# Extended Parallel Linear Algebra

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# Element decomposition

Assume: Finite element simulation over a complicated domain using an unstructured mesh.

- Main Strategy: Domain decomposition of the mesh.
  - Elements are distributed onto different processors
  - Graph partitioning problem: METIS/PARMETIS software
  - Communication setup **requires global node numbers** and local element connectivity

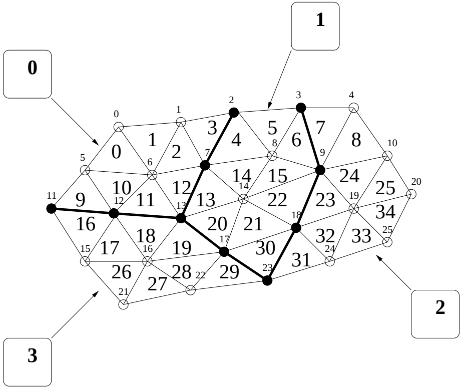



Figure: Finite element mesh distributed to four processing nodes with global element and

# General Parallelization - Operators

The linear system of equations

$$K_h \cdot \underline{u}_h = \underline{f}_h$$

resulting from a (FEM-) discretization of an elliptic spd. operator, i.e..

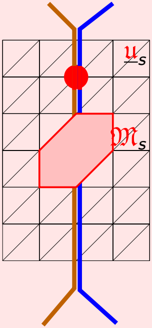
- $K_h$ is unstructured, sparse and has a **large dimension**,
- $K_h$ is symmetric, i.e., $K_h = K_h^T$,
- $K_h$ positive definite, i.e., $(K_h \cdot \underline{v}_h, \underline{v}_h) > 0 \qquad \forall \underline{v}_h \in \mathbb{R}^{N_h}\,,\ \underline{v}_h \neq 0$

has to be solved in parallel:

- iterative solvers
- multigrid as preconditioner
- AMG as preconditioner

# General Parallelization - Operators

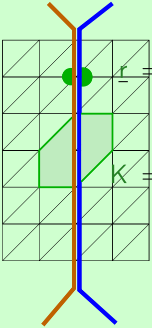
**accumulated**



$$\underline{\mathfrak{u}}_s = A_s \underline{\mathfrak{u}}$$

$$\mathfrak{M}_s = A_s \mathfrak{M} A_s^T$$

**distributed**

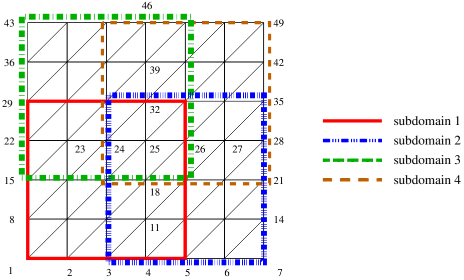


$$\underline{\mathsf{r}} = \sum_{s=1}^{P} A_s^T \underline{\mathsf{r}}_s$$

$$\mathsf{K} = \sum_{s=1}^{P} A_s^T \mathsf{K}_s^{\mathrm{FEM}} A_s$$

Parallel linear algebra **works great** for distributed matrix $\mathsf{K}$ stored as $\mathsf{K}_s$.
**What about** operations with an accumulated Matrix $\mathfrak{M}$ stored as $\mathfrak{M}_{\mathfrak{s}}$?

# Some definitions: classes of nodes




Set of subdomains among node $x^{[i]}$ is shared:

$$\sigma^{[i]} = \{s : x^{[i]} \in \overline{\Omega}_s\}$$

Set of indices/nodes for a set of subdomains:

$$\omega(\sigma) \quad := \quad \{i \in \omega : \sigma^{[i]} = \sigma\}$$

Ex.:

$$\sigma^{[11]} = \{1, 2\} \qquad \omega(\sigma^{[11]}) = \{3, 4, 5, 10, 11, 12\}$$

$$\sigma^{[27]} = \{2, 4\} \qquad \omega(\sigma^{[27]}) = \{20, 21, 27, 28, 34, 35\}$$

$$\sigma^{[14]} = \{2\} \qquad \omega(\{2\}) = \{6, 7, 13, 14\}$$

# Matrix Patterns and their Application

The following operations can be performed in parallel without any communication:

$$\underline{\mathsf{f}} \;=\; \mathsf{K} \cdot \underline{\mathfrak{u}}$$

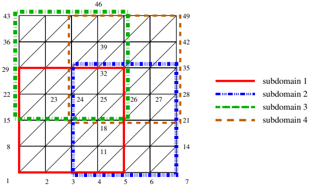
Matrix $\mathfrak{M}$ fulfills the pattern condition:

$$\forall i,j \in \omega \;:\; \sigma^{[i]} \not\subseteq \sigma^{[j]} \;\Longrightarrow\; \mathfrak{M}^{[i,j]} = 0 \qquad i \not\longleftarrow j$$

$$\begin{aligned} \underline{\mathfrak{u}} &= \mathfrak{M} \cdot \underline{\mathfrak{w}} \\ \underline{\mathsf{f}} &= \mathfrak{M}^T \cdot \underline{\mathsf{r}} \end{aligned}$$

$$\boxed{\mathsf{K}^H \;=\; \mathfrak{M}^T \cdot \mathsf{K} \cdot \mathfrak{M}}$$

Theorems/Proofs **[Haase]**




Ex.:

$$\sigma^{[11]} \not\subseteq \sigma^{[27]} \Longrightarrow \mathfrak{M}^{[11,27]} = 0 \qquad 11 \not\longleftarrow 27$$

$$\sigma^{[27]} \not\subseteq \sigma^{[11]} \Longrightarrow \mathfrak{M}^{[27,11]} = 0 \qquad 27 \not\longleftarrow 11$$

$$\sigma^{[11]} \not\subseteq \sigma^{[14]} \Longrightarrow \mathfrak{M}^{[11,14]} = 0 \qquad 11 \not\longleftarrow 14$$

$$\sigma^{[27]} \not\subseteq \sigma^{[14]} \Longrightarrow \mathfrak{M}^{[27,14]} = 0$$

# Admissible Matrix Operations

$$\mathfrak{M} = \begin{pmatrix} \mathfrak{M}_V & 0 & 0 \\ \mathfrak{M}_{EV} & \mathfrak{M}_E & 0 \\ \mathfrak{M}_{IV} & \mathfrak{M}_{IE} & \mathfrak{M}_I \end{pmatrix} = \mathfrak{M}_L + \mathfrak{M}_D \quad \Longrightarrow \quad \underline{\mathfrak{u}} = \mathfrak{M} \cdot \underline{\mathfrak{w}}$$

Pattern condition $\boxed{\sigma^{[i]} \not\subseteq \sigma^{[j]} \Longrightarrow \mathfrak{M}^{[i,j]} = 0}$ has to be fulfilled in all submatrices!!
Allows operations as (Parallel ADI **[DouHaa]**)

$$\underline{\mathfrak{w}} = \mathfrak{M} \cdot \underline{\mathfrak{u}} := (\mathfrak{M}_L + \mathfrak{M}_D) \cdot \underline{\mathfrak{u}} + \sum_{s=1}^{P} A_s^T \mathfrak{M}_{U,s} R_s^{-1} \cdot \underline{\mathfrak{u}}_s$$

or, for $\mathfrak{M} = \mathfrak{L}^{-1} \cdot \mathfrak{U}^{-1}$ (**[Haase]**)

$$\underline{\mathfrak{w}} = \mathfrak{L}^{-1} \mathfrak{U}^{-1} \cdot \underline{\mathfrak{r}} := \mathfrak{L}^{-1} \sum_{s=1}^{P} A_s^T \mathfrak{U}_s^{-1} \cdot \underline{\mathfrak{r}}_s$$

# Parallel Iteration Schemes to Solve

- Richardson iteration:

$$\underline{\mathfrak{u}}_s^{k+1} := \underline{\mathfrak{u}}_s^{k} + \tau \sum_{q=1}^{P} \left( \underline{\mathsf{f}} - \mathsf{K} \cdot \underline{\mathfrak{u}}^{k} \right)_q$$

- Jacobi iteration with $\mathfrak{D} = \sum\limits_{s=1}^{P} \operatorname{diag}\{K_s\}$:

$$\underline{\mathfrak{u}}_s^{k+1} := \underline{\mathfrak{u}}_s^{k} + \omega \mathfrak{D}_s^{-1} \sum_{q=1}^{P} \left( \underline{\mathsf{f}} - \mathsf{K} \cdot \underline{\mathfrak{u}}^{k} \right)_q$$

- Incomplete factorization $\mathfrak{K} = \mathfrak{U} \cdot \mathfrak{L} + \mathfrak{R}$:

$$\underline{\mathfrak{u}}_s^{k+1} := \underline{\mathfrak{u}}_s^{k} + \mathfrak{L}_s^{-1} \sum_{q=1}^{P} \mathfrak{U}_q^{-1} \left( \underline{\mathsf{f}} - \mathsf{K} \cdot \underline{\mathfrak{u}}^{k} \right)_q$$

# Parallel Multigrid : $\mathrm{PMG}(\mathsf{K}, \underline{\mathfrak{u}}, \underline{\mathsf{f}}, \ell)$

**if** $\ell == 1$ **then**

$\sum_{s=1}^{P} A_s^T \mathsf{K} A_s \cdot \underline{\mathfrak{u}} \;=\; \underline{\mathsf{f}}$ ▷ Coarsest grid solver

**else**

$\widetilde{\underline{\mathfrak{u}}} \leftarrow \textsc{Smooth}(\mathsf{K}, \underline{\mathfrak{u}}, \underline{\mathsf{f}}, \nu)$ ▷ e.g., Jacobi iteration

$\underline{\mathsf{d}} \leftarrow \underline{\mathsf{f}} - \mathsf{K} \cdot \underline{\mathfrak{u}}$ ▷ Defect

$\underline{\mathsf{d}}^H \leftarrow \mathfrak{P}^T \cdot \underline{\mathsf{d}}$

$\underline{\mathfrak{w}}^H \leftarrow 0$

$\mathrm{PMG}^{\gamma}(\mathsf{K}^H, \underline{\mathfrak{w}}^H, \underline{\mathsf{d}}^H, \ell - 1)$ ▷ Get the correction

$\underline{\mathfrak{w}} \leftarrow \mathfrak{P} \cdot \underline{\mathfrak{w}}^H$ ▷ Interpolation

$\widehat{\underline{\mathfrak{u}}} \leftarrow \widetilde{\underline{\mathfrak{u}}} + \underline{\mathfrak{w}}$

$\underline{\mathfrak{u}} \leftarrow \textsc{Smooth}(\mathsf{K}, \widehat{\underline{\mathfrak{u}}}, \underline{\mathsf{f}}, \nu)$ ▷ e.g., Jacobi iteration

**end if**

# Recall: Main Parts of AMG

(i) Coarsening:

$$\omega^h = \omega_C^h \cup \omega_F^h \quad .$$

(ii) Interpolation weights:

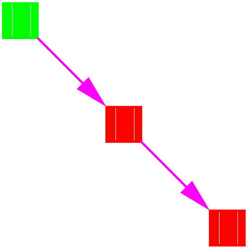
$$P = \{\alpha_{ij}\}_{i\in\omega^h, j\in\omega_C^h} \; : \mathbb{R}^H \mapsto \mathbb{R}^h \quad .$$

(iii) Coarse matrix:

$$K^H = P^T \cdot K \cdot P$$

(iv) (i)-(iii) recursively applicable.

(v) Apply the standard MG-procedure.



# Idea for Parallelizing of AMG setup

parallel MG $\Leftrightarrow$ interpolation $\mathfrak{P}$ has to fulfill the pattern condition

$$\Downarrow$$

$\mathsf{K}^H = \mathfrak{P}^T \cdot \mathsf{K} \cdot \mathfrak{P}$ can be used in $\mathrm{PMG}(\mathsf{K}^H, \underline{\mathfrak{u}}^H, \underline{\mathfrak{f}}^H, \ell - 1)$.

$$\Downarrow$$

**Idea**

Control of coarsening and interpolation
such that the pattern condition is fulfilled for $\mathfrak{P}$.

**[Haase]**

# Coarsening in sequential AMG

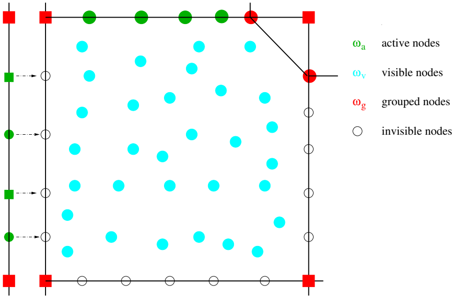
$$(\omega_C, \omega_F) \leftarrow \textsc{Coarse}(\{S_h^{i,T}\}, \omega_C, \omega_F)$$

**while** $\omega_C \cup \omega_F \not\supseteq \omega_h$ **do**
    $k \leftarrow \textsc{Pick}(\{S_h^{i,T}\},\, \omega_h \setminus (\omega_C \cup \omega_F)),$
    **if** $|S_h^{k,T}| + |S_h^{k,T} \cap \omega_F| = 0$ **then**
        $\omega_F \leftarrow \omega_F \cup (\omega_h \setminus \omega_C)$
    **else**
        $\omega_C \leftarrow \omega_C \cup \{k\}$
        $\omega_F \leftarrow \omega_F \cup (S_h^{k,T} \setminus \omega_C)$
    **end if**
**end while**

# Controlling the Pattern in the Coarsening Process

$$\textsc{CoarseP}\ (\{S^i\}, \omega_m, \omega_a, \omega_v, \omega_C, \omega_F)$$

- $\mathsf{K}_i \xrightarrow{\text{pattern}} \widetilde{\mathsf{K}}_i$
- active nodes:
  $\omega_a := \{ i \in \omega_s \, : \, \sigma^{[i]} \equiv \sigma_k \}$,
- visible nodes:
  $\omega_v := \{ i \in \omega_s \, : \, \sigma^{[i]} \subset \sigma_k \}$,
- grouped nodes:
  $\omega_g := \{ i \in \omega_s \, : \, \sigma^{[i]} \supset \sigma_k \}$,
- invisible nodes :
  $\omega_s \setminus (\omega_a \cap \omega_v \cap \omega_g)$ .



# Coarsening on Subsets of Nodes in Parallel AMG

$$(\omega_C, \omega_F) \leftarrow \textsc{CoarseP}(\{S_h^{i,T}\}, \omega_m, \omega_a, \omega_v, \omega_C, \omega_F)$$

**while** $\omega_C \cup \omega_F \not\supseteq \omega_a$ **do**
  $k \leftarrow \textsc{Pick}(\{S_h^{i,T}\}, \omega_m \setminus (\omega_C \cup \omega_F), \omega_F)$
  **if** $|S_h^{k,T}| + |S_h^{k,T} \cap \omega_F| = 0$ **then**
    $\omega_F \leftarrow \omega_F \cup (\omega_a \setminus \omega_C)$
  **else**
    $\omega_C \leftarrow \omega_C \cup \{k\}$
    $\omega_F \leftarrow \omega_F \cup ((S_h^{k,T} \setminus \omega_C) \cap \omega_v)$
  **end if**
**end while**

# ParCoarse$(\{S^{i,T}\}, \omega_s, \omega_C, \omega_F)$

Determine the list $[\sigma_k]_{k=1,\ldots,\mathrm{mc}_s}$
(= Communication groups).

**for all** $k = 1, \ldots, \mathrm{mc}_s$ **do**

$\omega_a \leftarrow \{i \in \omega_s \ : \ \sigma^{[i]} \equiv \sigma_k\}$

$\omega_v \leftarrow \{i \in \omega_s \ : \ \sigma^{[i]} \subset \sigma_k\}$

$\omega_g \leftarrow \{i \in \omega_s \ : \ \sigma^{[i]} \supset \sigma_k\}$

**if** $s == \textsc{Root}(\sigma_k)$ **then**

$\textsc{CoarseP}\ (\{S^{i,T}\}, \omega_a, \omega_a, \omega_v, \omega_C, \omega_F)$

$\omega_m \leftarrow \omega_C \cap \omega_a$

**end if**

**end for** $\longrightarrow$

**for all** $k = 1, \ldots, \mathrm{mc}_s$ **do**

$\omega_m \leftarrow \textsc{Broadcast}(\sigma_k, \omega_m)$

**if** $s! = \textsc{Root}(\sigma_k)$ **then**

$\textsc{CoarseP}\ (\{S^{i,T}\}, \omega_m, \omega_a, \omega_v, \omega_C, \omega_F)$

**end if**

**for all** $i \in \omega_a$ **do**

**if** $i \in \omega_F$ **then**

$S^{i,T} \leftarrow S^{i,T} \cap \ (\omega_a \cup \omega_g)$

**else**

$S^{i,T} \leftarrow S^{i,T} \cap \ (\omega_a \cup \omega_v)$

**end if**

**end for**

**end for**

# Parallel AMG: $\{\mathsf{K}^j, \mathfrak{P}^j\}_{j=1}^{\textsc{CoarseLevel}} \leftarrow \textsc{ParSetup}(\mathsf{K}_h, \omega_h, \ell)$

**if** $\|\omega\| > \textsc{CoarseGrid}$ **then**

$\mathsf{K}^\ell \leftarrow \mathsf{K}_h$

$\widetilde{\mathfrak{K}} \Leftarrow \left(\sum_{s=1}^{P} (A_s^\ell)^T \widetilde{\mathsf{K}}_s A_s^\ell\right)$ pattern

$\{S^{i,T}\}^\ell \leftarrow \textsc{GetStrong}(\widetilde{\mathfrak{K}})$

$(\{S^{i,T}\}^\ell, \omega_C, \omega_F) \Leftarrow \textsc{ParCoarse}(\{S^{i,T}\}^\ell, \omega^\ell)$

$\mathfrak{P}^\ell \leftarrow \textsc{Weights}(\left\{S^{i,T}\right\}^\ell, \widetilde{\mathfrak{K}}, \omega_C, \omega_F)$

$\mathsf{K}_H \leftarrow (\mathfrak{P}^\ell)^T \mathsf{K}^\ell \mathfrak{P}^\ell$

$\omega_H \leftarrow \omega_C$

$\textsc{ParSetup}(\mathsf{K}_H, \omega_H, \ell + 1)$

**else**

$\textsc{CoarseLevel} \leftarrow \ell$

**end if**