Geometric Multigrid

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Considered Problem Classes

$$\boxed{\begin{aligned}\text{Find } u: \quad Lu(x) &= f(x) \quad \forall x \in \Omega \\ lu(x) &= g(x) \quad \forall x \in \partial\Omega\end{aligned}}$$

variational $\Downarrow$ formulation

$$\boxed{\text{Find} \quad u \in \mathbb{V}: \quad a(u,v) = \langle F, v\rangle \quad \forall v \in \mathbb{V}}$$

FEM, FDM $\Downarrow$ FVM, FIT

$$\boxed{\text{Solve} \quad K_h \cdot \underline{u}_h = \underline{f}_h \qquad \underline{u}_h \in \mathbb{R}^{N_h}}$$

- (linear) 2^{nd} order problem.
  - Poisson equation (temperature)
  - Lamé equation (deformation)
  - Maxwell's equations (magnetic field)
- Matrix K_h is sparse, positive definite (symmetric, large dimension)
- non-linear and time-dependent problems.

The Principle of Multigrid

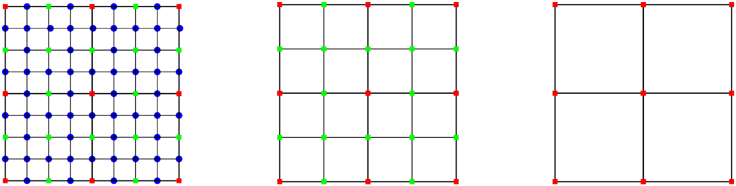
Solve the system of equations

$$K_h u_h = f_h$$

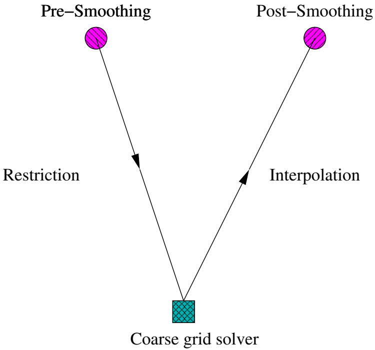
by a sequence of different discretizations and

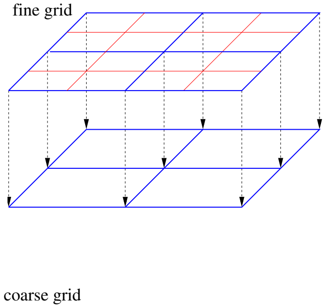
Combination of different accuracy levels.

$K_h u_h = f_h$ $K_{2h} u_{2h} = f_{2h}$ $K_{4h} u_{4h} = f_{4h}$



Two grid method





Gain: optimal iterative solver (memory, CPU-time).
Remark: A recursive application leads to a multigrid cycle.

(Classical) Multigrid V(ν_F,ν_B)-cycle: **MG(u_l, f_l,l)**

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if $l = COARSELEVEL$ then
    $\mathbf{u_l} = (\mathbf{K_l})^{-1}\mathbf{f_l}$ is solved by a direct solver
else
    Smooth $\nu_F$-times with $\mathbf{K_l u_l} = \mathbf{f_l}$
    Calculate defect $\mathbf{d_l} = \mathbf{f_l} - \mathbf{K_l u_l}$
    Restrict defect $\mathbf{d_{l+1}} = \mathbf{P_l^T d_l}$
    Set $\mathbf{w_{l+1}} \equiv \mathbf{0}$
    MG($w_{l+1}$,$d_{l+1}$,l+1)
    Interpolate correction $\mathbf{w_l} = \mathbf{P_l w_{l+1}}$
    Update solution $\mathbf{u_l} = \mathbf{u_l} + \mathbf{w_l}$
    Smooth $\nu_B$-time with $\mathbf{K_l u_l} = \mathbf{f_l}$
end if
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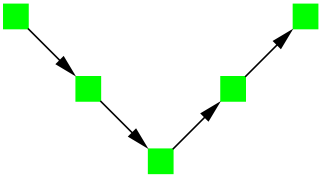
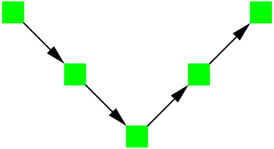


Strategies in Multigrid

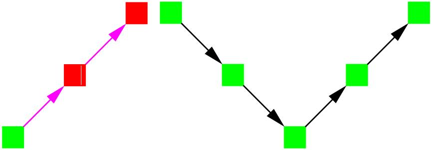
Each MG iteration requires on each discretization level l:

- Matrices K_l
- smoother (e.g., Jacobi iteration)
- interpolation P_l, (restriction $R_l = P_l^T$)
- **Classical MG**:
  Meshes/grids, Matrices, interpolation are given for all levels.

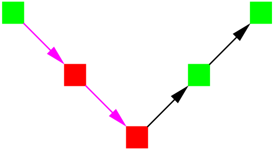


Strategies in Multigrid

- **Full MG - adaptive MG**:
  Coarse mesh, coarse matrix and interpolation are given.



- **Algebraic MG (AMG)**:
  Only the fine mesh matrix is known (+ fine mesh).



Triple Product and Approximation

Exact Interpolation with **C**oarse and **F**ine nodes:

$$K_h = \begin{pmatrix} K_C & K_{CF} \\ K_{FC} & K_F \end{pmatrix} = \begin{pmatrix} I_C & K_{CF}K_F^{-1} \\ 0 & I_F \end{pmatrix} \cdot \begin{pmatrix} \widehat{S}_C & 0 \\ 0 & K_F \end{pmatrix} \cdot \begin{pmatrix} I_C & 0 \\ K_F^{-1}K_{FC} & I_F \end{pmatrix}$$

with $\widehat{S}_C = K_C - K_{CF}K_F^{-1}K_{FC}$.

$$\Uparrow \qquad \Downarrow$$

$$\begin{pmatrix} \widehat{S}_C & 0 \\ 0 & K_F \end{pmatrix} = \begin{pmatrix} I_C & -K_{CF}K_F^{-1} \\ 0 & I_F \end{pmatrix} \cdot \begin{pmatrix} K_C & K_{CF} \\ K_{FC} & K_F \end{pmatrix} \cdot \begin{pmatrix} I_C & 0 \\ -K_F^{-1}K_{FC} & I_F \end{pmatrix}$$

Approximation: $B_{FC} \approx -K_F^{-1}K_{FC}$

$$\begin{pmatrix} S & (B_{CF}+K_{CF}K_F^{-1})K_F \\ K_F(B_{FC}+K_F^{-1}K_{FC}) & K_F \end{pmatrix} = \begin{pmatrix} I_C & B_{CF} \\ 0 & I_F \end{pmatrix} \cdot \begin{pmatrix} K_C & K_{CF} \\ K_{FC} & K_F \end{pmatrix} \cdot \begin{pmatrix} I_C & 0 \\ B_{FC} & I_F \end{pmatrix}$$

with $S = \widehat{S}_C + T_C$ and $T_C = (B_{CF} + K_{CF}K_F^{-1})K_F(B_{FC} + K_F^{-1}K_{FC})$.

Coarse grid operator and prolongation

Exact:

$$\begin{pmatrix} \widehat{S}_C & 0 \\ 0 & K_F \end{pmatrix} = \begin{pmatrix} I_C & -K_{CF}K_F^{-1} \\ 0 & I_F \end{pmatrix} \cdot \begin{pmatrix} K_C & K_{CF} \\ K_{FC} & K_F \end{pmatrix} \cdot \begin{pmatrix} I_C & 0 \\ -K_F^{-1}K_{FC} & I_F \end{pmatrix}$$

$$\widehat{P} := \begin{pmatrix} I_C \\ -K_F^{-1}K_{FC} \end{pmatrix}, \; \widehat{R} := \widehat{P}^T \qquad \Rightarrow \qquad \widehat{S}_C = \widehat{R} \cdot K_h \cdot \widehat{P}$$

Approximation: $B_{FC} \approx -K_F^{-1}K_{FC}$

$$\begin{pmatrix} S & {\scriptstyle (B_{CF}+K_{CF}K_F^{-1})K_F} \\ {\scriptstyle K_F(B_{FC}+K_F^{-1}K_{FC})} & K_F \end{pmatrix} = \begin{pmatrix} I_C & B_{CF} \\ 0 & I_F \end{pmatrix} \cdot \begin{pmatrix} K_C & K_{CF} \\ K_{FC} & K_F \end{pmatrix} \cdot \begin{pmatrix} I_C & 0 \\ B_{FC} & I_F \end{pmatrix}$$

$$P := \begin{pmatrix} I_C \\ B_{FC} \end{pmatrix}, \; R := P^T \Longrightarrow K_H := S = R \cdot K_h \cdot P \quad \text{(Galerkin approach)}$$

Angle between spaces

Exact: Solve transformed system with matrix $\widehat{K_h} := \begin{pmatrix} \widehat{S}_C & 0 \\ 0 & K_F \end{pmatrix}$

$$\mathbb{V}_h = \mathbb{V}_I \cup \mathbb{V}_F \quad \xrightarrow{\begin{pmatrix} I_C & 0 \\ -K_F^{-1} K_{FC} & I_F \end{pmatrix}} \quad \widehat{\mathbb{V}}_h = \mathbb{V}_F \cup \widehat{\mathbb{V}}_C \quad , \; \measuredangle(\mathbb{V}_F, \widehat{\mathbb{V}}_C) = \pi/2$$

Approximation: Solve transformed system with matrix $C_h := \begin{pmatrix} S & 0 \\ 0 & K_F \end{pmatrix}$

$$\mathbb{V}_h = \mathbb{V}_I \cup \mathbb{V}_F \quad \xrightarrow{\begin{pmatrix} I_C & 0 \\ B_{FC} & I_F \end{pmatrix}} \quad \widehat{\mathbb{V}}_h = \mathbb{V}_F \cup \widetilde{\mathbb{V}}_C \quad , \; 0 \le \measuredangle(\mathbb{V}_F, \widetilde{\mathbb{V}}_C) \le \pi/2$$

Angle and condition number

With

- $\widehat{S}_C = K_C - K_{CF} K_F^{-1} K_{FC}$,
- $T_C = (K_{CF} K_F^{-1} + {B_{FC}}^T) K_F (K_F^{-1} K_{FC} + B_{FC})$
- $\mu = \varrho({\widehat{S}_C}^{-1} T_C)$

we get

$$\boxed{\cos\measuredangle(\mathbb{V}_F, \widetilde{\mathbb{V}}_C) = \sup_{\substack{w \in \widetilde{\mathbb{V}}_C \setminus \{0\} \\ v \in \mathbb{V}_I \setminus \{0\}}} \frac{|a(w,v)|}{\sqrt{a(w,w)}\sqrt{a(v,v)}} = \sqrt{\frac{\mu}{1+\mu}}}$$

$$\boxed{\kappa({C_h}^{-1} \widehat{K}_h) \;=\; \left(\sqrt{\mu} + \sqrt{1+\mu}\right)^2}$$

Main ingredients of AMG

(i) coarsening :

$$\omega^h = \omega_C^h \cup \omega_F^h \quad .$$

(ii) interpolation weights :

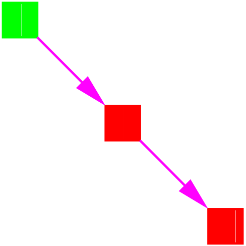
$$P = \{\alpha_{ij}\}_{i\in\omega^h,\, j\in\omega_C^h} \; : \mathbb{R}^H \mapsto \mathbb{R}^h \quad .$$

(iii) coarse mesh matrix:

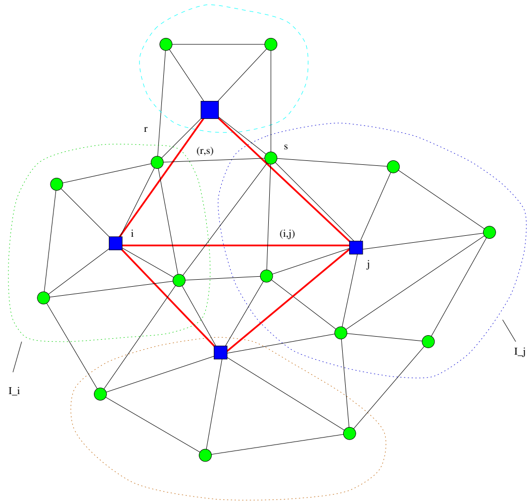
$$K^H = P^T \cdot K \cdot P$$

(iv) (i)-(iii) have to be applied recursively.

(v) apply the standard MG-procedure.



Coarsening



Prolongation $P : \mathbb{R}^{N_H}(\omega_C) \mapsto \mathbb{R}^{N_h}(\omega_C \cup \omega_F)$

Exakt factorization of $K_h = \begin{pmatrix} K_C & K_{CF} \\ K_{FC} & K_F \end{pmatrix}$ and inversion of the factors:

$$K_h^{-1} = \underbrace{\begin{pmatrix} I_C & 0 \\ -K_F^{-1}K_{FC} & I_F \end{pmatrix}}_{\approx P} \cdot \begin{pmatrix} S_C^{-1} & 0 \\ 0 & K_F^{-1} \end{pmatrix} \cdot \begin{pmatrix} I_C & -K_{CF}K_F^{-1} \\ 0 & I_F \end{pmatrix}$$

The Twogrid method is

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The Twogrid method is an approximation of the factors:

$$C_h^{-1} = P \cdot \begin{pmatrix} (P^T K_h P)^{-1} & 0 \\ 0 & (I_F - M_F)K_F^{-1} \end{pmatrix} \cdot P^T$$

The prolongation is

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The prolongation is the local approximation of the global discrete harmonic extension from the coarse space into the fine space:

$$\boxed{\underline{u}_h = P \cdot \underline{u}_H \quad \approx \quad \begin{array}{lrll} \text{Find } u_h \in \mathbb{V}_h : & L_h u_h(x) = & 0 & \forall x \in \omega^h \setminus \omega^H \\ & u_h(x) = & u_H(x) & \forall x \in \omega^H \end{array}}$$