Variationsformulierung 1D

Funktionsräume

$$L_2(a,b) = \left\{ u(x) : \exists \int_a^b (u(x))^2 \, dx < \infty \right\}$$

example: $u(x) = \mathrm{sgn}(x) = \begin{cases} 1 & x \in (0,2] \\ 0 & x = 0 \\ -1 & x \in [-2,0) \end{cases}$

remark: Two functions $u, \hat{u} \in L_2(a,b)$ are equivalent

$$\Longleftrightarrow \int_a^b (u(x) - \hat{u}(x))^2 \, dx = 0$$

$\Leftrightarrow$ (equal almost everywhere)

L_2 is linear vector space

(i) $\|v\| \geq 0$ $\quad \|v\| = 0 \Longleftrightarrow v = 0$

(ii) $\|\lambda v\| = |\lambda| \cdot \|v\| \qquad \forall v, \forall \lambda \in \mathbb{R}$

(iii) $\|u+v\| \leq \|u\| + \|v\| \qquad \forall u, v$

$$\|u\|_{2,(a,b)} = \sqrt{\int_a^b (u(x))^2 \, dx}$$

Cauchy-Schwarz inequality

$$\left|\langle u,v\rangle_2\right| = \left|\int_a^b u(x)v(x)\,dx\right| \le \sqrt{\int_a^b (u(x))^2\,dx}\,\sqrt{\int_a^b (v(x))^2\,dx}$$

$$= \|u\|_2 \cdot \|v\|_2$$

generalized derivaty

Let $u \in C^1[a,b]$ (exists $u'(x)$) and $\varphi(x) \in C^1[a,b]$

with $\varphi(a) = \varphi(b) = 0$ }

partial integration

$$\int_a^b u(x)\varphi'(x) = -\int_a^b u'\varphi(x) + \underbrace{u(x)\varphi(x)\Big|_a^b}_{=0}$$

weak derivative (Sobolev)

Let $u(x), w(x) \in L_2(a,b)$ and we get

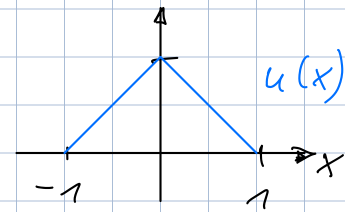
$$\int_a^b u(x)\varphi'(x)\,dx = -\int_a^b w(x)\varphi(x)\,dx$$

$\forall \varphi \in C^1[a,b]$ with $\varphi(a) = \varphi(b) = 0$

Then $w(x)$ is the generalized/weak derivative of $u(x)$.

Example:

$$u(x) = \begin{cases} x+1 & x \in [-1, 0] \\ 1-x & x \in (0, 1) \end{cases}$$



$\varphi \in C^1$
$\varphi(-1) = \varphi(1) = 0$

$$\int_{-1}^{1} u(x)\varphi'(x)\,dx = \int_{-1}^{0} (x+1)\varphi'(x)\,dx + \int_{0}^{1} (1-x)\varphi'(x)\,dx$$

$$\overset{P.I.}{=} -\int_{-1}^{0} 1\cdot\varphi(x)\,dx + (x+1)\varphi(x)\Big|_{-1}^{0}$$

$$-\int_{0}^{1} (-1)\varphi(x)\,dx + (1-x)\varphi(x)\Big|_{0}^{1}$$

$$= -\left[\int_{-1}^{0} 1\cdot\varphi(x)\,dx + \int_{0}^{1} (-1)\varphi(x)\,dx\right] + \underbrace{\varphi(0) - \varphi(0)}_{=0}$$

and we get $u'(x)$ in the weak sense

$$u'(x) = \begin{cases} 1 & x \in [-1, 0] \\ -1 & x \in (0, 1] \end{cases}$$

Definition: Sobolev space $H^1(a,b)$

$$H^1(a,b) = \{ u \in L_2(a,b) : \exists \text{ weak derivative } u' \in L_2(a,b) \}$$

with norm $$\|u\|_1 = \sqrt{\int_a^b (u(x))^2 dx + \int_a^b (u'(x))^2 dx}$$

Weak formulation for 1D heat equation

strong formulation: $\lambda \in C^1,\ f \in L_\infty,\ u \in C^2(\Omega)$, $\Omega = (a,b)$

$$-(\lambda(x) u'(x))' = f(x) \qquad x \in \Omega$$

$$u(x) = g_1(x) \qquad x \in \Gamma_1$$

$$\left(\lambda \frac{\partial u}{\partial x}\right) \cdot \vec{n} = \lambda \frac{\partial u(x)}{\partial \vec{n}} = g_2(x) \qquad x \in \Gamma_2$$

$$\lambda \frac{\partial u(x)}{\partial \vec{n}} = \alpha(x) (g_3(x) - u(x)) \qquad x \in \Gamma_3$$

$\downarrow \int_a^b \_\_ \cdot v(x) dx$ with $v(x) \in V_0 := H_0^1(\Omega) = \{ v \in H^1(\Omega) : v(x) = 0\ \forall x \in \Gamma_1 \}$

$$-\int_a^b (\lambda(x) u'(x))' v(x) dx = \int_a^b f(x) v(x) dx \qquad \forall v \in V_0$$

partial integration with $w(x) := -\lambda(x) u'(x)$

$$\int_a^b \lambda(x) u'(x) v'(x) dx - \lambda(x) \underbrace{u'(x)}_{\frac{\partial u}{\partial \vec{n}}} v(x) \Big|_a^b = \int_a^b f(x) v(x) dx$$

incorporation of boundary conditions

$$\Gamma_1 = \{a\}: \quad u \in V_g(\Omega) := \{ u \in H^1(\Omega) : u(x) = g_1(x) \quad \forall x \in \Gamma_1 \}$$

$$\Gamma_3 = \{b\} \quad -\lambda \frac{\partial u(b)}{\partial \vec{n}} = -\lambda \frac{\partial u(b)}{\partial x} = \alpha_b \left(u(b) - g_3(b) \right)$$

we get $\qquad \forall v \in V_0$

$$\int_a^b \lambda(x)\, u'(x)\, v'(x)\, dx + \alpha_b \left(u(b) - g_3(b) \right) v(b) = \int_a^b f(x)\, v(x)\, dx$$

$$\Updownarrow$$

$$\int_a^b \lambda(x)\, u'(x)\, v'(x)\, dx + \alpha_b\, u(b)\, v(b) = \int_a^b f(x)\, v(x)\, dx + \alpha_b\, g_3(b)\, v(b)$$

Definition:

bilinear form $\quad a(u,v) := \int_a^b \lambda(x)\, u'(x)\, v'(x)\, dx + \alpha_b\, u(b)\, v(b)$

linear form $\quad \langle F, v \rangle := \int_a^b f(x)\, v(x)\, dx + \alpha_b\, g_3(b)\, v(b)$

- bilinear form: $a(c_1 u_1 + c_2 u_2, v) = c_1 a(u_1, v) + c_2 a(u_2, v) \quad \forall u_1, u_2, v \in V, \; c_1, c_2 \in \mathbb{R}$

  $a(u, c_1 v_1 + c_2 v_2) = c_1 a(u, v_1) + c_2 a(u, v_2) \quad \forall u, v_1, v_2 \in V, \; c_1, c_2 \in \mathbb{R}$

linear form: $\langle F, c_1 v_1 + c_2 v_2 \rangle = c_1 \langle F, v_1 \rangle + c_2 \langle F, v_2 \rangle \quad \forall v_1, v_2 \in V, \; c_1, c_2 \in \mathbb{R}$

▶ PDE in variational form

find $u \in V_g = \{ u \in H^1(\Omega) : u(a) = g_1(a) \}$ such that

$$a(u,v) = \langle F, v \rangle$$

holds for all $v \in V_0 = \{ v \in H^1(\Omega) : v(a) = 0 \}$.

Now in 3D

PDE: $$-\nabla^T \Lambda \nabla u(\vec{x}) = f(\vec{x}) \qquad \vec{x} \in \Omega \subset \mathbb{R}^3$$
$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

$$u(x) = g_1(\vec{x}) \qquad \vec{x} \in \Gamma_1$$

$$(\Lambda \nabla u(\vec{x})) \cdot \vec{n} = \Lambda \frac{\partial u(\vec{x})}{\partial \vec{n}} = g_2(\vec{x}) \qquad \vec{x} \in \Gamma_2$$

$$\Lambda \frac{\partial u(x)}{\partial \vec{n}} = \alpha(\vec{x}) \left(g_3(\vec{x}) - u(\vec{x}) \right) \qquad \vec{x} \in \Gamma_3$$

$$\downarrow \cdot v(\vec{x}) \in V_0 := \{ v \in H^1(\Omega) : v(\vec{x}) = 0 \quad \forall \vec{x} \in \Gamma_1 \}$$

$$\int_\Omega \quad v(x)\,dx$$

\+ partial integration

$$\int_\Omega (\nabla v)^T \Lambda \nabla u \, dx - \int_{\Gamma_2 \cup \Gamma_3} \Lambda \frac{\partial u}{\partial \vec{n}} v \, ds = \int_\Omega f v \, dx$$

$\downarrow$ b.c.

$$\underbrace{\int_\Omega (\nabla v)^T \Lambda \nabla u \, dx + \int_{\Gamma_3} \alpha u \cdot v \, ds}_{=: a(u,v)} = \underbrace{\int_\Omega f v \, dx + \int_{\Gamma_2} g_2(x) v \, ds + \int_{\Gamma_3} \alpha g_3 v \, ds}_{=: \langle F, v \rangle}$$

Find $u(\vec{x}) \in V_g$:

$$a(u,v) = \langle F, v \rangle \qquad \forall\, v(\vec{x}) \in V_0$$