Mathematische Strukturen in Sage

Sehr viele mathematische Strukturen sind bereits in Sage imlementiert. Wir sehen uns hier exemplarisch drei Strukturen an: Gruppen, am Beispiel der symmetrischen Gruppe S_n, Graphen und elliptische Kurven.

Symmetrische Gruppe

In [78]:
```
S5=SymmetricGroup(5)
```

In [79]:
```
S5
```

Out[79]:
```
Symmetric group of order 5! as a permutation group
```

Angabe von Permutationen $\sigma\in S_n$ in Listenschreibweise: $[\sigma(1), \sigma(2),\ldots,\sigma(n)]$

In [80]:
```
p1=S5([2,1,4,5,3])
p1
```

Out[80]:
```
(1,2)(3,4,5)
```

Intern werden Permutationen in **Zyklenschreibweise** umgewandelt. Ein **Zyklus** $ \sigma=(a_1,a_2,\ldots,a_3)$ ist die Permutation mit $\sigma(a_1)=a_2,\ \sigma(a_2)=a_3,\ldots, \sigma(a_{k-1})=a_k$ und $\sigma(a_k)=a_1$. Jede Permutation kann als Proeukt disjunkter Zyklen geschrieben werden.

In [81]:
```
S5([2,4,5,6,1,3])
```

```
---------------------------------------------------------------------------
-
AssertionError                            Traceback (most recent call las
t)
Cell In[81], line 1
----> 1 S5([Integer(2),Integer(4),Integer(5),Integer(6),Integer(1),Integer
(3)])

File /opt/sagemath/sage-10.7/src/sage/structure/parent.pyx:900, in sage.st
ructure.parent.Parent.__call__()
    898 if mor is not None:
    899     if no_extra_args:
--> 900         return mor._call_(x)
    901     else:
    902         return mor._call_with_args(x, args, kwds)

File /opt/sagemath/sage-10.7/src/sage/structure/coerce_maps.pyx:164, in sa
ge.structure.coerce_maps.DefaultConvertMap_unique._call_()
    162             print(type(C), C)
    163             print(type(C._element_constructor), C._element_constru
ctor)
--> 164         raise
    165
    166 cpdef Element _call_with_args(self, x, args=(), kwds={}):

File /opt/sagemath/sage-10.7/src/sage/structure/coerce_maps.pyx:159, in sa
ge.structure.coerce_maps.DefaultConvertMap_unique._call_()
    157 cdef Parent C = self._codomain
    158 try:
--> 159     return C._element_constructor(x)
    160 except Exception:
    161     if print_warnings:

File /opt/sagemath/sage-10.7/src/sage/groups/perm_gps/permgroup.py:900, in
PermutationGroup_generic._element_constructor_(self, x, check)
    896     if compatible_domains and (isinstance(self, SymmetricGroup)
    897                                or x.gap() in self.gap()):
    898         return self.element_class(x, self, check=False)
--> 900 return self.element_class(x, self, check=check)

File /opt/sagemath/sage-10.7/src/sage/groups/perm_gps/permgroup_element.py
x:480, in sage.groups.perm_gps.permgroup_element.PermutationGroupElement._
_init__()
    478         self._set_list_cycles(g, convert)
    479     else:
--> 480         self._set_list_images(g, convert)
    481 elif isinstance(g, str):
    482     self._set_string(g)

File /opt/sagemath/sage-10.7/src/sage/groups/perm_gps/permgroup_element.py
x:568, in sage.groups.perm_gps.permgroup_element.PermutationGroupElement._
set_list_images()
    566 """
    567 cdef int i, j, vn = len(v)
--> 568 assert vn <= self.n
    569 if convert:
    570     convert_dict = self._parent._domain_to_gap

AssertionError:
```

In [82]:
```
S5([1,1,2,3,4])
```

```
---------------------------------------------------------------------------
ValueError                                Traceback (most recent call last)
Cell In[82], line 1
----> 1 S5([Integer(1),Integer(1),Integer(2),Integer(3),Integer(4)])

File /opt/sagemath/sage-10.7/src/sage/structure/parent.pyx:900, in sage.structure.parent.Parent.__call__()
    898 if mor is not None:
    899     if no_extra_args:
--> 900         return mor._call_(x)
    901     else:
    902         return mor._call_with_args(x, args, kwds)

File /opt/sagemath/sage-10.7/src/sage/structure/coerce_maps.pyx:164, in sage.structure.coerce_maps.DefaultConvertMap_unique._call_()
    162             print(type(C), C)
    163             print(type(C._element_constructor), C._element_constructor)
--> 164         raise
    165
    166 cpdef Element _call_with_args(self, x, args=(), kwds={}):

File /opt/sagemath/sage-10.7/src/sage/structure/coerce_maps.pyx:159, in sage.structure.coerce_maps.DefaultConvertMap_unique._call_()
    157 cdef Parent C = self._codomain
    158 try:
--> 159     return C._element_constructor(x)
    160 except Exception:
    161     if print_warnings:

File /opt/sagemath/sage-10.7/src/sage/groups/perm_gps/permgroup.py:900, in PermutationGroup_generic._element_constructor_(self, x, check)
    896     if compatible_domains and (isinstance(self, SymmetricGroup)
    897                                or x.gap() in self.gap()):
    898         return self.element_class(x, self, check=False)
--> 900 return self.element_class(x, self, check=check)

File /opt/sagemath/sage-10.7/src/sage/groups/perm_gps/permgroup_element.pyx:518, in sage.groups.perm_gps.permgroup_element.PermutationGroupElement.__init__()
    516 # a valid permutation (else segfaults, infinite loops may occur).
    517 if not is_valid_permutation(self.perm, self.n):
--> 518     raise ValueError("invalid data to initialize a permutation")
    519
    520 # This is more expensive

ValueError: invalid data to initialize a permutation
```

Auch die Eingabe kann in Zyklenschreibweise, als Liste der Zyklen, erfolgen.

In [83]:
```
p2=S5([(2,3),(1,4)])
p2
```

Out[83]: (1,4)(2,3)

Achtung: Multiplikation von Permutationen erfolgt in Sage mit der etwas unüblichen Konvention(fg)(x):=g(f(x)), anstatt des üblicheren (fg)(x):=f(g(x))!

In [89]:
```
p1,p2
```

Out[89]:
```
((1,2)(3,4,5), (1,4)(2,3))
```

In [90]:
```
p1*p2
```

Out[90]:
```
(1,3)(2,4,5)
```

In [91]:
```
p2*p1
```

Out[91]:
```
(1,5,3)(2,4)
```

In [92]:
```
p1^-1
```

Out[92]:
```
(1,2)(3,5,4)
```

In [93]:
```
p1
```

Out[93]:
```
(1,2)(3,4,5)
```

Analog zu Vektorräumen: die von Elementen erzeugte Untergruppe ist die kleinste Untergruppe, die alle gegebenen Elemente enthält.

In [94]:
```
G=S5.subgroup([p1])
G
```

Out[94]:
```
Subgroup generated by [(1,2)(3,4,5)] of (Symmetric group of order 5! as
a permutation group)
```

In [95]:
```
list(G)
```

Out[95]:
```
[(), (1,2)(3,4,5), (3,5,4), (1,2), (3,4,5), (1,2)(3,5,4)]
```

In [96]:
```
len(G)
```

Out[96]:
```
6
```

Verknüpfungstabelle

In [97]:
```
G.cayley_table()
```

Out[97]:
```
*  a b c d e f
 +------------
a| a b c d e f
b| b c a e f d
c| c a b f d e
d| d e f a b c
e| e f d b c a
f| f d e c a b
```

In [98]:
```
_.translation()
```

Out[98]:
```
{'a': (),
 'b': (3,4,5),
 'c': (3,5,4),
 'd': (1,2),
 'e': (1,2)(3,4,5),
 'f': (1,2)(3,5,4)}
```

In [99]:
```
G.cayley_table(names='elements')
```

Out[99]:
```
           *           ()      (3,4,5)      (3,5,4)        (1,2) (1,2)
(3,4,5) (1,2)(3,5,4)

+-------------------------------------------------------------------------
-------
          ()|           ()      (3,4,5)      (3,5,4)        (1,2) (1,2)
(3,4,5) (1,2)(3,5,4)
     (3,4,5)|      (3,4,5)      (3,5,4)           () (1,2)(3,4,5) (1,2)
(3,5,4)        (1,2)
     (3,5,4)|      (3,5,4)           ()      (3,4,5) (1,2)(3,5,4)
(1,2) (1,2)(3,4,5)
       (1,2)|        (1,2) (1,2)(3,4,5) (1,2)(3,5,4)           ()
(3,4,5)      (3,5,4)
(1,2)(3,4,5)| (1,2)(3,4,5) (1,2)(3,5,4)        (1,2)      (3,4,5)
(3,5,4)           ()
(1,2)(3,5,4)| (1,2)(3,5,4)        (1,2) (1,2)(3,4,5)      (3,5,4)
()      (3,4,5)
```

In [100…
```
G.is_abelian()
```

Out[100…
```
True
```

In [101…
```
S5.is_abelian()
```

Out[101…
```
False
```

Liste aller Untergruppen

In [102…
```
S5ss=S5.subgroups()
len(S5ss)
```

Out[102…
```
156
```

In [103…
```
S5ss
```

Out[103… [Subgroup generated by [()] of (Symmetric group of order 5! as a permuta
tion group),
 Subgroup generated by [(4,5)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(3,4)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(3,5)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(2,3)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(2,4)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(2,5)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(1,2)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(1,3)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(1,4)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(1,5)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(2,3)(4,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(2,4)(3,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(2,5)(3,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,2)(4,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,2)(3,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,2)(3,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,3)(4,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,3)(2,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,3)(2,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,4)(3,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,4)(2,3)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,4)(2,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,5)(3,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,5)(2,3)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,5)(2,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(3,4,5)] of (Symmetric group of order 5! as a pe
rmutation group),
 Subgroup generated by [(2,3,4)] of (Symmetric group of order 5! as a pe
rmutation group),
 Subgroup generated by [(2,3,5)] of (Symmetric group of order 5! as a pe
rmutation group),
 Subgroup generated by [(2,4,5)] of (Symmetric group of order 5! as a pe
rmutation group),

Subgroup generated by [(1,2,3)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(1,2,4)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(1,2,5)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(1,3,4)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(1,3,5)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(1,4,5)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(2,5)(3,4), (2,4)(3,5)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,5)(2,4), (1,4)(2,5)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,4)(2,3), (1,3)(2,4)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,2)(3,5), (1,3)(2,5)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,3)(4,5), (1,4)(3,5)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(4,5), (2,3)(4,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(4,5), (1,2)(4,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(4,5), (1,3)(4,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(3,4), (2,5)(3,4)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,2), (1,2)(3,4)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(3,4), (1,5)(3,4)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(2,4), (2,4)(3,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,2), (1,2)(3,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(3,5), (1,4)(3,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(2,3), (1,4)(2,3)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,5), (1,5)(2,3)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,3), (1,3)(2,4)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,5), (1,5)(2,4)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,3), (1,3)(2,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(2,5), (1,4)(2,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(2,4,3,5)] of (Symmetric group of order 5! as a
permutation group),
Subgroup generated by [(2,3,4,5)] of (Symmetric group of order 5! as a
permutation group),
Subgroup generated by [(2,4,5,3)] of (Symmetric group of order 5! as a
permutation group),
Subgroup generated by [(1,4,2,5)] of (Symmetric group of order 5! as a
permutation group),

Subgroup generated by [(1,4,2,3)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,5,2,3)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,2,3,4)] of (Symmetric group of order 5! as a permutation group),
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Subgroup generated by [(1,2,5,3)] of (Symmetric group of order 5! as a permutation group),
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Subgroup generated by [(1,3,4,5)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,4,5,3)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,2,3,4,5)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,3,4,2,5)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,5,2,3,4)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,2,4,3,5)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,3,4,5,2)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,3,2,4,5)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,2,3), (4,5)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,2,5), (3,4)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(3,4,5), (1,2)] of (Symmetric group of order 5! as a permutation group),
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Subgroup generated by [(1,4,5), (2,3)] of (Symmetric group of order 5! as a permutation group),
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Subgroup generated by [(1,3,4), (2,5)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(3,4,5), (4,5)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(2,4,5), (4,5)] of (Symmetric group of order 5! as a permutation group),
Subgroup generated by [(1,4,5), (1,5)] of (Symmetric group of order 5! as a permutation group),

Subgroup generated by [(2,3,4), (3,4)] of (Symmetric group of order 5!
as a permutation group),
Subgroup generated by [(1,3,4), (3,4)] of (Symmetric group of order 5!
as a permutation group),
Subgroup generated by [(2,3,5), (2,5)] of (Symmetric group of order 5!
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Subgroup generated by [(1,3,5), (1,5)] of (Symmetric group of order 5!
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Subgroup generated by [(1,2,3), (2,3)] of (Symmetric group of order 5!
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Subgroup generated by [(1,2,4), (1,2)] of (Symmetric group of order 5!
as a permutation group),
Subgroup generated by [(1,2,5), (1,2)] of (Symmetric group of order 5!
as a permutation group),
Subgroup generated by [(3,4,5), (1,2)(4,5)] of (Symmetric group of orde
r 5! as a permutation group),
Subgroup generated by [(1,2,3), (2,3)(4,5)] of (Symmetric group of orde
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Subgroup generated by [(1,4,5), (1,5)(2,3)] of (Symmetric group of orde
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Subgroup generated by [(2,3,4), (1,5)(3,4)] of (Symmetric group of orde
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Subgroup generated by [(2,3,5), (1,4)(2,5)] of (Symmetric group of orde
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Subgroup generated by [(2,4,5), (1,3)(4,5)] of (Symmetric group of orde
r 5! as a permutation group),
Subgroup generated by [(1,2,4), (1,2)(3,5)] of (Symmetric group of orde
r 5! as a permutation group),
Subgroup generated by [(1,3,5), (1,5)(2,4)] of (Symmetric group of orde
r 5! as a permutation group),
Subgroup generated by [(1,2,5), (1,2)(3,4)] of (Symmetric group of orde
r 5! as a permutation group),
Subgroup generated by [(1,3,4), (2,5)(3,4)] of (Symmetric group of orde
r 5! as a permutation group),
Subgroup generated by [(4,5), (2,4)(3,5), (2,5)(3,4)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(1,2), (1,4)(2,5), (1,5)(2,4)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(4,5), (1,4)(3,5), (1,5)(3,4)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(2,5), (2,4)(3,5), (2,3)(4,5)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(3,4), (1,3)(2,4), (1,4)(2,3)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(1,5), (1,4)(3,5), (1,3)(4,5)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(3,5), (2,3)(4,5), (2,5)(3,4)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(1,2), (1,3)(2,5), (1,5)(2,3)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(1,4), (1,5)(3,4), (1,3)(4,5)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(2,3), (1,3)(2,4), (1,2)(3,4)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(2,3), (1,3)(2,5), (1,2)(3,5)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(1,5), (1,4)(2,5), (1,2)(4,5)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(2,4), (1,2)(3,4), (1,4)(2,3)] of (Symmetric gro
up of order 5! as a permutation group),

Subgroup generated by [(2,5), (1,5)(2,4), (1,2)(4,5)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(1,3), (1,5)(2,3), (1,2)(3,5)] of (Symmetric gro
up of order 5! as a permutation group),
Subgroup generated by [(1,5,2,3,4), (2,3)(4,5)] of (Symmetric group of
order 5! as a permutation group),
Subgroup generated by [(1,3,4,5,2), (1,5)(3,4)] of (Symmetric group of
order 5! as a permutation group),
Subgroup generated by [(1,3,2,4,5), (1,2)(4,5)] of (Symmetric group of
order 5! as a permutation group),
Subgroup generated by [(1,3,4,2,5), (1,3)(4,5)] of (Symmetric group of
order 5! as a permutation group),
Subgroup generated by [(1,2,4,3,5), (1,5)(2,3)] of (Symmetric group of
order 5! as a permutation group),
Subgroup generated by [(1,2,3,4,5), (2,5)(3,4)] of (Symmetric group of
order 5! as a permutation group),
Subgroup generated by [(2,5)(3,4), (2,4)(3,5), (2,3,4)] of (Symmetric g
roup of order 5! as a permutation group),
Subgroup generated by [(1,3)(4,5), (1,4)(3,5), (3,4,5)] of (Symmetric g
roup of order 5! as a permutation group),
Subgroup generated by [(1,4)(2,3), (1,3)(2,4), (1,2,3)] of (Symmetric g
roup of order 5! as a permutation group),
Subgroup generated by [(1,2)(3,5), (1,3)(2,5), (1,2,5)] of (Symmetric g
roup of order 5! as a permutation group),
Subgroup generated by [(1,5)(2,4), (1,4)(2,5), (1,4,5)] of (Symmetric g
roup of order 5! as a permutation group),
Subgroup generated by [(3,4,5), (4,5), (1,2)(4,5)] of (Symmetric group
of order 5! as a permutation group),
Subgroup generated by [(1,2,3), (2,3), (2,3)(4,5)] of (Symmetric group
of order 5! as a permutation group),
Subgroup generated by [(1,4,5), (1,5), (1,5)(2,3)] of (Symmetric group
of order 5! as a permutation group),
Subgroup generated by [(2,4,5), (4,5), (1,3)(4,5)] of (Symmetric group
of order 5! as a permutation group),
Subgroup generated by [(2,3,4), (3,4), (1,5)(3,4)] of (Symmetric group
of order 5! as a permutation group),
Subgroup generated by [(1,2,5), (1,2), (1,2)(3,4)] of (Symmetric group
of order 5! as a permutation group),
Subgroup generated by [(1,3,4), (3,4), (2,5)(3,4)] of (Symmetric group
of order 5! as a permutation group),
Subgroup generated by [(2,3,5), (2,5), (1,4)(2,5)] of (Symmetric group
of order 5! as a permutation group),
Subgroup generated by [(1,2,4), (1,2), (1,2)(3,5)] of (Symmetric group
of order 5! as a permutation group),
Subgroup generated by [(1,3,5), (1,5), (1,5)(2,4)] of (Symmetric group
of order 5! as a permutation group),
Subgroup generated by [(1,5,2,3,4), (2,4,3,5)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,3,4,5,2), (1,3,5,4)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,3,2,4,5), (1,5,2,4)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,3,4,2,5), (1,4,3,5)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,2,4,3,5), (1,3,5,2)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,2,3,4,5), (2,3,5,4)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(2,3,4), (4,5), (2,5)(3,4)] of (Symmetric group
of order 5! as a permutation group),

```
 Subgroup generated by [(3,4,5), (1,5), (1,3)(4,5)] of (Symmetric group
of order 5! as a permutation group),
 Subgroup generated by [(1,4,5), (1,2), (1,5)(2,4)] of (Symmetric group
of order 5! as a permutation group),
 Subgroup generated by [(1,2,3), (3,4), (1,4)(2,3)] of (Symmetric group
of order 5! as a permutation group),
 Subgroup generated by [(1,2,5), (2,3), (1,2)(3,5)] of (Symmetric group
of order 5! as a permutation group),
 Subgroup generated by [(1,5,2,3,4), (2,3,4)] of (Symmetric group of ord
er 5! as a permutation group),
 Subgroup generated by [(1,5,2,3,4), (2,5)] of (Symmetric group of order
5! as a permutation group)]
```

Wir filtern diese Liste nach Abelschen Untergruppen.

In [104…
```
S5ab=[G for G in S5ss if G.is_abelian()]
```

In [105…
```
len(S5ab)
```

Out[105… 87

In [106…
```
S5ab
```

Out[106… [Subgroup generated by [()] of (Symmetric group of order 5! as a permuta
tion group),
 Subgroup generated by [(4,5)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(3,4)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(3,5)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(2,3)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(2,4)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(2,5)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(1,2)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(1,3)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(1,4)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(1,5)] of (Symmetric group of order 5! as a perm
utation group),
 Subgroup generated by [(2,3)(4,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(2,4)(3,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(2,5)(3,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,2)(4,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,2)(3,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,2)(3,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,3)(4,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,3)(2,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,3)(2,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,4)(3,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,4)(2,3)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,4)(2,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,5)(3,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,5)(2,3)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,5)(2,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(3,4,5)] of (Symmetric group of order 5! as a pe
rmutation group),
 Subgroup generated by [(2,3,4)] of (Symmetric group of order 5! as a pe
rmutation group),
 Subgroup generated by [(2,3,5)] of (Symmetric group of order 5! as a pe
rmutation group),
 Subgroup generated by [(2,4,5)] of (Symmetric group of order 5! as a pe
rmutation group),

Subgroup generated by [(1,2,3)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(1,2,4)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(1,2,5)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(1,3,4)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(1,3,5)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(1,4,5)] of (Symmetric group of order 5! as a pe
rmutation group),
Subgroup generated by [(2,5)(3,4), (2,4)(3,5)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,5)(2,4), (1,4)(2,5)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,4)(2,3), (1,3)(2,4)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,2)(3,5), (1,3)(2,5)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(1,3)(4,5), (1,4)(3,5)] of (Symmetric group of o
rder 5! as a permutation group),
Subgroup generated by [(4,5), (2,3)(4,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(4,5), (1,2)(4,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(4,5), (1,3)(4,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(3,4), (2,5)(3,4)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,2), (1,2)(3,4)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(3,4), (1,5)(3,4)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(2,4), (2,4)(3,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,2), (1,2)(3,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(3,5), (1,4)(3,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(2,3), (1,4)(2,3)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,5), (1,5)(2,3)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,3), (1,3)(2,4)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,5), (1,5)(2,4)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(1,3), (1,3)(2,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(2,5), (1,4)(2,5)] of (Symmetric group of order
5! as a permutation group),
Subgroup generated by [(2,4,3,5)] of (Symmetric group of order 5! as a
permutation group),
Subgroup generated by [(2,3,4,5)] of (Symmetric group of order 5! as a
permutation group),
Subgroup generated by [(2,4,5,3)] of (Symmetric group of order 5! as a
permutation group),
Subgroup generated by [(1,4,2,5)] of (Symmetric group of order 5! as a
permutation group),

```
 Subgroup generated by [(1,4,2,3)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,5,2,3)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,2,3,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,5,3,2)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,3,4,2)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,5,4,2)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,2,5,3)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,2,5,4)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,4,3,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,3,4,5)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,4,5,3)] of (Symmetric group of order 5! as a
permutation group),
 Subgroup generated by [(1,2,3,4,5)] of (Symmetric group of order 5! as
a permutation group),
 Subgroup generated by [(1,3,4,2,5)] of (Symmetric group of order 5! as
a permutation group),
 Subgroup generated by [(1,5,2,3,4)] of (Symmetric group of order 5! as
a permutation group),
 Subgroup generated by [(1,2,4,3,5)] of (Symmetric group of order 5! as
a permutation group),
 Subgroup generated by [(1,3,4,5,2)] of (Symmetric group of order 5! as
a permutation group),
 Subgroup generated by [(1,3,2,4,5)] of (Symmetric group of order 5! as
a permutation group),
 Subgroup generated by [(1,2,3), (4,5)] of (Symmetric group of order 5!
as a permutation group),
 Subgroup generated by [(1,2,5), (3,4)] of (Symmetric group of order 5!
as a permutation group),
 Subgroup generated by [(3,4,5), (1,2)] of (Symmetric group of order 5!
as a permutation group),
 Subgroup generated by [(1,2,4), (3,5)] of (Symmetric group of order 5!
as a permutation group),
 Subgroup generated by [(1,4,5), (2,3)] of (Symmetric group of order 5!
as a permutation group),
 Subgroup generated by [(2,3,4), (1,5)] of (Symmetric group of order 5!
as a permutation group),
 Subgroup generated by [(2,3,5), (1,4)] of (Symmetric group of order 5!
as a permutation group),
 Subgroup generated by [(1,3,5), (2,4)] of (Symmetric group of order 5!
as a permutation group),
 Subgroup generated by [(2,4,5), (1,3)] of (Symmetric group of order 5!
as a permutation group),
 Subgroup generated by [(1,3,4), (2,5)] of (Symmetric group of order 5!
as a permutation group)]
```

Graphen

Ein Graph besteht aus einer Menge von Knoten und einer Menge von Kanten, die

zwischen je zwei Knoten verlaufen.

In [107…
```
g=Graph() # leerer Graph
```

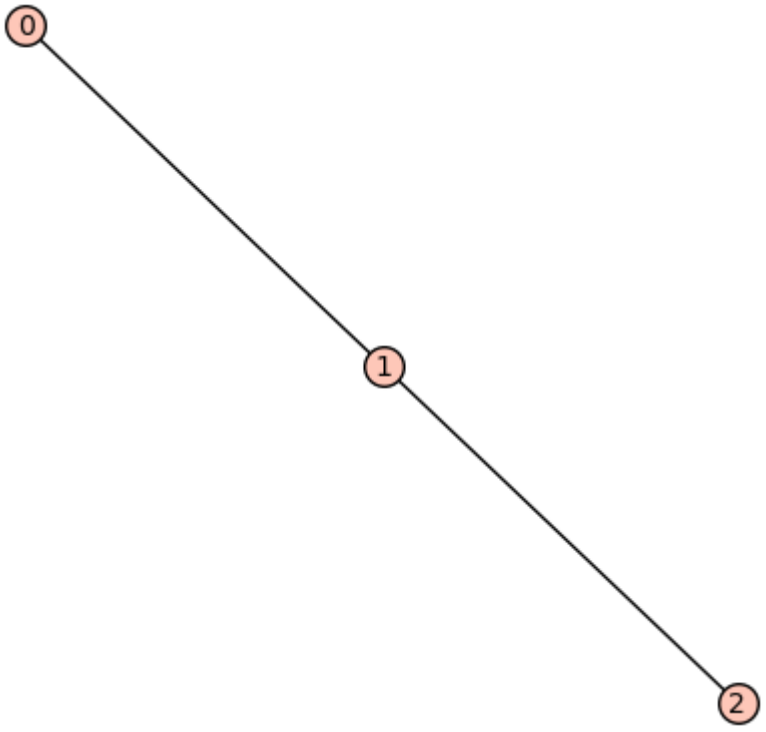
In [108…
```
g.add_vertex(0)
g
```

Out[108…

Graph on 1 vertex

In [109…
```
g.show()
```

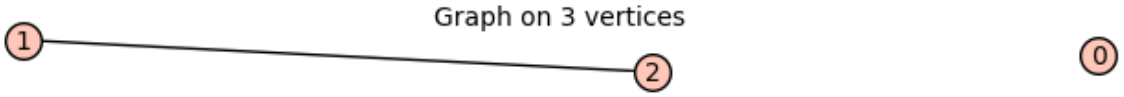
In [110…
```
g.add_edge(1,2)
g.add_edge(1,0)
g.show()
```



In [111…
```
g.delete_edge(1,0)
```

In [112…
```
g
```

Out[112…



Als Bezeichnungen der Knoten können fast beliebige Objekte genommen werden.

Wichtig ist nur, dass die Objekte **immutable**, also unveränderbar sind.

Matrizen sind z.B. standardmäßig nicht immutable, da ihre Einträge verändert werden können.

In [113…
```
a=matrix(2,2,[1,x,2,3])
a
```

Out[113…
```
[1 x]
[2 3]
```

In [114…
```
g.add_edge(1,a)
```

```
---------------------------------------------------------------------------
TypeError                                 Traceback (most recent call last
t)
Cell In[114], line 1
----> 1 g.add_edge(Integer(1),a)

File /opt/sagemath/sage-10.7/src/sage/graphs/generic_graph.py:12629, in Ge
nericGraph.add_edge(self, u, v, label)
  12626         except Exception:
  12627             pass
> 12629 self._backend.add_edge(u, v, label, self._directed)

File /opt/sagemath/sage-10.7/src/sage/graphs/base/c_graph.pyx:2334, in sag
e.graphs.base.c_graph.CGraphBackend.add_edge()
   2332         self.add_edge(u, v, l, directed)
   2333
-> 2334 cpdef add_edge(self, object u, object v, object l, bint directed):
   2335     """
   2336     Add the edge ``(u,v)`` to ``self``.

File /opt/sagemath/sage-10.7/src/sage/graphs/base/c_graph.pyx:2412, in sag
e.graphs.base.c_graph.CGraphBackend.add_edge()
   2410
   2411         cdef int u_int = self.check_labelled_vertex(u, False)
-> 2412         cdef int v_int = self.check_labelled_vertex(v, False)
   2413
   2414         cdef CGraph cg = self.cg()

File /opt/sagemath/sage-10.7/src/sage/graphs/base/c_graph.pyx:1643, in sag
e.graphs.base.c_graph.CGraphBackend.check_labelled_vertex()
   1641 cdef CGraph G = self.cg()
   1642
-> 1643 cdef int u_int = self.get_vertex(u)
   1644 if u_int != -1:
   1645     if not bitset_in(G.active_vertices, u_int):

File /opt/sagemath/sage-10.7/src/sage/graphs/base/c_graph.pyx:1600, in sag
e.graphs.base.c_graph.CGraphBackend.get_vertex()
   1598 cdef CGraph G = self.cg()
   1599 cdef long u_long
-> 1600 if u in vertex_ints:
   1601     return vertex_ints[u]
   1602 try:

File /opt/sagemath/sage-10.7/src/sage/matrix/matrix0.pyx:6123, in sage.mat
rix.matrix0.Matrix.__hash__()
   6121 """
   6122 if not self._is_immutable:
-> 6123     raise TypeError("mutable matrices are unhashable")
   6124 if self.hash != -1:
   6125     return self.hash

TypeError: mutable matrices are unhashable
```

Wir machen die Matrix immutable.

In [115…

```
a.set_immutable()
```

In [116…
```
a[0,0]=2
```

```
---------------------------------------------------------------------------
ValueError                                Traceback (most recent call last)
Cell In[116], line 1
----> 1 a[Integer(0),Integer(0)]=Integer(2)

File /opt/sagemath/sage-10.7/src/sage/matrix/matrix0.pyx:1460, in sage.matrix.matrix0.Matrix.__setitem__()
   1458 # If the matrix is immutable, check_mutability will raise an
   1459 # exception.
-> 1460 self.check_mutability()
   1461
   1462 if type(key) is tuple:

File /opt/sagemath/sage-10.7/src/sage/matrix/matrix0.pyx:422, in sage.matrix.matrix0.Matrix.check_mutability()
    420 """
    421 if self._is_immutable:
--> 422     raise ValueError("matrix is immutable; please change a copy instead (i.e., use copy(M) to change a copy of M).")
    423 else:
    424     self._cache = None

ValueError: matrix is immutable; please change a copy instead (i.e., use copy(M) to change a copy of M).
```

In [117…
```
B=a
```

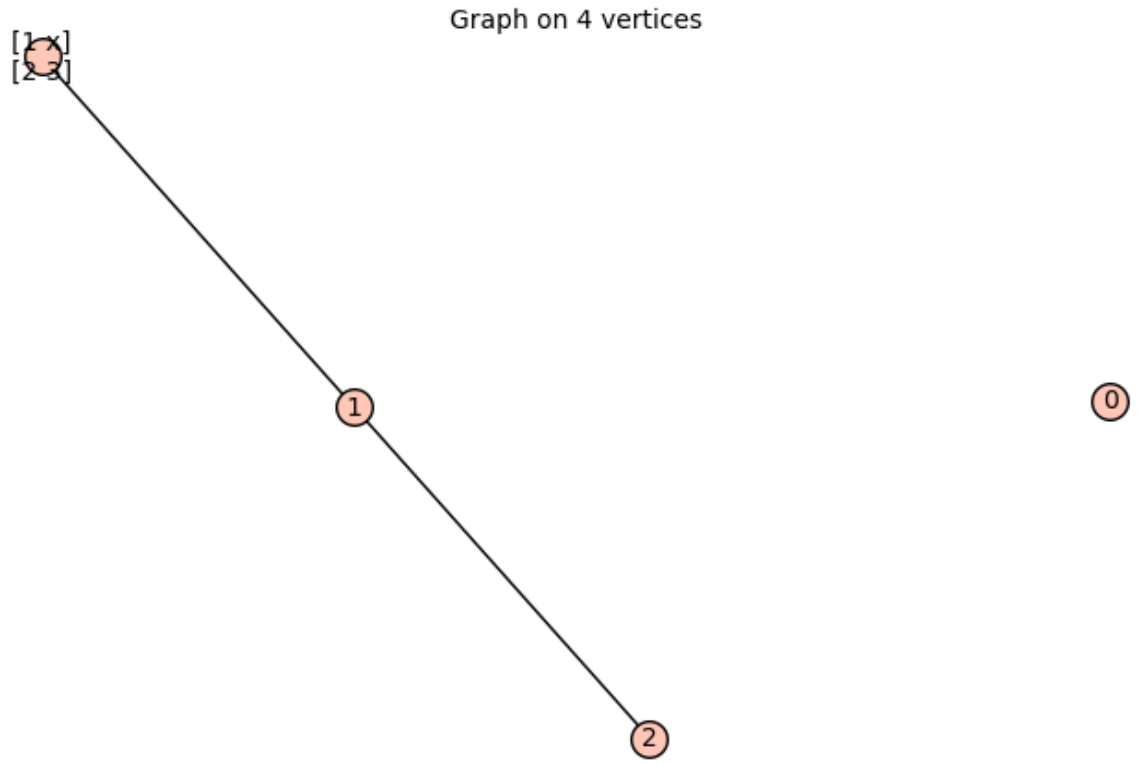
In [118…
```
B[0,0]=2
```

```
---------------------------------------------------------------------------
ValueError                                Traceback (most recent call last)
Cell In[118], line 1
----> 1 B[Integer(0),Integer(0)]=Integer(2)

File /opt/sagemath/sage-10.7/src/sage/matrix/matrix0.pyx:1460, in sage.matrix.matrix0.Matrix.__setitem__()
   1458 # If the matrix is immutable, check_mutability will raise an
   1459 # exception.
-> 1460 self.check_mutability()
   1461
   1462 if type(key) is tuple:

File /opt/sagemath/sage-10.7/src/sage/matrix/matrix0.pyx:422, in sage.matrix.matrix0.Matrix.check_mutability()
    420 """
    421 if self._is_immutable:
--> 422     raise ValueError("matrix is immutable; please change a copy instead (i.e., use copy(M) to change a copy of M).")
    423 else:
    424     self._cache = None

ValueError: matrix is immutable; please change a copy instead (i.e., use copy(M) to change a copy of M).
```

In [119…
```
g.add_edge(1,a)
```

In [120…
```
g
```

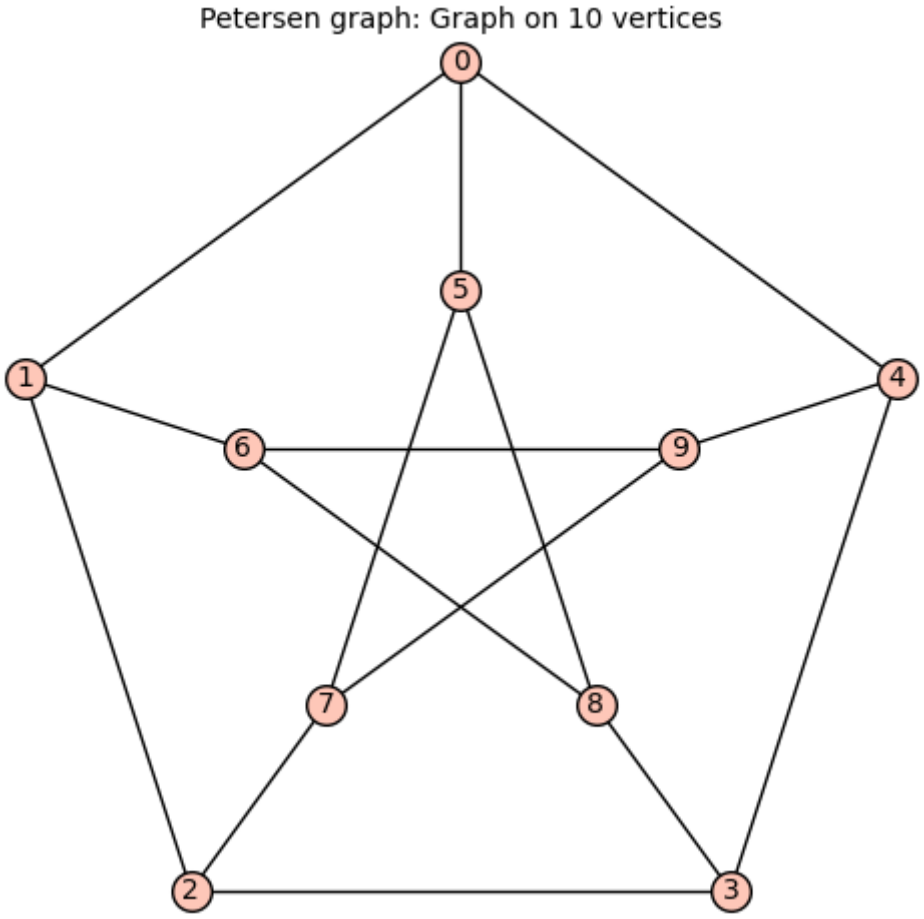
Out[120…



Sage hat eine große Bibliothek an vordefinierten Graphen.

In [121…
```
p=graphs.PetersenGraph()
p
```

Out[121…



Viele Graphentheoretische Algorithmen sind in Sage implementiert. Hier ein paar Beispiele.

Kann der Graph gezeichnet werden, ohne dass sich Kanten überkreuzen?

In [122…
```
p.is_planar()
```

Out[122… False

In [123…
```
p.neighbors(1)
```

Out[123… [0, 2, 6]

Wie viele Farben braucht man, um die Knoten so einzufärben, dass benachbarte Knoten verschiedene Farben haben? (vgl. Vierfarbensatz)

In [124…
```
p.chromatic_number()
```

Out[124… 3

Eine konkrete Färbung.

In [125…
```
p.coloring()
```

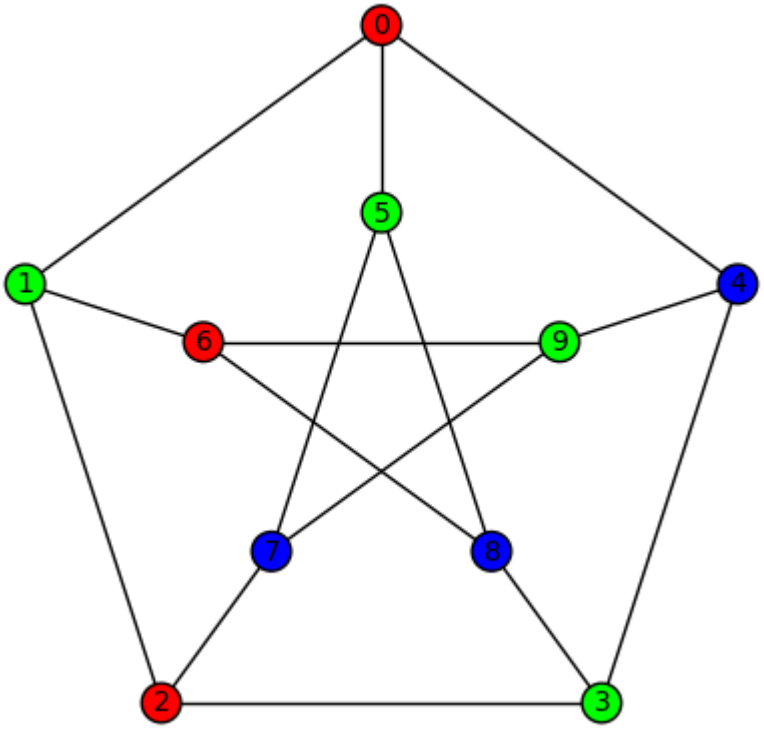
Out[125… [[0, 2, 6], [1, 3, 5, 9], [4, 7, 8]]

Wir wollen die Färbung direkt im Bild des Graphen anzeigen.

In [127…
```
c=p.coloring(hex_colors=True)
c
```

Out[127…
```
{'#ff0000': [0, 2, 6], '#00ff00': [1, 3, 5, 9], '#0000ff': [4, 7, 8]}
```

In [128…
```
p.show(vertex_colors=c)
```



Wir wollen nun die Knoten so umbenennen, dass jeweils die inneren mit den äußeren Knoten Plätze tauschen. Dass können wir über eine Permutation der Knoten erreichen.

In [129…
```
S10=SymmetricGroup([0..9])
S10
```

Out[129…
```
Symmetric group of order 10! as a permutation group
```

In [130…
```
u=S10([5,6,7,8,9,0,1,2,3,4])
u
```

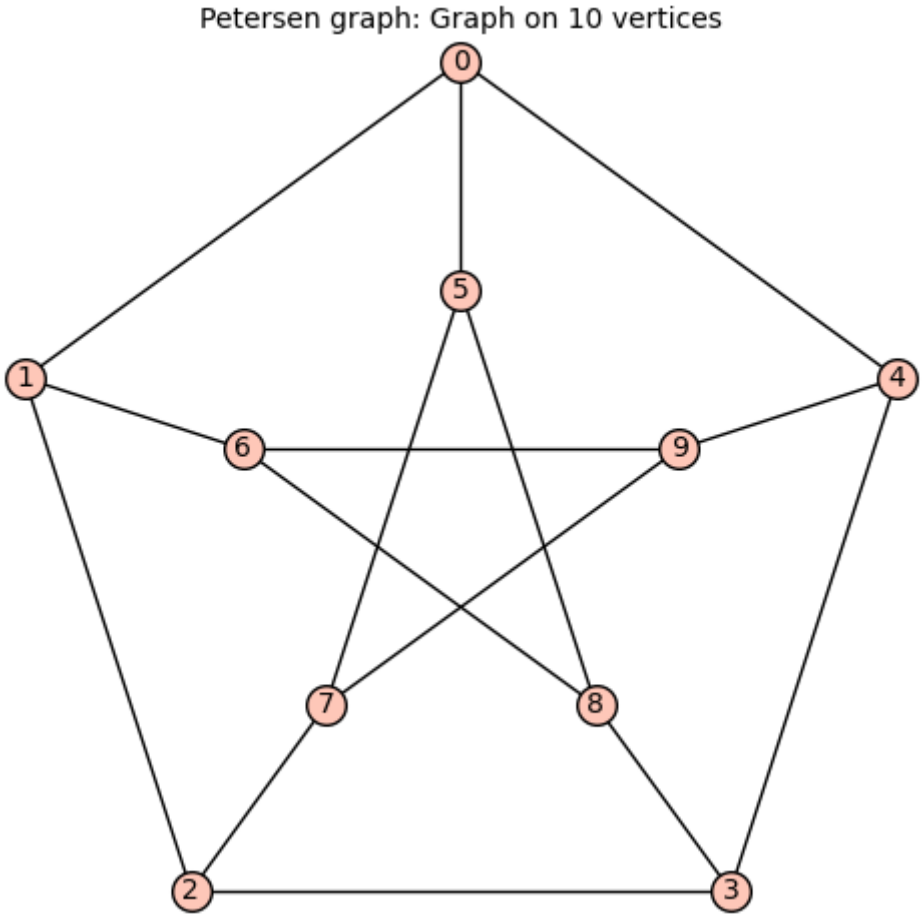
Out[130…
```
(0,5)(1,6)(2,7)(3,8)(4,9)
```

Umbenennen verändert den Graphen, also arbeiten wir lieber mit einer Kopie

In [133…
```
p2=p.copy()
```

In [134…
```
p2
```

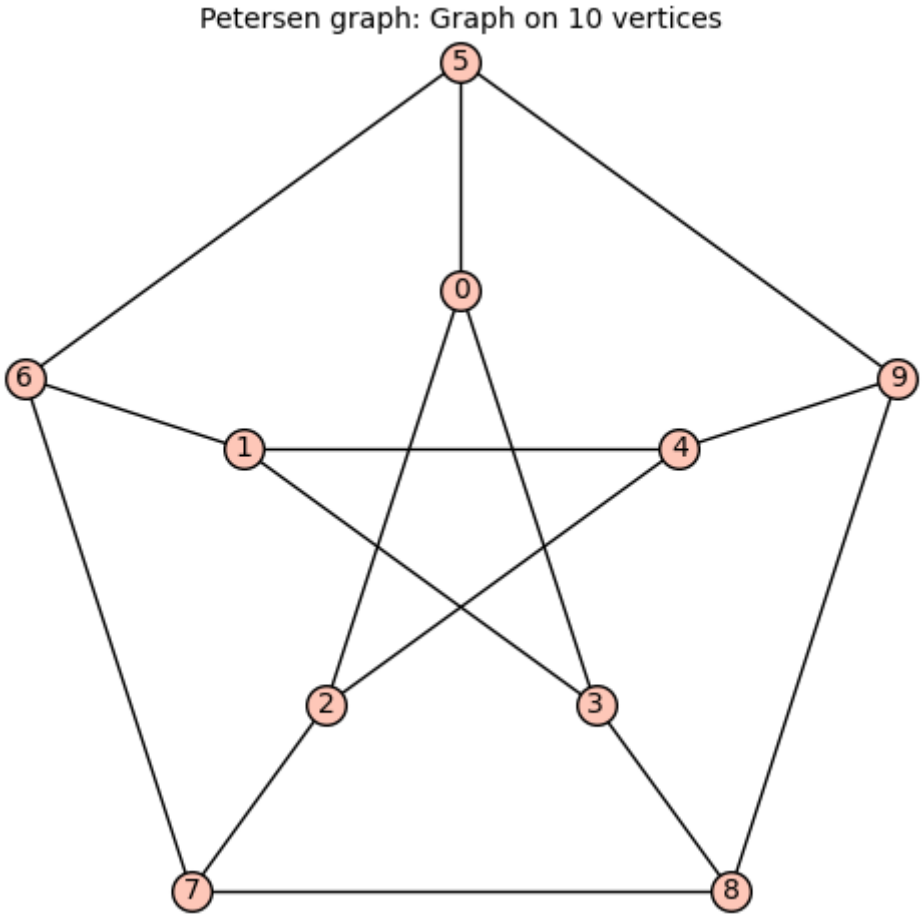
Out[134…



In [135…
```
p2.relabel(u)
```

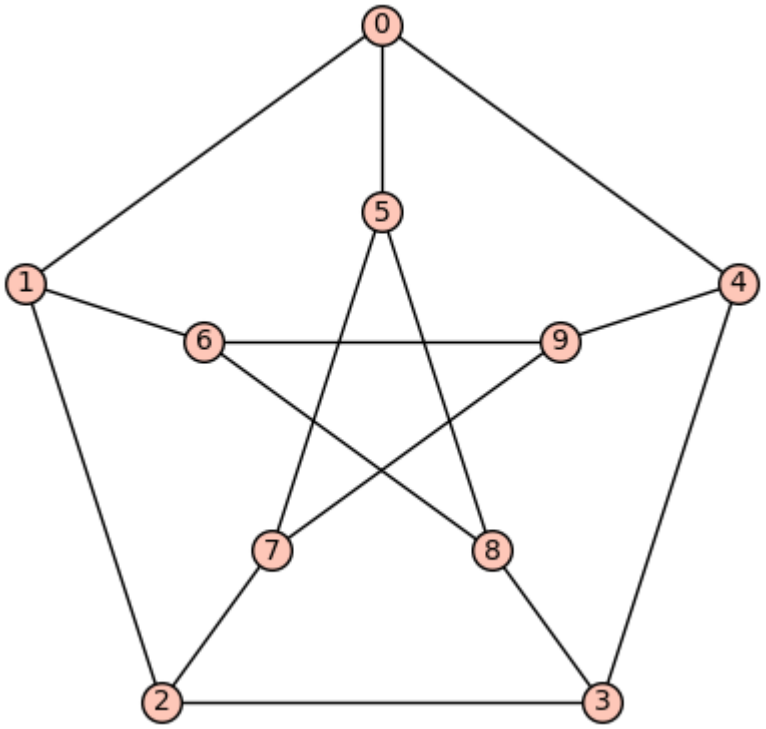
In [136…
```
p2
```

Out[136…



In [137…

```
p.show()
```



Sind zwei Graphen gleich bis auf Umbenennen der Knoten?

In [138…

```
p.is_isomorphic(p2)
```

Out[138… True

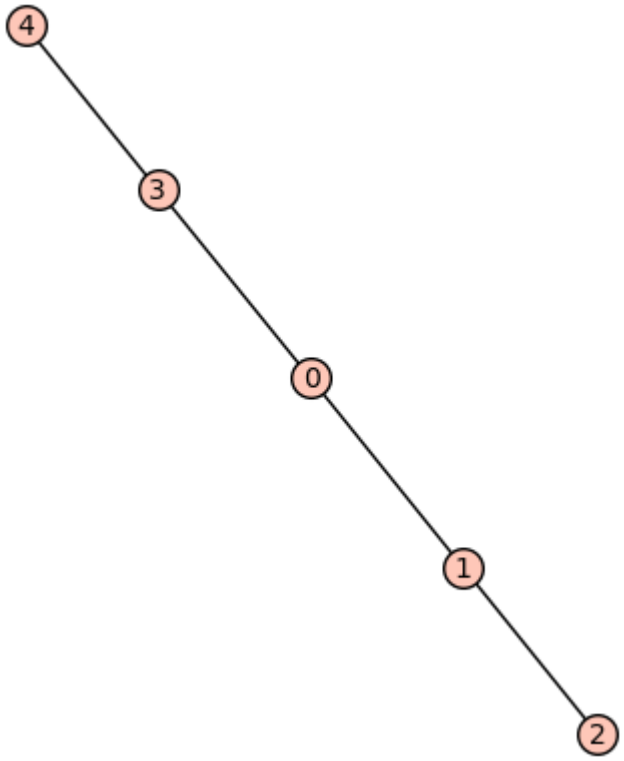
Bäume sind Graphen, die keine Kreise enthalten, also keine Pfade, die im selben Knoten beginnen und enden.

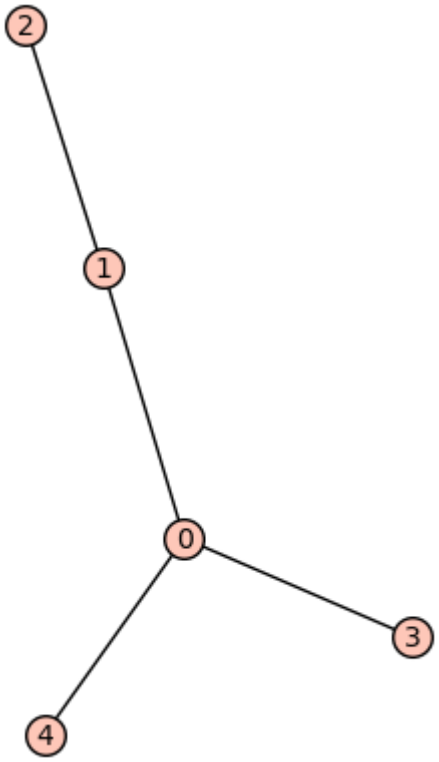
Wir erzeugen hier alle Bäume mit 5 Knoten.

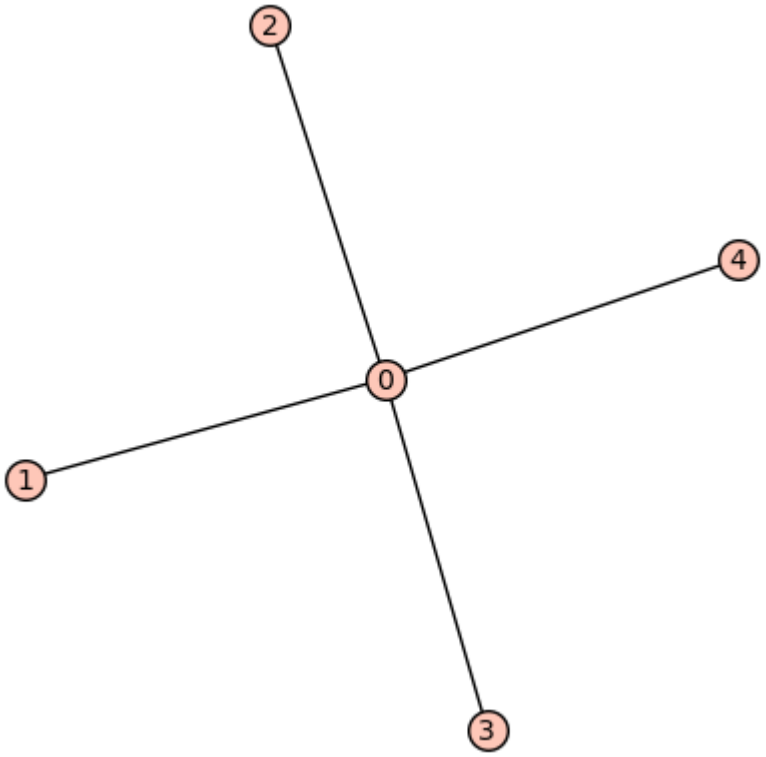
In [139…
```
t5=graphs.trees(5)
t5
```

Out[139… `<sage.graphs.trees.TreeIterator object at 0x7f8055308bc0>`

In [140…
```
for t in t5:
    t.show()
```



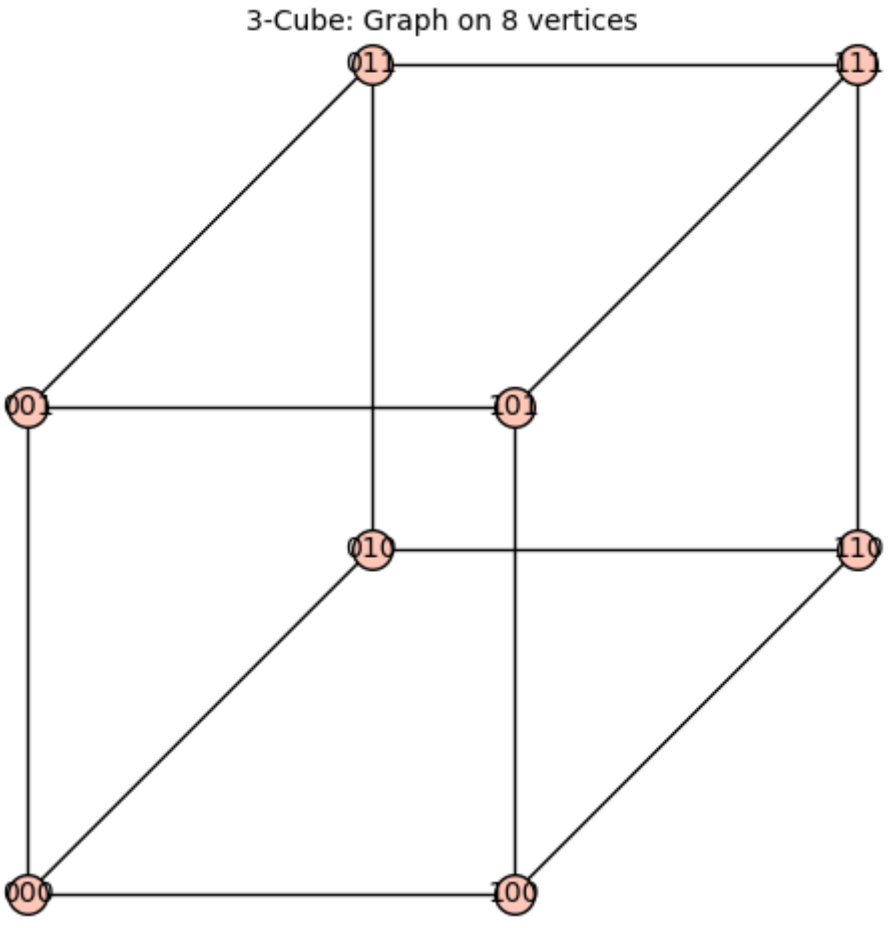




Der Graph bestehend aus Ecken und Kanten eines Würfel.

In [243…
```
g6=graphs.CubeGraph(3,embedding=3)
g6
```

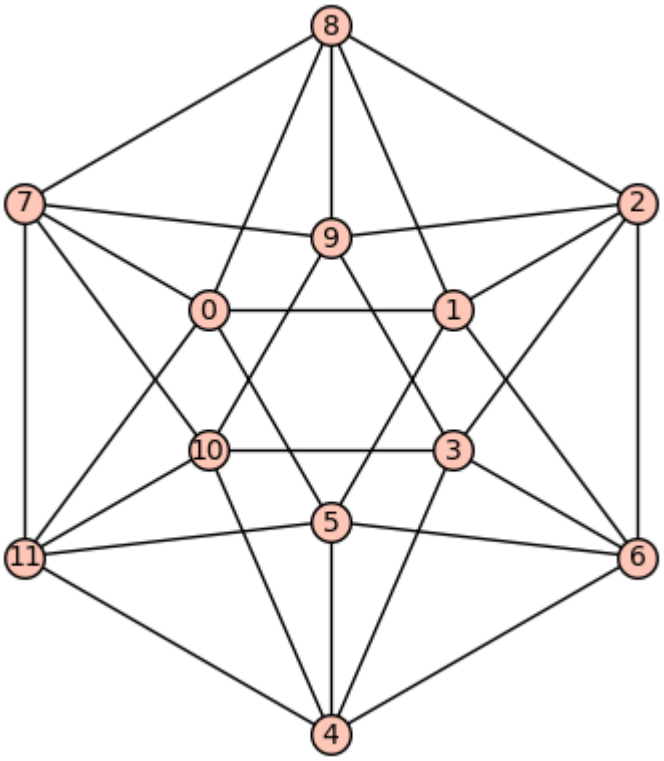
Out[243…



Dasselbe mit einem Ikosaeder statt Würfel.

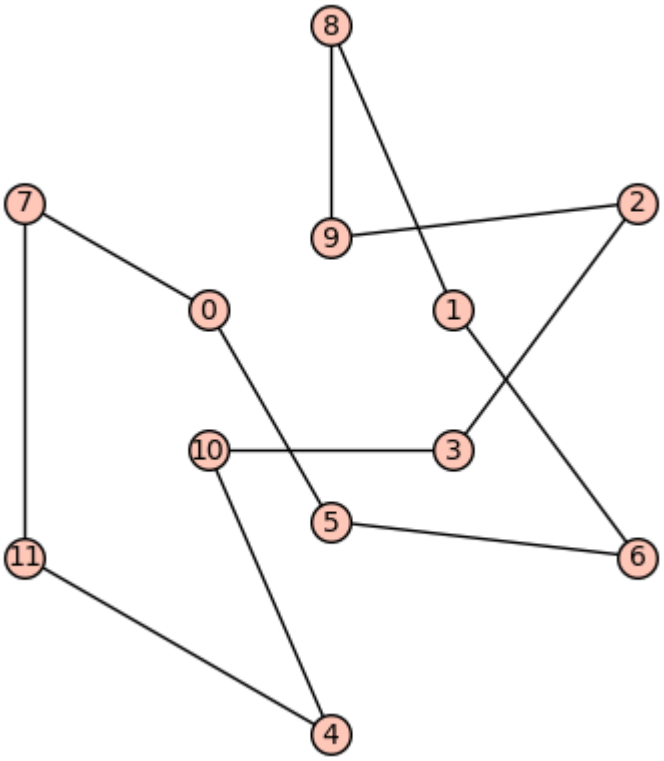
In [142…
```
g20=graphs.IcosahedralGraph()
```

```
g20.show()
```



Hamiltonscher Kreis: Ein Kreis, der alle Knoten genau einmal enthält.

In [143…
```
g20.hamiltonian_cycle().show()
```



Eulertour: ein Kreis, der alle Kanten genau einmal durchläuft.

In [144…
```
g20.eulerian_circuit()
```

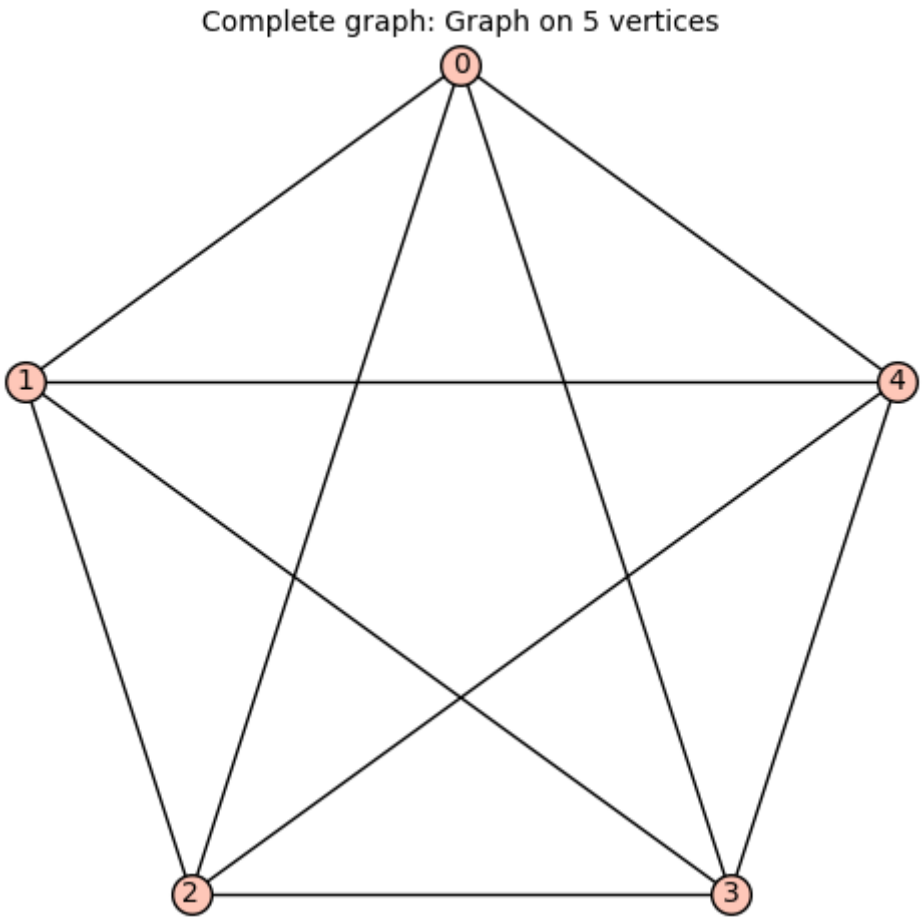
Out[144… False

Der Ikosaedergraph hat also keine Eulertour.

Der vollständige Graph mit 5 Knoten. (D.h. alle möglichen Kanten existieren in dem Graphen)

In [145…
```
k5=graphs.CompleteGraph(5)
k5
```

Out[145…



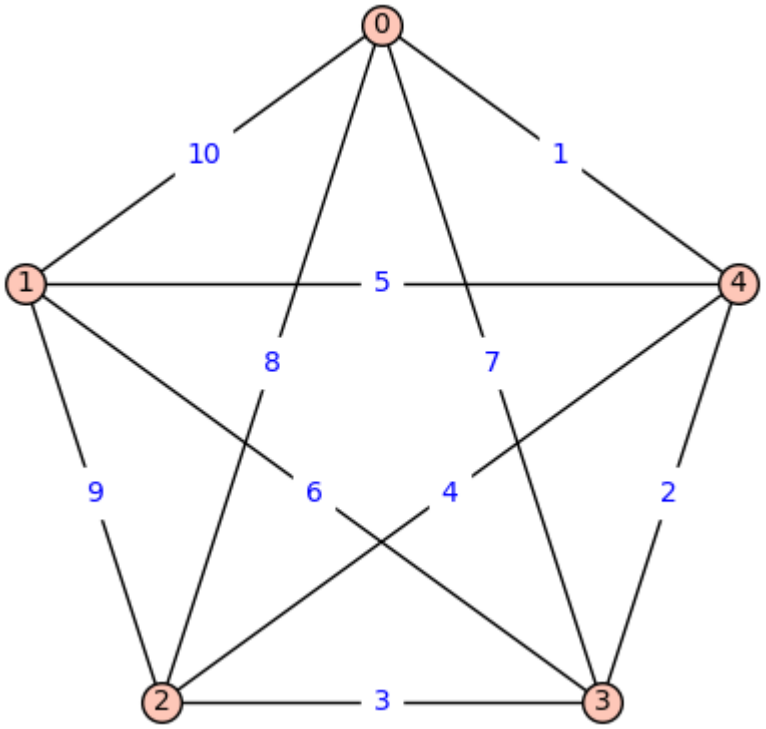
In [146…
```
k5.eulerian_circuit()
```

Out[146…
```
[(0, 4, None),
 (4, 3, None),
 (3, 2, None),
 (2, 4, None),
 (4, 1, None),
 (1, 3, None),
 (3, 0, None),
 (0, 2, None),
 (2, 1, None),
 (1, 0, None)]
```

Darstellung der Eulertour im Bild des Graphen

In [147…
```
for e,i in zip(k5.eulerian_circuit(),(1..)):
    k5.set_edge_label(e[0],e[1],i)
```

In [148…
```
k5.show(edge_labels=True)
```



Elliptische Kurven

In [149…
```
191.is_prime()
```

Out[149…
```
True
```

In [150…
```
E=EllipticCurve(GF(191),[-1,-1])
E
```

Out[150…
```
Elliptic Curve defined by y^2 = x^3 + 190*x + 190 over Finite Field of s
ize 191
```

In [151…
```
E.order()
```

Out[151…
```
194
```

In [152…
```
E.order().factor()
```

Out[152…
```
2 * 97
```

In [153…
```
E.abelian_group()
```

Out[153…
```
Additive abelian group isomorphic to Z/194 embedded in Abelian group of
points on Elliptic Curve defined by y^2 = x^3 + 190*x + 190 over Finite
Field of size 191
```

In [154…
```
P=E.random_element()
Q=E.random_element()
P,Q
```

Out[154…
```
((189 : 152 : 1), (115 : 50 : 1))
```

In [155…
```
P+Q
```

Out[155… (22 : 104 : 1)

In [156…
```
2*P
```

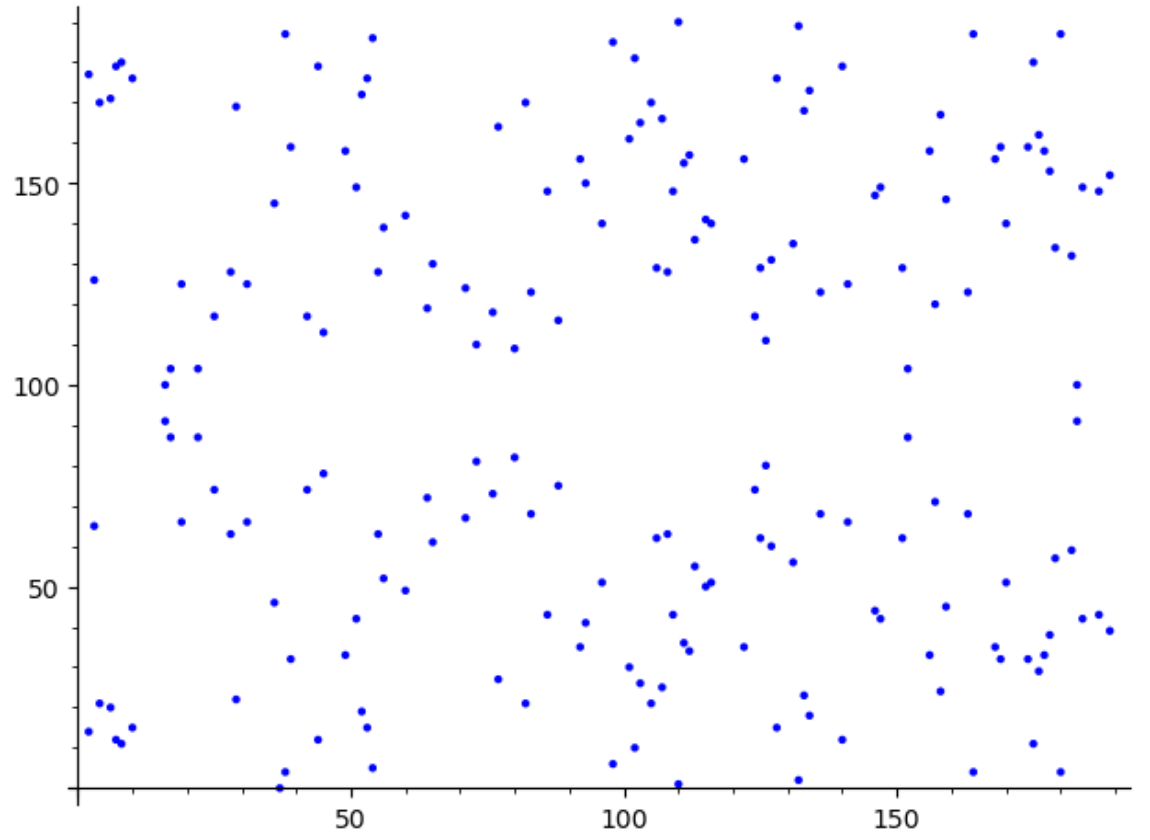
Out[156… (102 : 181 : 1)

In [157…
```
g=E.random_element()
g, g.order()
```

Out[157… ((108 : 128 : 1), 194)

In [158…
```
cp=E.plot()
cp
```

Out[158…



In [159…
```
@cached_function
def ntimesg(n):
    return cp + point([ZZ(c) for c in (n*g).xy()],pointsize=100,rgbcolor=
```

In [160…
```
@interact
def _(n=slider(1,g.order()-1,step_size=1)):
    show(ntimesg(n),xmin=0,xmax=191,ymin=0,ymax=191)
```

```
Interactive function <function _ at 0x7f80549dc900> with 1 widget
  n: TransformIntSlider(value=1, description='n', max=193, min=1)
```

In [161…
```
# Hier ist ein Beispiel einer derzeit tatsächlich in der Kryptographie ve
# Curve25519
```

In [162…
```
p=2^255-19
p.is_prime(), p
```

Out[162… (True,
 578960446186580977117854925043439539266349923328202820197287920039565648199 49)

In [163…
```
C=EllipticCurve(GF(p),[0,486662,0,1,0])
C
```

Out[163… Elliptic Curve defined by y^2 = x^3 + 486662*x^2 + x over Finite Field of size 578960446186580977117854925043439539266349923328202820197287920039565648199 49

In [164…
```
C.order().factor()
```

Out[164… 2^3 * 72370055773322622139731865630429942408571163593799076060019509382854542509 89

In [165…
```
(C.order()).log(2).n()
```

Out[165… 255.000000000000

In [168…
```
P=C.lift_x(9)
P
```

Out[168… (9 : 14781619447589544791020593568409986887264606134616475288964881837755586237401 : 1)

In [169…
```
ord=P.order()
ord
```

Out[169… 72370055773322622139731865630429942408571163593799076060019509382854542509 89

In [170…
```
ord.is_prime()
```

Out[170… True

In [171…
```
ord.log(2).n()
```

Out[171… 252.000000000000

In [172…
```
2*P
```

Out[172… (148472771456354834839633725375570916347109851328257810888871408905975963522 51 : 89146130912291478312779354720486430668800678992518404188551817939385055942 11 : 1)

In [173…
```
4563452*P
```

Out[173… (313716339399131546970523989042467626254998469360570956737515695468079329073 53 : 20391317136630991215898914908497483156190294406040533569992798838533997490 39 : 1)