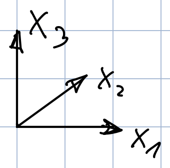
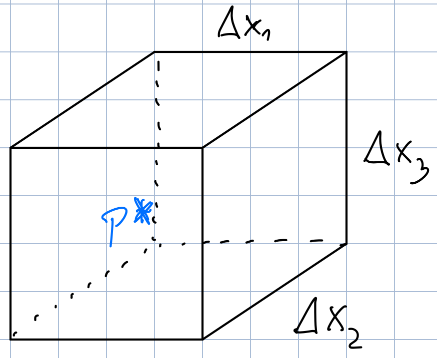
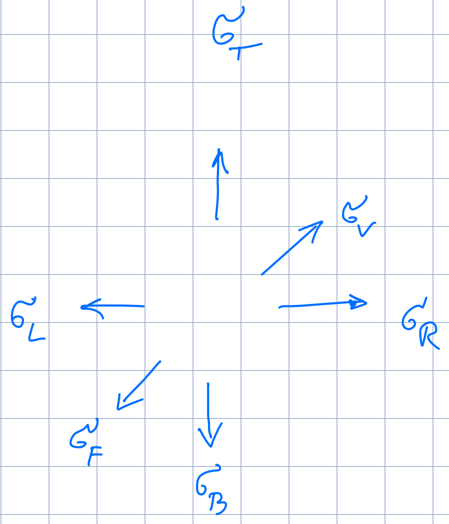
Regarding modelling

Box $\subset \Omega \subset \mathbb{R}^3$







$P(x_1, x_2, x_3)$

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \qquad \nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}, \qquad \frac{\partial u}{\partial \vec{n}} = \vec{n}^T \nabla u = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} n_i$$

$\|\vec{n}\| = 1$, outward

Fourier law for orthotrop material $\quad \sigma_i = -\lambda_i \frac{\partial u}{\partial x_i}$

or in general

$$\sigma = -\Lambda \nabla u \qquad (*)$$

[on boundary: $\sigma_n = n^T \sigma$, orthogonal to edge/surface]

diff. formulation energy balance

$$\longrightarrow \quad -\overset{\text{div}}{\nabla^T} \Lambda \overset{\text{grad}}{\nabla u} = f(x) \quad \text{in } \Omega \qquad + \text{B.C} \qquad (**)$$

$$\Lambda = Id \quad \Rightarrow \quad \nabla^T \nabla = \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i}$$

Laplace operator

Integral formulation Energy balance

(***)

$$\underbrace{-\int_{\partial\Omega_B}\sigma_{B,\vec{n}}-\int_{\partial\Omega_T}\sigma_{T,\vec{n}}-\int_{\partial\Omega_L}\sigma_{L,\vec{n}}-\int_{\partial\Omega_R}\sigma_{R,\vec{n}}-\int_{\partial\Omega_F}\sigma_{F,\vec{n}}-\int_{\partial\Omega_V}\sigma_{V,\vec{n}}}_{\text{energy outflow}}+\underbrace{\int_{\Omega}f(\vec{x})\,dx}_{\text{internal energy production}}=0$$

$\downarrow$ (*) $\qquad\qquad\qquad$ $\downarrow$ (**)

$$\int_{\partial\Omega}\underbrace{(\Lambda\nabla u)}_{\vec{F}}\cdot\vec{n}\,ds-\int_{\Omega}\nabla^T\underbrace{\Lambda\nabla u}_{\vec{F}}\,dx=0$$

$\downarrow$

$$\boxed{\underline{\int_{\partial\Omega}\vec{F}\cdot\vec{n}\,ds}=\underline{\int_{\Omega}\underbrace{\nabla^T}_{div}\vec{F}\,dx}}$$

Theorem by Gauß-Ostrogradski

|||

energy balance

(***) is the basis for Finite Volume Methods (FDM)

Instationary heat conductivity problem

$$c(x)\,\varrho(x)\,\frac{\partial u(x,t)}{\partial t} - \nabla^T \Lambda \nabla u(x,t) = f(x,t)$$

specific heat capacity

density

specific heat conductivity

$(x,t) \in \Omega \times T = Q_T$

space time cylinder

+ b.c., e.g. $u(x,t) = g_1(x,t)$

+ initial condition $u(x,0) = u_0(x)$

! Compatibility between b.c. and i.c. ✓

- $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}\left((\Omega \cup \Gamma_2 \cup \Gamma_3) \times (0,T)\right) \cap C(\bar{Q}_T)$

- additional Terms:
  - advection $+ q(x,t) \cdot u(x,t)$
  - convection $+ \vec{v}(x,t)\,\nabla u(x,t)$
  - radiation in b.c. $-(\Lambda \nabla u) n = \varkappa\left(u - u_{out}\right)^4$