus assume from now on that $P \notin r\text{-max}(D)$.

We consider the r-overmonoid $E = (P\!:\!P)$ and set $q = r[E]$. By Lemma 6.5, the monoid E is a q-Prüfer monoid and thus $q = t(E)$. If $\Omega = \{N \in r\text{-spec}(D) \mid N_q \neq E\}$, then there is a bijective map $\theta\colon q\text{-spec}(E) \to \Omega$ such that $\theta(\overline{N}) = \overline{N} \cap D$ for all $\overline{N} \in q\text{-spec}(E)$ and $\theta^{-1}(N) = N_q$ for all $N \in \Omega$. Moreover, it follows that $\Omega = \{N \in r\text{-spec}(D) \mid E \subset D_N\}$. If $N \in \Omega$, then $N_q = D_N \cap E$, and if $\overline{N} \in E$, then $E_{\overline{N}} = D_{\overline{N} \cap D}$. If $\Omega_{\max}$ denotes the set of all maximal elements of Ω, then $\theta(q\text{-max}(E)) = \Omega_{\max}$.

Since $PE \neq E$, it follows that $P_q = P_r = P \in q\text{-spec}(E)$, and since all $N \in r\text{-spec}(D)$ with $N \subset P$ belong to Ω, it follows that P is also branched in E. We claim that $P \in q\text{-max}(E)$, and for this we must prove that $P \in \Omega_{\max}$. Assume to the contrary that there is some $Q \in \Omega$ such that $P \subsetneq Q$. By Lemma 6.4.3 there exists some r-locally principal r-ideal J such that $P \subsetneq J \subset Q$, and by assumption J is r-finite, say $J = Y_r$ for some finite set $Y \subset J$. Therefore it follows that $(J_q)_{v(E)} = Y_{v(E)} = Y_{t(E)} = Y_q = J_q$, and $P \subsetneq J_q \subset Q_q \subsetneq E$ (in fact, it can be proved that $Q_q = Q$, but we do not need this). Since $P \notin q\text{-max}(E)$, Lemma 6.4 (applied for E) implies $(E\!:\!P) = E$, and we obtain $E = P_{v(E)} \subset J_{v(E)} = J_q \subset Q_q \subsetneq E$, a contradiction.

We show now that $\Omega_{\max} = \{P\} \cup \{N \in r\text{-max}(D) \mid P \not\subset N\}$. If $N \in r\text{-max}(D)$ and $P \not\subset N$, then $E \subset D_N$ by Lemma 6.4 and thus $N \in \Omega_{\max}$. To prove the converse, assume that $N \in \Omega_{\max} \setminus \{P\}$, and let $N' \in r\text{-max}(D)$ be such that $N \subset N'$. Since N and P are incomparable, it follows that $P \not\subset N'$, hence $N' \in \Omega$ (as we have just proved) and $N = N' \in r\text{-max}(D)$. Applying the bijection θ, we obtain $\{\overline{N} \cap D \mid \overline{N} \in q\text{-max}(E),\ \overline{N} \neq P\} = \{N \in r\text{-max}(D) \mid P \not\subset N\}$. Since P is branched in E, we obtain

$$P \supsetneq \bigcup_{\substack{\overline{N} \in q\text{-max}(E) \\ \overline{N} \neq P}} P \wedge_r \overline{N}\,,$$

$$\text{and if} \quad a \in P \setminus \bigcup_{\substack{\overline{N} \in q\text{-max}(E) \\ \overline{N} \neq P}} P \wedge_r \overline{N}\,, \quad \text{then} \quad a^{-1} \in \bigcap_{\substack{\overline{N} \in q\text{-max}(E) \\ \overline{N} \neq P}} E_{\overline{N}} \setminus E_P = \bigcap_{\substack{N \in r\text{-max}(D) \\ P \not\subset N}} D_N \setminus D_P\,. \quad \square$$

An r-maximal r-ideal $M \in r\text{-max}(D)$ is called *r-essential* if

$$\bigcap_{\substack{N \in r\text{-}\max D \\ N \neq M}} D_N \not\subset D_M \quad \Big[\text{equivalently,} \quad D \subsetneq \bigcap_{\substack{N \in r\text{-}\max D \\ N \neq M}} D_N\Big]\,.$$

Lemma 6.8. *Let $M \in r$-max(D).*

1. *If M is r-essential, then there exists some $I \in \mathcal{I}_{r,\mathsf{f}}(D)$ such that r-max$(D, I) = \{M\}$.*
2. *M is r-essential if and only if*

$$M \supsetneq \bigcup_{\substack{N \in r\text{-}\max D \\ N \neq M}} (N \wedge_r M)\,.$$

Proof. 1. By Lemma 6.6, applied with $P = M$.

2. Assume first that there is some $x \in M$ such that $x \notin N$ for all $N \in r$-max$(D) \setminus \{M\}$. Then it follows that $x^{-1} \in D_N$ for all $N \in r$-max$(D) \setminus \{M\}$ and yet $x^{-1} \notin D_M$.

To prove the converse, assume to the contrary that M is r-essential and yet

$$M = \bigcup_{\substack{N \in r\text{-}\max D \\ N \neq M}} (N \wedge_r M)\,.$$

By 1. there exists some $I \in \mathcal{I}_{r,\mathsf{f}}(D)$ such that r-max$(D, I) = \{M\}$, say $I = Y_r$ for some finite subset $Y \subset D$. Since $\Omega = \{N \wedge M \mid N \in r\text{-}\max(D)\,,\ N \neq M\}$ is a chain, there is some $N \in \Omega$ such that $Y \subset N$, hence $I \subset N$ and thus $M = N$, a contradiction. □

Lemma 6.9. *Suppose that every r-locally principal r-ideal of D is r-finite. Let $x \in D^\bullet$, $(M_\lambda)_{\lambda \in \Lambda}$ a family in r-max(D), and for each $\lambda \in \Lambda$, let $J_\lambda \in \mathcal{I}_{r,\mathsf{f}}(D)$ be such that $x \in J_\lambda \subset M_\lambda$. Assume further that for each $N \in r$-max(D) there is at most one $\lambda \in \Lambda$ such that $J_\lambda \subset N$. Then Λ is finite.*

Proof. For each $\lambda \in \Lambda$ we have $xJ_\lambda^{-1} \subset D$, and thus it follows that

$$B = \Big(\bigcup_{\lambda \in \Lambda} J_\lambda^{-1} \Big)_r \in \mathcal{F}_r(D)\,.$$

If $N \in r\text{-}\max(D)$, then $J_\lambda^{-1} D_N = (D_N \colon J_\lambda D_N)$ is principal (since D_N is a valuation monoid and each J_λ is r-finite), and $J_\lambda^{-1} D_N = D_N$ if $J_\lambda \not\subset N$. Therefore

$$B_N = \bigcup_{\lambda \in \Lambda} J_\lambda^{-1} D_N$$

either coincides with D_N or with $J_\mu^{-1} D_N$, it N contains some (necessarily unique) J_μ. Therefore B_N is principal for all $N \in r\text{-}\max(D)$, hence, by assumption, B is r-finite. Hence there exist a finite subset $L \subset \Lambda$ such that

$$B = \bigcup_{l \in L} (J_l^{-1})_r, \quad \text{whence} \quad B^{-1} = \bigcap_{l \in L} J_l\,, \quad [\text{since } (J_l^{-1})^{-1} = (J_l)_{v(D)} = (J_l)_r = J_l\,].$$

For each $\lambda \in \Lambda$, $J_\lambda^{-1} \subset B$ implies $B^{-1} \subset J_\lambda \subset M_\lambda$ and therefore $J_l \subset M_\lambda$ for some $\lambda \in \Lambda$. Since $J_l \subset M_l$, we obtain $\lambda = l$, and thus $\Lambda = L$ is finite. □

Lemma 6.10. *Suppose that every r-locally principal r-ideal of D is r-finite. Then every r-maximal r-ideal of D is r-essential.*

Proof. Assume to the contrary that $M \in r\text{-}\max(D)$ is not r-essential and $0 \neq x \in M$. By Lemma 6.8 we obtain

$$M = \bigcup_{\substack{N \in r\text{-}\max(D) \\ N \neq M}} N \wedge_r M\,.$$

If $\{N \in r\text{-}\max(D) \mid N \neq M,\ x \in N\} = \{N_\lambda \mid \lambda \in \Lambda\}$, then $\{N \wedge_r M \mid N \in r\text{-}\max(D),\ N \neq M\}$ is a chain, and therefore

$$M = \bigcup_{\lambda \in \Lambda} P_\lambda\,, \quad \text{where} \quad P_\lambda = N_\lambda \wedge_r M \in r\text{-spec}(D) \quad \text{and} \quad P_\lambda \subsetneq M \quad \text{for all} \quad \lambda \in \Lambda\,.$$

We may assume that $P_\lambda \neq P_\mu$ for all $\lambda,\, \mu \in \Lambda$ with $\lambda \neq \mu$. For each $\lambda \in \Lambda$ we have $P_\lambda \subsetneq N_\lambda$, and by Lemma 6.1.2 there exist $P'_\lambda,\, Q_\lambda \in r\text{-spec}(D)$ such that $P_\lambda \subset P'_\lambda \subsetneq Q_\lambda \subset N_\lambda$ and there is no prime r-ideal strictly between P'_λ and Q_λ. We shall prove:

A. If $\lambda,\, \mu \in \Lambda$ and $\lambda \neq \mu$, then Q_λ and Q_μ are incomparable.

Proof of **A.** Assume to the contrary that there exist $\lambda,\, \mu \in \Lambda$ with $P_\lambda \subsetneq P_\mu$ such that Q_λ and Q_μ are comparable. If $Q_\mu \subset Q_\lambda$, then $P_\mu \subset N_\lambda \wedge_r M = P_\lambda$, a contradiction. Hence we have $Q_\lambda \subset Q_\mu$, and thus Q_λ and P_μ are comparable (since both lie in N_μ). If $Q_\lambda \subset P_\mu$, then $Q_\lambda \subset N_\lambda \wedge_r M = P_\lambda$, a contradiction. If $P_\mu \subset Q_\lambda$, then $P_\mu \subset N_\lambda \wedge_r M = P_\lambda$, which again is impossible. This completes the proof of **A.**

For $\lambda \in \Lambda$, Lemma 6.7 implies the existence of some $J_\lambda \in \mathcal{I}_{r,\mathsf{f}}(D)$ such that $P_\lambda \subsetneq J_\lambda \subset Q_\lambda$ and $r\text{-max}(D, J_\lambda) = r\text{-max}(D, Q_\lambda)$. We assert now that for every $N \in r\text{-max}(D)$ there is at most one $\lambda \in \Lambda$ such that $J_\lambda \subset N$. Then Lemma 6.9 implies that Λ is finite which is impossible.

Assume to the contrary that $N \in r\text{-max}(D)$ and $\lambda,\, \mu \in \Lambda$ are such that $\lambda \neq \mu$ and $J_\lambda \cup J_\mu \subset N$. Then $N \in r\text{-max}(D, J_\lambda) = r\text{-max}(D, Q_\lambda)$ and also $N \in r\text{-max}(D, J_\mu) = r\text{-max}(D, Q_\mu)$, whence Q_λ and Q_μ are comparable, a contradiction. □

Theorem 6.11. *Suppose that every r-locally principal r-ideal of D is r-finite. Then D is a monoid of Krull type. In particular, if D is Clifford r-regular, then D is a monoid of Krull type.*

Proof. By Lemma 4.4 it suffices to prove the first assertion.

Let $x \in D^\bullet$, and set $\{M_\lambda \mid \lambda \in \Lambda\} = r\text{-max}(D, \{x\})$ such that $M_\lambda \neq M_\mu$ if $\lambda \neq \mu$. For each $\lambda \in \Lambda$, M_λ is r-essential by Lemma 6.10, and by Lemma 6.8.1 there exists some $J_\lambda \in \mathcal{I}_{r,\mathsf{f}}(D)$ such that $r\text{-max}(D, J_\lambda) = \{M_\lambda\}$. For $N \in r\text{-max}(D)$ and $\lambda \in \Lambda$ we have $J_\lambda \subset N$ if and only if $N = M_\lambda$. Hence we may apply Lemma 6.9 and conclude that Λ is finite. □

Proposition 6.12. *Suppose that $\mathcal{F}_{r,\mathsf{f}}(D)^\bullet$ is cancellative and every r-finite r-ideal is r-regular. Then D is an r-Prüfer monoid.*

In particular, every t-Clifford regular v-domain is a Prüfer v-multiplication domain (*and even a domain of Krull type*).

Proof. By Proposition 4.2, every $J \in \mathcal{F}_{r,\mathsf{f}}(D)^\bullet$ is N-regular and cancellative and thus it is invertible. Therefore $\mathcal{F}_{r,\mathsf{f}}(D)^\bullet$ is a group and thus D is an r-Prüfer monoid.

If D is a v-domain (that means, D is a v-Prüfer monoid), then $\mathcal{I}_{t,\mathsf{f}}(D)^\bullet = \mathcal{I}_{v,\mathsf{f}}(D)^\bullet$ is cancellative by [17, Theorem 19.2]. Hence $\mathcal{F}_{t,\mathsf{f}}(D)^\bullet$ is cancellative, whence D is a t-Prüfer monoid (that means, D is a Prüfer v-multiplication domain). That it is even a domain of Krull type follows by Theorem 6.11. □

Note added in proof. We are indebted to the referee for pointing out, that Bazzoni's conjecture (that is, Theorem 6.11 in the case of Prüfer domains) has only recently been proved (using basically different techniques) by W. C. Holland, J. Martinez, W. Wm. McGovern and M. Tesemma in the paper "Bazzoni's conjecture", accepted for publication in the Journal of Algebra.

References

[1] D.F. Anderson, *A general theory of class groups*, Commun. Algebra **16** (1988), 805 – 847.
[2] S. Bazzoni, *Class semigroups of Prüfer domains*, J. Algebra **184** (1996), 613 – 631.
[3] ———, *Idempotents of the class semigroup of a Prüfer domain of finite character*, Abelian gGroups, Module Theory, and Topology, Lect. Notes Pure Appl. Math., vol. 201, Marcel Dekker, 1998, pp. 79 – 89.
[4] ———, *Groups in the class semigroup of a Prüfer domain of finite character*, Commun. Algebra **28** (2000), 5157 – 5167.
[5] ———, *Clifford regular domains*, J. Algebra **238** (2001), 703 – 722.
[6] S. Bazzoni, L. Fuchs, and L. Salce, *The hierarchy of uniserial modules over a valuation domain*, Forum Math. **7** (1995), 247 – 277.
[7] S. Bazzoni and L. Salce, *Groups in the class semigroups of valuation domains*, Israel J. Math. **95** (1996), 135 – 155.
[8] N. Bourbaki, *Commutative Algebra*, Springer, 1988.
[9] A. Bouvier and M. Zafrullah, *On some class groups of an integral domain*, Bull. Greek Math. Soc. **29** (1988), 45 – 59.
[10] L.W. Cohen and C. Goffman, *The topology of ordered abelian groups*, Trans. Am. Math. Soc. **67** (1949), 310 – 319.
[11] E.C. Dade, O. Taussky, and H. Zassenhaus, *On the theory of orders, in particular on the semigroup of ideal classes and genera on an order in an algebraic number field*, Math. Ann. **148** (1962), 31 – 64.
[12] A. Foroutan, F. Halter-Koch, and W.A. Schmid, *On the dual of an ideal and ideal transforms*, Bull. Greek Math. Soc. **48** (2003), 19 – 40.
[13] L. Fuchs, *Teilweise geordnete algebraische strukturen*, Vandenhoeck & Ruprecht, 1966.
[14] L. Fuchs and L. Salce, *Modules over valuation domains*, Pure and Applied Mathematics, vol. 97, Marcel Dekker, 1985.
[15] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, 2006.
[16] R. Gilmer, *Multiplicative Ideal Theory*, vol. 90, Queen's Papers, 1992.
[17] F. Halter-Koch, *Ideal Systems. An Introduction to Multiplicative Ideal Theory*, Marcel Dekker, 1998.
[18] ———, *Ideal semigroups of noetherian domains and Ponizovski decompositions*, J. Pure Appl. Algebra **209** (2007), 763 – 770.
[19] S. Kabbaj and A. Mimouni, *Class semigroups of integral domains*, J. Algebra **264** (2003), 620 – 640.
[20] ———, *t-class semigroups of integral domains*, J. Reine Angew. Math. **612** (2007), 213 – 229.
[21] P. Ribenboim, *Sur les groupes totalement ordonnés et l'arithmétique des anneaux de valuation*, Summa Brasiliensis Math. **4** (1958), no. 1, 1 – 65.
[22] P. Zanardo, *The class semigroup of local one-dimensional domains*, Preprint.
[23] P. Zanardo and U. Zannier, *The class semigroup of orders in number fields*, Math. Proc. Camb. Philos. Soc. **115** (1994), 379 – 391.

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