• Min - Max principle (for elliptic PDEs)

$$Lu = -\sum_{i,j} a_{ij} u_{x_i} u_{x_j} + \sum_i b_i u_{x_i} + c u$$

a, b, c functions of x

L is coerciv

$\Rightarrow$ weak maximum principle

Peclet problem

A) $c(x) = 0 \quad \forall x \in \Omega$

$$Lu \leq 0 \Rightarrow \max_{\Omega} u = \max_{\partial\Omega} u$$

$$Lu \geq 0 \Rightarrow \min_{\Omega} u = \min_{\partial\Omega} u$$

$\Rightarrow Lu = 0 \Rightarrow$ Min/Max of u is on $\partial\Omega$

B) $c(x) \geq 0 \quad \forall x \in \Omega$

$$Lu \leq 0 \Rightarrow \max_{\Omega} u \leq \max_{\partial\Omega} u^+$$

$$Lu \geq 0 \Rightarrow \min_{\Omega} u \geq -\max_{\partial\Omega} u^-$$

$-u'' + pu' = 0 \Rightarrow$ Minimumum and Maximum of $u(x)$ are located on $\Gamma = \partial\Omega$

Matrices [Meurant, p. 22ff, § 1.4]

- Matrix A is (generalized) strictly diagonally dominant if there exists a vector d, $d_i > 0$:
$$|a_{i,i}|\, d_i \overset{\geq}{>} \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|\, d_j$$
strictly $\Leftrightarrow$ $\exists\, i$ such that $>$ holds

- A is strictly diagonally dominant $\Longrightarrow$ A is non-singular and $a_{ii} \neq 0 \; \forall i$

- If $a_{i,j} \overset{\geq}{>} 0$ then A is a strictly positive matrix
denoted by $A \overset{\geq}{>} 0$

- A is monotone $\Longleftrightarrow$ A is non-singular and $A^{-1} \geq 0$
$\uparrow$
Finite dimensional (discrete) version of Maximum principle
($b \geq 0 \Longrightarrow x \geq 0$ as solution of $Ax = b$)

- If we can write $A = \sigma I - B$ with $\sigma > 0$ and $B \geq 0$ such that $\sigma(B) \leq \sigma$
is named an M-matrix.
If A is non-singular then $\sigma(B) < \sigma$ holds.

Remark: $a_{ii} > 0 \; \forall i$ in an M-matrix

- A is a non-singular M-matrix
$\Longleftrightarrow$ $a_{i,j} \leq 0 \; \forall i \neq j$ and A is strictly diagonally dominant

Remark: $a_{ii} \geq -\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij} \; \forall i$ and $\exists\, i$: $a_{ii} > -\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}$

Min-Max principle in context of FEM

- $a(u,v)$ coerciv $\Longrightarrow$ Min/Max principle holds in weak formulation

- Galerkin approach

$$a(u_h, v_h) = \langle F, v_h \rangle \qquad V_h \subset H^1$$

discretization

FEM isomorphism

$$K\underline{u} = \underline{f} \qquad \mathbb{R}^{N_h}$$

- discrete Version of Min-Max principle has to be preserved by the FEM in matrix K

- Violation of the discrete maximum principle manifests itself in an unphysical discrete solution

- Kondition of stiffness matrix

$$\varkappa(K) \leq \frac{\lambda_{max}(K)}{\lambda_{min}(K)} \leq \frac{\mu_2}{\mu_1}$$

- finite support of basis functions and test functions
  $\longrightarrow$ sparse matrix K (sparsity pattern)

Storage schemes for K:

e.g: CRS compressed row storage