Iterative Solution Techniques in the FEM on Massively Parallel Computers

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Abstract

The non-overlapping Domain Decomposition (DD) approach provides a varity of techniques for developing parallel solution strategies for large-scale systems of finite element (f.e.) equations. In the present paper, the use of local multigrid methods in various Dirichlet DD preconditioners is studied. The numerical experiments carried out on a MULTICLUSTER–II-transputer system show that the use of non-zero initial guesses for the local multigrid methods leads to quite efficient parallel PCG solvers.

1 Introduction

In some foregoing papers [6, 8, 10, 9, 11], the authors and their colleague A. Meyer proposed the parallelization and the preconditioning of the Conjugate Gradient (CG) method on the basis of a non-overlapping Domain Decomposition (DD) approach. In Sections 2 and 3 we review the results of these papers. The DD preconditioner proposed contains three components which can be chosen in order to adapt the preconditioner to the problem under consideration as well as possible. One component is a (modified) Schur-complement preconditioner that has been studied by the DD communitiy very intensively [3, 2]. Another component is a preconditioner for the local homogeneous Dirichlet problems arising in each subdomain. The most sensitive part is the basis transformation matrix transforming the nodal f.e. basis on the interfaces into the approximate discrete harmonic basis [11]. In order to construct the two last components, we can use local multigrid methods. In the foregoing papers, we have already used local multigrid methods with zero initial guess for constructing the Dirichlet problem preconditioner as well as for the basis transformation. In the first case this is certainly sufficiently efficient. However, in the basis transformation case, a specially chosen non-zero initial guess for the local multigrid methods can improve the DD preconditioner very much as is shown by the numerical experiments presented in Section 5. In Section 4 we propose to use either the Full-Multigrid-Method [13] that has been also studied in the numerical experiments of Section 5 or Nepomnyaschikh's norm preserving extension technique [14] for constructing initial guesses on the finest grid.

2 The Nonoverlapping Domain Decomposition FEM

We consider the abstract symmetric, $\mathbf{V}_0$–elliptic and $\mathbf{V}_0$–bounded variational problem

$$\text{Find } u \in \mathbf{V}_0 \, : \, a(u,v) \, = < F, v > \ \forall v \in \mathbf{V}_0 \, , \tag{2.1}$$

arising from the weak formulation of a scalar second–order, symmetric and uniformly elliptic boundery value problem (b.v.p.) given in a d–dimensional bounded domain $\Omega \subset \mathbf{R}^d$ $(d = 2, 3)$ with a piecewise smooth boundary $\Gamma = \partial\Omega$.

As in the finite element substructuring technique, we decompose Ω into p non-overlapping subdomains Ω_i $(i = 1, 2, \ldots, p)$ such that $\overline{\Omega} = \bigcup\limits_{i=1}^{p} \overline{\Omega}_i$, and each subdomain Ω_i into finite elements δ_r such that this discretization process results in a conform triangulation of Ω . In the following, the indices "C" and "I" correspond to the nodes belonging to the coupling boundaries (interfaces) $\Gamma_C = \bigcup\limits_{i=1}^{p} \partial\Omega_i \setminus \Gamma_D$ and to the interior $\Omega_I = \bigcup\limits_{i=1}^{p} \Omega_i$ of the subdomains, respectively, where Γ_D is that part of $\partial\Omega$ where Dirichlet–type boundary conditions are given.

In Section 5, we consider the homogenious Dirichlet problem $(u = 0$ on $\Gamma_D = \Gamma)$ for the potential equation ($-\mathrm{div}\,(a\nabla u) = f$ in $\Omega = (0, 1)^2$) as a model problem for testing our algorithms. The variational formulation (2.1) of this problem consists in finding some function $u \in \mathbf{V}_0 = \overset{\circ}{\mathbf{H}}{}^1(\Omega)$ such that

$$\int_\Omega a(x)\, \nabla^T u(x)\, \nabla v(x)\, \mathrm{dx} \;=\; \int_\Omega f(x)\, v(x)\, \mathrm{dx} \qquad \forall v \in \mathbf{V}_0\, .$$

We decompose Ω into 16 subdomains Ω_i ($i = 1, 2, \ldots, 16$) and each subdomain into three–node triangular finite elements as is shown in Fig.1. We suppose that $a(x) = a_i = \mathrm{const} > 0 \quad \forall x \in \overline{\Omega}_i$.

Define the usual f.e. nodal basis

$$\Phi = \left[\varphi_1, \cdots, \varphi_{N_C}, \varphi_{N_C+1}, \cdots, \varphi_{N_C+N_{I,1}}, \cdots, \varphi_{N=N_C+N_I}\right] , \tag{2.2}$$

where the first N_C basis functions belong to Γ_C, the next $N_{I,1}$ to Ω_1, the next $N_{I,2}$ to Ω_2 and so on such that $N_I = N_{I,1} + N_{I,2} + \ldots + N_{I,p}$. The f.e. subspace

$$\mathbf{V} \;=\; \mathbf{V}_h \;=\; \mathrm{span}(\Phi) \;=\; \mathrm{span}(\Phi V) \;=\; \mathbf{V}_C + \mathbf{V}_I \subset \mathbf{V}_0 \tag{2.3}$$

can be represented as direct sum of the subspaces $\mathbf{V}_C = \mathrm{span}(\Phi V_C)$ and $\mathbf{V}_I = \mathrm{span}(\Phi V_I)$

$$V \;=\; (V_C\; V_I) \;=\; I \;=\; \begin{pmatrix} I_C & O \\ O & I_I \end{pmatrix}_{N\times N} , V_C \;=\; \begin{pmatrix} I_C \\ O \end{pmatrix}_{N\times N_C} \quad \text{and} \quad V_I = \begin{pmatrix} O \\ I_I \end{pmatrix}_{N\times N_I} .$$

The f.e. isomorphism between the f.e. function $u \in \mathbf{V}$ and the corresponding vector $\underline{u} = (\underline{u}_C^T, \underline{u}_I^T)^T \in \mathbf{R}^N$ of the nodal parameters is given by

$$\mathbf{V} \ni u \;=\; \Phi V \underline{u} \;=\; \Phi \underline{u} \quad \overset{\Phi}{\longleftrightarrow} \quad \underline{u} \in \mathbf{R}^N \quad . \tag{2.4}$$

Once the basis Φ is chosen, the f.e. approximation

$$\text{Find} \quad u = \Phi V \underline{u} \in \mathbf{V} \;\;:\;\; a(\Phi V\underline{u}, \Phi V \underline{v}) = < F, \Phi V \underline{u} > \quad \forall v = \Phi V \underline{v} \in \mathbf{V}\, , \tag{2.5}$$

leads to a large–scale sparse system

$$K\,\underline{u} \;=\; \underline{f} \tag{2.6}$$

of finite element equations with the symmetric and positiv definite stiffness matrix K. Because of the arrangement of the basis functions made above, the system (2.6) can be rewritten in the block form

$$\begin{pmatrix} K_C & K_{CI} \\ K_{IC} & K_I \end{pmatrix} \begin{pmatrix} \underline{u}_C \\ \underline{u}_I \end{pmatrix} = \begin{pmatrix} \underline{f}_C \\ \underline{f}_I \end{pmatrix} \quad , \tag{2.7}$$

where $K_I = \mathrm{blockdiag}\,(K_{I,i})_{i=1,2,\ldots,p}$ is blockdiagonal.

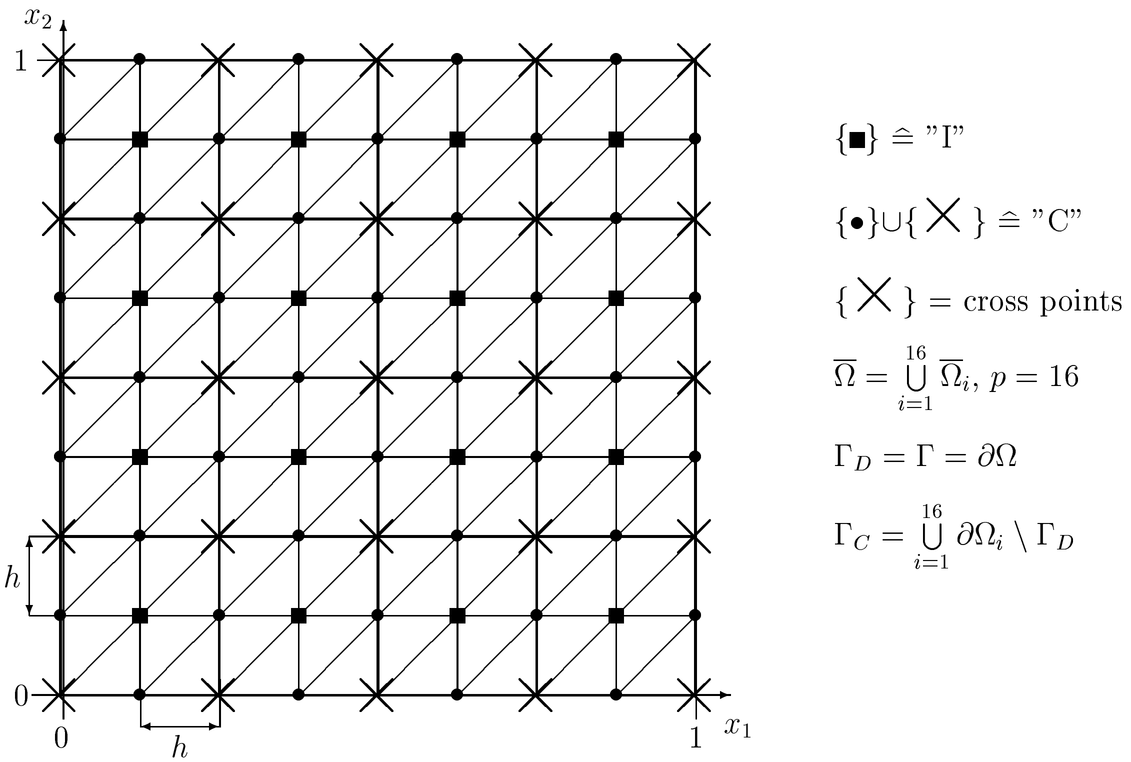


Figure 1: 4×4 Domain decomposition and level-1-triangulation of our model problem
Grid 1 : $N_{I,i} = 1$ (1 unknown per Ω_i), $N = N_C + N_I = 49$ unknowns
Grid 2 : $N_{I,i} = 65.025$, $N = 1.046.529$ unknowns

3 Some DD Preconditioners

Now we use the Parallel Preconditioned Conjugate Gradiant (ParPCG) for solving (2.6)–(2.7) on massively parallel computers. The data distribution and the parallelization of the CG–method is described in [11] in detail. The crucial point is the preconditioning equation

$$C\,\underline{w} \;=\; \underline{r} \tag{3.1}$$

which must fit into the DD parallelization concept proposed by the authors earlier [10, 11].

In [10, 11] the authors presented the ASM–DD preconditioner

$$C = \begin{pmatrix} I_C & K_{CI}B_I^{-T} \\ O & I_I \end{pmatrix} \begin{pmatrix} C_C & O \\ O & C_I \end{pmatrix} \begin{pmatrix} I_C & O \\ B_I^{-1}K_{IC} & I_I \end{pmatrix}\,, \tag{3.2}$$

containing the three components C_C, $C_I = \operatorname{diag}(C_{I,i})_{i=1,2,\ldots,p}$ and $B_I = \operatorname{diag}(B_{I,i})_{i=1,2,\ldots,p}$. These block matrices can be freely chosen in order to adapt the preconditioner to the specialities of the problem under consideration.

In [10, 11] the authors proved the following result.

THEOREM 3.1 *Let the symmetric and positive definite block preconditioners C_C and C_I satisfy the spectral equivalence inequalities*

$$\underline{\gamma}_C C_C \;\leq\; S_C + T_C \;\leq\; \overline{\gamma}_C C_C \quad \text{and} \quad \underline{\gamma}_I C_I \;\leq\; K_I \;\leq\; \overline{\gamma}_I C_I \tag{3.3}$$

with some positive constants $\underline{\gamma}_C$, $\overline{\gamma}_C$, $\underline{\gamma}_I$ and $\overline{\gamma}_I$. Then the ASM-DD preconditioner (3.2) satisfies the spectral equivalence inequalities

$$\underline{\gamma} C \;\leq\; K \;\leq\; \overline{\gamma} C \tag{3.4}$$

with the spectral equivalence constants

$$\underline{\gamma} = \min\{\underline{\gamma}_C, \underline{\gamma}_I\} \left(1 - \sqrt{\frac{\mu}{1+\mu}}\right) \qquad \text{and} \qquad \overline{\gamma} = \max\{\overline{\gamma}_C, \overline{\gamma}_I\} \left(1 + \sqrt{\frac{\mu}{1+\mu}}\right) \tag{3.5}$$

where $\mu = \rho(S_C^{-1} T_C)$ denotes the spectral radius of $S_C^{-1} T_C$, with $S_C = K_C - K_{CI} K_I^{-1} K_{IC}$ and $T_C = K_{CI}(K_I^{-1} - B_I^{-T}) K_I (K_I^{-1} - B_I^{-1}) K_{IC}$. For the spectral condition number $\kappa(C^{-1}K)$ of $C^{-1}K$, the two–side estimate

$$\left(\underline{\gamma}_1 \,/\, \overline{\gamma}_1\right) \left(\sqrt{\mu} + \sqrt{1+\mu}\right)^2 \;\leq\; \kappa(C^{-1}K) \;\leq\; \left(\overline{\gamma}_1 \,/\, \underline{\gamma}_1\right) \left(\sqrt{\mu} + \sqrt{1+\mu}\right)^2 \tag{3.6}$$

holds with $\underline{\gamma}_1 = \min\{\underline{\gamma}_C, \underline{\gamma}_I\}$ and $\overline{\gamma}_1 = \max\{\overline{\gamma}_C, \overline{\gamma}_I\}$.

The matrix B_I, which is not supposed to be neither symmetric nor positive definite, can be interpreted as a part of the basis transformation matrix

$$\widetilde{V} \;=\; \left(\widetilde{V}_C \; \widetilde{V}_I\right) \;=\; \begin{pmatrix} I_C & O \\ -B_I^{-1} K_{IC} & I_I \end{pmatrix} \tag{3.7}$$

transforming the nodal basis Φ in the approximate discrete harmonic basis $\widetilde{\Phi} = \Phi \widetilde{V}$ and μ can be defined by the angle between $\widetilde{\mathbf{V}}_C = \operatorname{span}(\Phi \widetilde{\mathbf{V}}_C)$ and $\widetilde{\mathbf{V}}_I = \mathbf{V}_I = \operatorname{span}(\Phi \widetilde{V}_I)$, [10, 11]. More precisely,

$$\sqrt{\frac{\mu}{1+\mu}} \;=\; \cos \measuredangle \left(\widetilde{\mathbf{V}}_C, \widetilde{\mathbf{V}}_I\right) \;. \tag{3.8}$$

An alternative interpretation follows from the inequalies

$$\begin{aligned}
(\,S_C \underline{u}_C, \underline{u}_C\,) \;&=\; \inf_{\underline{v}_I \in \mathbf{R}^{N_I}} \left(K \begin{pmatrix} \underline{u}_C \\ \underline{v}_I \end{pmatrix}, \begin{pmatrix} \underline{u}_C \\ \underline{v}_I \end{pmatrix}\right) \;=\; \inf_{\substack{u \in \mathbf{V} \\ u = u_C \text{ on } \Gamma_C}} a(\,u, u\,) \\
&=\; \left(K \begin{pmatrix} \underline{u}_C \\ -K_I^{-1} K_{IC} \underline{u}_C \end{pmatrix}, \begin{pmatrix} \underline{u}_C \\ -K_I^{-1} K_{IC} \underline{u}_C \end{pmatrix}\right) \\
&\leq\; \left(K \begin{pmatrix} \underline{u}_C \\ -B_I^{-1} K_{IC} \underline{u}_C \end{pmatrix}, \begin{pmatrix} \underline{u}_C \\ -B_I^{-1} K_{IC} \underline{u}_C \end{pmatrix}\right) \\
&=\; (\,(S_C + T_C) \underline{u}_C, \underline{u}_C\,) \;\leq\; (1+\mu)\,(\,S_C \underline{u}_C, \underline{u}_C\,) \qquad \forall \underline{u}_C \in \mathbf{R}^{N_C} \;,
\end{aligned} \tag{3.9}$$

namely, that the function $\Phi\begin{pmatrix} \underline{u}_C \\ -B_I^{-1} K_{IC} \underline{u}_C \end{pmatrix}$ should be a norm–preserving extension of the function $u_C = \Phi V_C \underline{u}_C$ on Γ_C in the sense of the energy norm induced by the Schur–complement S_C.

S.V. Nepomuyaschikh proved in [14] that one can construct a norm–preserving extension $\Phi\binom{\underline{u}_C}{E_{IC}\underline{u}_C}$ of u_C such that the inequality

$$\left\| \Phi \begin{pmatrix} \underline{u}_C \\ E_{IC}\underline{u}_C \end{pmatrix} \right\|_{H^1(\Omega)} \leq c_E \; \| u_C \|_{H^{\frac{1}{2}}(\Gamma_C)} \tag{3.10}$$

holds for all $u_C = \Phi V_C \underline{u}_C$ and $\underline{u}_C \in \mathbf{R}^{N_C}$, where $E_{IC}: \mathbf{R}^{N_C} \to \mathbf{R}^{N_I}$ denotes the corresponding extension operator and c_E is an h–independent positive constant. Replacing $-B_I^{-1}K_{IC}\underline{u}_C$ by $E_{IC}\underline{u}_C$ we obtain an h–independent bound $1 + \mu = \tilde{c}_E{}^2$, where $\tilde{c}_E$ is defined by c_E and by the norm equivalence constants between the $\mathbf{H}^1(\Omega)$– and the K–energy norm on the one hand and the $\mathbf{H}^{\frac{1}{2}}(\Gamma_C)$ and the S_C–energy norm on the other hand. In Section 4, we will use norm–preserving extensions of the form $E_{IC}\underline{u}_C$ as initual guesses in the local multigrid methods defining $B_{I,i}$ for $i = 1, 2, \ldots, p$.

Let us return to some algorithmic aspects for solving the preconditioning equation (3.1) and to some modifications of the basic preconditioning algorithm (Algorithm 1) given by the ASM–DD preconditioner (3.2). The basic preconditioning Algorithm 1 can be rewritten in the form $\underline{w} = C^{-1}\underline{r}$ as

Algorithm 1 : The ASM-DD Preconditioner [10, 11]
$\underline{\mathbf{w}}_C \;=\; C_C^{-1} \sum_{i=1}^{p} A_{C,i}^T \left(\underline{r}_{C,i} - K_{CI,i} B_{I,i}^{-T} \underline{r}_{I,i} \right)$
$\underline{\mathbf{w}}_{I,i} \;=\; C_{I,i}^{-1} \underline{r}_{I,i} - B_{I,i}^{-1} K_{IC,i} \underline{\mathbf{w}}_{C,i} \qquad ; \; i = 1, 2, \ldots, p$
Determined by the user : $C_C = ?$, $C_I = ?$, $B_I = ?$

where $A_i = \begin{pmatrix} A_{C,i} & A_{CI,i} \\ A_{IC,i} & A_{I,i} \end{pmatrix}$ denotes the subdomain connectivity matrix which is used for a convenient notation only. The subdomain f.e. assembling process which is connected with nearest neighbour communication stands behind this notation [10, 11].

The first modification consists in choosing $B_I = C_I$ which simplifies Algorithm 1. The resulting preconditioner is called simple DD preconditioner [1, 10, 12].

Algorithm 2 : The simple DD Preconditioner [1, 6, 12]
$\underline{\mathbf{w}}_C \;=\; C_C^{-1} \sum_{i=1}^{p} A_{C,i}^T \left(\underline{r}_{C,i} - K_{CI,i} C_{I,i}^{-T} \underline{r}_{I,i} \right)$
$\underline{\mathbf{w}}_{I,i} \;=\; C_{I,i}^{-1} \left(\underline{r}_{I,i} - K_{IC,i} \underline{\mathbf{w}}_{C,i} \right) \qquad ; \; i = 1, 2, \ldots, p$
Determined by the user : $C_C = ?$, $C_I = ?$

Two more interesting modifications of Algorithm 1 will be discussed in the next section where we use local multigrid methods for defining the $K_{I,i}$–preconditioner $C_{I,i}$ and the "basis transformation matrix" $B_{I,i}$ for all subdomains $\Omega_i,\; i = 1, 2, \ldots, p$.

4 Local Multigrid Methods in the DD Preconditioning

In this section we use local multigrid methods for defining C_I and B_I. Because of the obvious spectral equivalence inequalities

$$S_C \;\leq\; S_C + T_C \;\leq\; (1+\mu)\, S_C \tag{4.1}$$

we can utilize Schur-Complement-Preconditioners in order to define C_C. In recent years various S_C-preconditioners have been developed by the DD community [3, 2]. In our numerical experiments presented in the next section we use the properly scaled BPS-preconditioner proposed in [2].

If one applies k symmetric multigrid steps to

$$K_{I,i}\, \underline{w}_{I,i} \;=\; \underline{r}_{I,i} \qquad , \; i = 1,2,\ldots,p\,, \tag{4.2}$$

then the symmetric multigrid preconditioner

$$C_I \;=\; K_I\, \left(I_I - M_I^k\right)^{-1} \tag{4.3}$$

is implicitly defined, where $M_I = \mathrm{diag}\,[M_{I,i}]_{i=1,2,\ldots,p}$ is again the multigrid iteration operator the iteration operator of the multigrid method applied [10, 11]. In the same way, we can use non-necessarily symmetric, local multigrid methods for the basis transformation, i.e.

$$B_I \;=\; K_I\, \left(I_I - \overline{M}_I^s\right)^{-1} \tag{4.4}$$

where $\overline{M}_I = \mathrm{diag}\left[\overline{M}_{I,i}\right]_{i=1,2,\ldots,p}$ and s denotes the corresponding multigrid iteration operator and the number of multigrid steps applied, respectively.

Before discussing some theoretical results, we will give two further modifications of Algorithm 1. Let $\overset{*}{M}_I$ be the adjoint operator to M_I with respect to the K_I-energy inner product. Choosing in Algorithm 1 $B_I \;=\; K_I\, \left(I_I - \overline{M}_I^s\right)^{-1}$ and $C_I \;=\; K_I\, \left(I_I - \overline{M}_I^s\, \overset{*}{\overline{M}}{}_I^s\right)^{-1}$, G. Haase proposed the following Algorithm 3 for the preconditioning equation [4, 5].

Algorithm 3 : Haase's DD Preconditioner [4, 5]
$\tilde{\underline{\mathbf{w}}}_{I,i} \;=\; \left(I_{I,i} - \overset{*}{\overline{M}}{}_{I,i}^s\right) K_{I,i}^{-1}\underline{r}_{I,i}$
$\underline{\mathbf{w}}_C \;=\; C_C^{-1} \sum A_{C,i}^T \left(\underline{r}_{C,i} - K_{CI,i}\tilde{\underline{\mathbf{w}}}_{I,i}\right)$
$\underline{\mathbf{w}}_{I,i} \;=\; \overline{M}_{I,i}\tilde{\underline{\mathbf{w}}}_{I,i} + \left(I_{I,i} - \overline{M}_{I,i}^s\right) K_{I,i}^{-1} \left(\underline{r}_{I,i} - K_{IC,i}\underline{\mathbf{w}}_C\right) \qquad ;\; i = 1,2,\ldots,p$
Determined by the user : $\quad C_C = ?,\ \overline{M}_I = ?$

In [7], the authors showed that the MSM (Multiplicative Schwarz Method) DD Preconditioner corresponding to the ASM preconditioner of Algorithm 1 is nothing else but an ASM preconditioner with the components $B_I \;=\; K_I\, \left(I_I - \overline{M}_I^s M_I^k\right)^{-1}$ and $C_I \;=\; K_I\, \left(I_I - M_I^{2k}\right)^{-1}$:

Algorithm 4 : The MSM-DD Preconditioner [7]	
1) $\underline{y}_{I,i} \; = \; (I_{I,i} - M^k_{I,i})\, K^{-1}_{I,i} \underline{r}_{I,i}$	$i = 1, 2, \ldots, p$
2) $\underline{\hat{w}}_{I,i} \; = \; B^{-T}_{I,i} \underline{r}_{I,i} \; = \overset{*}{\overline{M}}{}^s_{I,i} \underline{y}_{I,i} + (I_{I,i} - \overset{*}{\overline{M}}{}^s_{I,i})\, K^{-1}_{I,i} \underline{r}_{I,i}$	$i = 1, 2, \ldots, p$
3) $\underline{w}_C \; = \; C^{-1}_C \sum\limits_{i=1}^{p} A^T_{C,i} (\underline{r}_{C,i} - K_{CI,i} \underline{\hat{w}}_{I,i})$	$i = 1, 2, \ldots, p$
4) $\underline{z}_{I,i} \; = \; (I_{I,i} - \overline{M}^s_{I,i})\, K^{-1}_{I,i} (-K_{IC,i} \underline{w}_{C,i})$	$i = 1, 2, \ldots, p$
5) $\underline{\tilde{r}}_{I,i} \; = \; \underline{r}_{I,i} - K_{IC,i} \underline{w}_{C,i}$	$i = 1, 2, \ldots, p$
6) $\underline{\tilde{w}}_{I,i} \; = \; \underline{y}_{I,i} + \underline{z}_{I,i}$	$i = 1, 2, \ldots, p$
7) $\underline{w}_{I,i} \; = \; M^k_{I,i} \underline{\tilde{w}}_{I,i} + (I_{I,i} - M^k_{I,i})\, K^{-1}_{I,i} \underline{\tilde{r}}_{I,i}$	$i = 1, 2, \ldots, p$
Determined by the user : $\quad C_C = ?$, $M_I = ?$, $\overline{M}_I = ?$	

Let us now return to the estimate of the spectral constants $\underline{\gamma}_I$, $\overline{\gamma}_I$ and, especially, μ for second-order elliptic b.v.p. considered in Section 2 :

Lemma 4.1 *If* $M_I = \overset{*}{M}_I \geq 0$ *is selfadjoint and non-negative in the* K_I*-energy inner product, then the multigrid preconditioner (4.3) is symmetric and positive definite. Furthermore,* C_I *satisfies the spectral equivalence inequalities (3.3) with* $\underline{\gamma}_I = 1 - \eta^k$ *and* $\overline{\gamma}_I = 1$ *, where* $\eta = \rho(M_I) = \| M_I \|_{K_I} < 1$ *denotes the spectral radius of* M_I*.*

Lemma 4.2 *If* $\| M_I \|_{K_I} \leq \overline{\eta} < 1$ *, then* $\mu \leq \overline{\eta}^{2s}(\frac{c}{h} - 1)$ *where* h *denotes the usual discretization parameter and* c *is an* h*-independent positive constant.*

The proofs of both lemmas can be found in [10, 11]. ■

Lemma 4.2 implies an h-independent μ-estimate if and only if $\overline{\eta}^{2s} = \mathcal{O}(h)$, i.e. $s = \mathcal{O}(\ln h^{-1})$. However, even in the case $s = 1$, the condition $\overline{\eta}^2 \leq h^{-1}$ isn't restrictive in the range of practical applications. Imagine that $h = 1/256$ yields $\overline{\eta} = 1/16$. Such a multigrid rate can be obtained by a tuned V-cycle with 2 pre- and 2 post-smoothing Gauss-Seidel sweeps or by a W-cycle. But $h = 1/256$ means, roughly speaking, $N_i = h^{-2} = 65536$ and $N_i = h^{-3} = 16,8$ million unknowns per subdomain Ω_i (**!**) in the 2D- and 3D-case, respectively. The sensitivity of the multigrid methods used in the basis transformation will be illustrated in the numerical experiments presented in Section 5.

Nevertheless, we should think about the improvement of the basis transformation technique. Looking carefully on the right-hand side of the basis transformation equation

$$B_{I,i}\, \underline{\tilde{w}}_{I,i} \; = \; -K_{IC}\, \underline{w}_{C,i} \qquad i = 1, 2, \ldots, p\,, \tag{4.5}$$

we observe that the right-hand side belongs to the subspace $K_{IC}\mathbf{R}^{N_C}$. So, we must adapt the basis transformation technique to this subspace. This idea was discussed in [4, 5]. In particular, the frequency filtering technique of G. Wittum [15] was used for B_I. Another idea consists in using a non-zero initial guess in the local multigrid method defining B_I. We investigate two good canidates for such non-zero, adapted initial guesses, namely.

1. that one obtained by special Full-Multigrid-Methods
2. and that one obtained by Nepomnyaschikh's norm preserving extension technique.

Firstly, we discuss the Full-Multigrid-Technique. Now the lower index q behind the ";" denotes the level index in sequence of grids used, where q varies from 1 (coarsest grid) up to l (finest grid). The basis transformation equation

$$\underline{w}_{I;l}^{s_l} \;=\; -B_{I;l}^{-1} K_{IC;l}\, \underline{w}_{C;l} \tag{4.6}$$

means the application of s_l multigrid steps to the equation

$$K_{I;l}\, \underline{w}_{I;l} \;=\; -K_{IC;l}\, \underline{w}_{C;l} \tag{4.7}$$

with the (non-zero) initial guess

$$\underline{w}_{I;l}^{0} \;=\; I_{I;l-1}^{l}\, \underline{w}_{I;l-1}^{s_{l-1}} \tag{4.8}$$

obtained by interpolating the nested iterate $\underline{w}_{I;l-1}^{s_{l-1}}$ from the coarseer grid. In the two-grid case, we have

$$\begin{aligned} \underline{w}_{I;l}^{s_l} &= \overline{M}_{I;l}^{s_l} \underline{w}_{I;l}^{0} + \left(I_{I;l} - \overline{M}_{I;l}^{s_l}\right) K_{I;l}^{-1} \left(-K_{IC;l}\underline{w}_{C;l}\right) \\ &= \overline{M}_{I;l}^{s_l}\, I_{I;l-1}^{l}\, K_{I;l-1}^{-1}\, I_{I;l}^{l-1} \left(-K_{IC;l}\underline{w}_{C;l}\right) + \left(I_{I;l} - \overline{M}_{I;l}^{s_l}\right) K_{I;l}^{-1} \left(-K_{IC;l}\underline{w}_{C;l}\right) \\ &= \left[\left(I_{I;l} - \overline{M}_{I;l}^{s_l} \left(I_{I;l} - I_{I;l-1}^{l}\, K_{I;l-1}^{-1}\, I_{I;l}^{l-1}\, K_{I;l}\right)\right) K_{I;l}^{-1}\right] \left(-K_{IC;l}\underline{w}_{C;l}\right) \;, \end{aligned} \tag{4.9}$$

where $I_{I;q}^{q-1}$ and $I_{I;q-1}^{q}$ denote special restriction and interpolation operators, respectively. Therefore,

$$B_{I;l} \;=\; K_{I;l} \left(I_{I;l} - \overline{M}_{I;l}^{s_l} \left(I_{I;l} - I_{I;l-1}^{l}\, K_{I;l-1}^{-1}\, I_{I;l}^{l-1}\, K_{I;l}\right)\right)^{-1} \;. \tag{4.10}$$

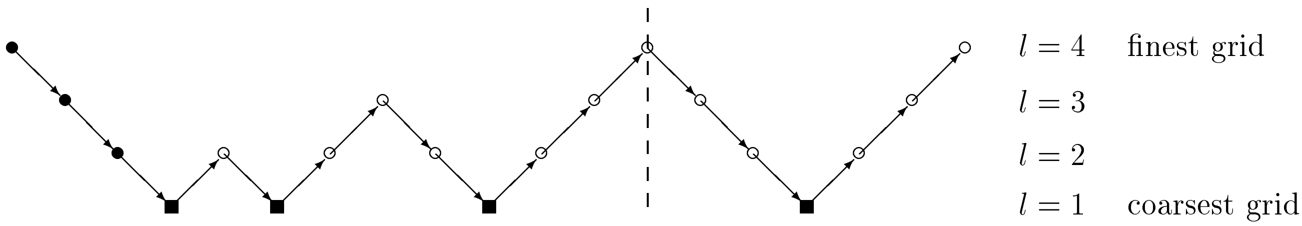
So, in the Full-Multigrid-case with l grids, $B_{I;l}$ is defined by the following recurrent relations :

$$\left.\begin{aligned} B_{I;2}^{-1} &= \left(I_{I;2} - \overline{M}_{I;2}^{s_2} \left(I_{I;2} - I_{I;1}^{2}\, K_{I;1}^{-1}\, I_{I;2}^{1}\, K_{I;2}\right)\right) K_{I;2}^{-1} \;, \\ B_{I;q}^{-1} &= \left(I_{I;q} - \overline{M}_{I;q}^{s_q} \left(I_{I;q} - I_{I;q-1}^{q}\, B_{I;q-1}^{-1}\, I_{I;q}^{q-1}\, K_{I;q}\right)\right) K_{I;q}^{-1} \;, \qquad q = 3, 4, \ldots, l \;. \end{aligned}\right\} \tag{4.11}$$

The operator $B_{I;l}^{-T}$ also needed in Algorithms 1, 3 and 4 is defined by the recurrancy :

$$\left.\begin{aligned} B_{I;2}^{-T} &= \left(I_{I;2} - \left(I_{I;2} - I_{I;1}^{2}\, K_{I;1}^{-1}\, I_{I;2}^{1}\, K_{I;2}\right) \overset{*}{\overline{M}}{}_{I;2}^{s_2}\right) K_{I;2}^{-1} \;, \\ B_{I;q}^{-T} &= \left(I_{I;q} - \left(I_{I;q} - I_{I;q-1}^{q}\, B_{I;q-1}^{-T}\, I_{I;q}^{q-1}\, K_{I;q}\right) \overset{*}{\overline{M}}{}_{I;q}^{s_q}\right) K_{I;q}^{-1} \;, \qquad q = 3, 4, \ldots, l \;. \end{aligned}\right\} \tag{4.12}$$

If we use 1 nested iteration for all grids, i.e. $s_q = 1 \;\; \forall\, q = 2, \ldots, l$, and the V-cycle in the multigrid solver, then the 4-grid-case ($l = 4$) gives the following pictogram for $B_{I;l}$:



The pictogram for $B_{I;l}^T$ is obtained from that above by mirroring it on the broken line and turning the arrows. Therefore, the pictogram for $B_{I;l}$ represents a F-cycle without smoothing ($\bullet$) in the first restriction steps and $B_{I;l}^T$ is an adjoint F-cycle without smoothing ($\bullet$) in the last interpolation steps. The numerical experiments presented in the next section show that the FMGM basis transformation technique yields the best results.

Secondly, we can use Nepomnyaschikh's norm preserving extension technique in order to construct the initial guess

$$\underline{w}_{I;l}^0 \;=\; E_{IC;l}\,\underline{w}_{C;l} \tag{4.13}$$

for the multigrid iteration (4.9). Using Nepomnyaschikh's result (3.10), we can prove the following μ-estimate.

Lemma 4.3 *If*

$$\|\,\overline{M}_I\,\|_{K_I} \;\le\; \overline{\eta} < 1 \qquad \textit{and} \tag{4.14}$$

$$\left\|\begin{pmatrix}\underline{v}_C\\ E_{IC}\underline{v}_C\end{pmatrix}\right\|_K \;\le\; \tilde{c}_E\;\|\,\underline{v}_C\,\|_{S_C} \quad \forall \underline{v}_C \in \mathbf{R}^{N_C}\,, \tag{4.15}$$

then

$$\mu \;\equiv\; \rho(S_C^{-1}T_C) \;\le\; \overline{\eta}^{2s}\,(1+\tilde{c}_E)^2\,. \tag{4.16}$$

Proof. Using the identity

$$\begin{aligned}\left(K\left[\begin{pmatrix}\underline{v}_C\\ E_{IC}\underline{v}_C\end{pmatrix} - \begin{pmatrix}\underline{v}_C\\ -K_I^{-1}K_{IC}\underline{v}_C\end{pmatrix}\right],\;\left[\begin{pmatrix}\underline{v}_C\\ E_{IC}\underline{v}_C\end{pmatrix} - \begin{pmatrix}\underline{v}_C\\ -K_I^{-1}K_{IC}\underline{v}_C\end{pmatrix}\right]\right)\\ =\;\|\,E_{IC}\underline{v}_C + K_I^{-1}K_{IC}\underline{v}_C\,\|_{K_I}^2 \qquad \forall \underline{v}_C \in \mathbf{R}^{N_C}\,,\end{aligned} \tag{4.17}$$

we can estimate

$$\begin{aligned}
\mu \;&=\; \rho(S_C^{-1}T_C) \;=\; \max_{\underline{v}_C\in\mathbf{R}^{N_C}\setminus\{0\}} \frac{\|\,K_I^{-1}K_{IC}\underline{v}_C - B_I^{-1}K_{IC}\underline{v}_C\,\|_{K_I}^2}{\|\,\underline{v}_C\,\|_{S_C}^2}\\
&=\; \max_{\underline{v}_C\in\mathbf{R}^{N_C}\setminus\{0\}} \frac{\left\|K_I^{-1}K_{IC}\underline{v}_C + \overline{M}_I^s E_{IC}\underline{v}_C + (I_I - \overline{M}_I^s)K_I^{-1}(-K_{IC}\underline{v}_C)\right\|_{K_I}^2}{\|\,\underline{v}_C\,\|_{S_C}^2}\\
&=\; \max_{\underline{v}_C\in\mathbf{R}^{N_C}\setminus\{0\}} \frac{\left\|\overline{M}_I^s(E_{IC}\underline{v}_C + K_I^{-1}K_{IC}\underline{v}_C)\right\|_{K_I}^2}{\|\,\underline{v}_C\,\|_{S_C}^2}\\
&\le\; \|\,\overline{M}_I\,\|_{K_I}^{2s} \max_{\underline{v}_C\in\mathbf{R}^{N_C}\setminus\{0\}} \frac{\|\,E_{IC}\underline{v}_C + K_I^{-1}K_{IC}\underline{v}_C\,\|_{K_I}^2}{\|\,\underline{v}_C\,\|_{S_C}^2}\\
&=\; \|\,\overline{M}_I\,\|_{K_I}^{2s} \left[\max_{\underline{v}_C\in\mathbf{R}^{N_C}\setminus\{0\}} \frac{\left\|\begin{pmatrix}\underline{v}_C\\ E_{IC}\underline{v}_C\end{pmatrix} - \begin{pmatrix}\underline{v}_C\\ -K_I^{-1}K_{IC}\underline{v}_C\end{pmatrix}\right\|_K}{\|\,\underline{v}_C\,\|_{S_C}}\right]^2\\
&\le\; \overline{\eta}^{2s} \left[\max_{\underline{v}_C\in\mathbf{R}^{N_C}\setminus\{0\}} \frac{\left\|\begin{pmatrix}\underline{v}_C\\ E_{IC}\underline{v}_C\end{pmatrix}\right\|_K + \left\|\begin{pmatrix}\underline{v}_C\\ -K_I^{-1}K_{IC}\underline{v}_C\end{pmatrix}\right\|_K}{\|\,\underline{v}_C\,\|_{S_C}}\right]^2\\
&\le\; \overline{\eta}^{2s}\,(\tilde{c}_E + 1)^2\,.
\end{aligned}$$

■

Combining the norm equivalence inequalities

$$\underline{c}_1^2 \;\| \Phi \underline{v} \|^2_{H^1(\Omega)} \leq \| \underline{v} \|^2_K = a(\Phi \underline{v}, \Phi \underline{v}) \leq \overline{c}_1^2 \;\| \Phi \underline{v} \|^2_{H^1(\Omega)} \qquad \forall \underline{v} \in \mathbf{R}^N$$

and

$$\underline{c}_2^2 \;\| \Phi V_C \underline{v}_C \|^2_{H^{\frac{1}{2}}(\Gamma_C)} \leq \| \underline{v}_C \|^2_{S_C} \leq \overline{c}_2^2 \;\| \Phi V_C \underline{v}_C \|^2_{H^{\frac{1}{2}}(\Gamma_C)} \qquad \forall \underline{v}_C \in \mathbf{R}^{N_C}$$

with inequality (3.10), we have

$$\tilde{c}_E = \overline{c}_1 \, c_E \, / \, \underline{c}_2 \, .$$

Because $\overline{\eta} < 1$ and these constants are independent of h, we arrive at the h-independent bound (4.16) of μ for each s including $s = 0$. Choosing s appropriately, we can force this bound to be arbitrary small. It seems that the complicated extension procedure proposed by S. Nepomnyaschikh [14] can be replaced by the simple hierarchical basis transformation procedure of H. Yserentant [16] (cf. also [9]).

5 Numerical Results

Let us return to the example described in Section 2, see also Fig. 1. Recall that we want to solve the Dirichlet problem for the Poisson-equation ($a(x) \equiv 1$) in the unit square which was decomposed into 16 subdomains ($p = 16$), and each subdomain was divided into three-nodes (linear) triangular elements. We use Algorithm 3 with an appropriately scaled BPS-preconditioner C_C [2] and the following multigrid methods defining B_I and C_I :

V11	:	V-cycle with 1 pre- and 1 post-smoothing sweep on all grids;
gV11	:	generalized V-cycle with 1 pre- and 1 post-smoothing sweep on all grids;
FV11	:	Full-Multigrid (FMG) V-cycle with 1 pre- and 1 post-smoothing sweep on all grids;
FgV11	:	FMG generalized V-cycle with 1 pre- and 1 post-smoothing sweep on all grids;
V22	:	V-cycle with 2 pre- and 2 post-smoothing sweeps on all grids;
V11(22)	:	V-cycle with 1 (2) pre- and 1 (2) post-smoothing sweeps on the grids $q = 2, \ldots, l-1$ (l) ;
gV11(22)	:	generalized V-cycle with 1 (2) pre- and 1 (2) post-smoothing sweeps on the grids $q = 2, \ldots, l-1$ (l) ;
FV11(22)	:	FMG V-cycle with 1 (2) pre- and 1 (2) post-smoothing sweeps on the grids $q = 2, \ldots, l-1$ (l) ;
3W22	:	3 W-cycles with 2 pre- and 2 post-smoothing sweeps on all grids, corresponds to the exact solution (cf. [2]).

We use lexicographically forward and backward Gauss-Seidel sweeps in the pre- and post-smoothing, respectively. The interpolation and the restriction operators are defined by the bilinear interpolation $I^{q}_{I;q-1}$ and by the so called full weighting restriction $I^{q-1}_{I;q} = (I^{q}_{I;q-1})^T$ for all $q = 2, 3, \ldots, l$. In the FMGM, we use only one nested iteration on all grids, i.e. $s_q = 1$ for all $q = 2, 3, \ldots, l$.

Tables 5.1 and 5.2 show the number $I(\varepsilon = 10^{-6})$ of iterations and the CPU-time in seconds, respectively. The PCG-iteration was stopped if the usual CG-relative accuracy $\varepsilon = 10^{-6}$, i.e. $(\underline{w}^k, \underline{r}^k) \leq \varepsilon\, (\underline{w}^0, \underline{r}^0)$, had been attained. In all cases, we used the preconditioning Algorithm 3 presented in the preceding section. The numerical experiments were carried out on a MULTICLUSTER–II-system with 32 T805 transputers (16 processors were used in our experiments) under the operating system PARIX 1.1. Recall that the 8-grid problem contains **1.046.529** unknowns !

$B =$ \ # grids	3	4	5	6	7	8
$V11$	14	18	23	33	51	>
$gV11$	14	16	19	26	35	50
$FV11$	14	14	17	22	29	41
$FgV11$	14	14	17	21	28	41
$V22$	13	14	16	18	21	28
$V11(22)$	14	14	18	24	34	49
$gV11(22)$	13	14	15	17	20	24
$FV11(22)$	13	13	14	16	17	19
$3\,W22$	12	12	13	14	16	17

Table 5.1 Number $I(\varepsilon = 10^{-6})$ of iterations for the preconditioning Algorithm 3

$B =$ \ # grids	3	4	5	6	7	8
$V11$	1.45	2.96	9.06	42.3	234	>
$gV11$	1.45	2.72	7.96	35.7	180	986
$FV11$	1.50	2.80	8.66	36.3	186	1022
$FgV11$	1.52	2.85	9.72	42.8	219	1273
$V22$	1.40	2.65	7.40	27.9	121	624
$V11(22)$	1.46	2.48	8.00	35.4	188	1025
$gV11(22)$	1.38	2.56	7.11	26.9	120	562
$FV11(22)$	1.44	2.66	7.80	31.2	124	547
$3\,W22$	1.77	4.15	15.00	64.3	284	1204

Table 5.2 CPU-time in seconds for the preconditioning Algorithm 3

The FV11(22) is the best method in the preconditioning Algorithm 3 for finer grids. The number of iterations differ from those where the subdomain problem with the system matrix $K_{I,i}$ is de facto solved exactly (cf. also [2]) by 1 or, at most, 2 iterations. In these cases the number of iterations grows like $\ln h^{-1}$ caused by the BPS interface preconditioner [2]. If we solve the same problem in the rectangle $\Omega = (0,2) \times (0,1)$ decomposed into two squares, then we can use Dryja's interface preconditioner as an optimal Schur-complement preconditioner [3]. The FV11(22) yields $I(\varepsilon = 10^{-6}) = 5$ iteration for all $q = 2, \ldots, 8$ as in the case of exact solution simulated by 3W22 cycles. On the other hand, one can see the sensitivity of the method against the subdomain solvers. The V11-cycle is too poor for finer grids because, for the 8th grid, we have $\eta_{V11} \approx 0.25 \gg 1/16$ (cf. discussion in Section 4 and [5]).

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