Numerical solution of fractional PDEs

Reza Mokhtari
Department of Mathematical Sciences
Isfahan University of Technology, Isfahan, Iran

Iranian-Austrian summer school

University of Graz

September 2019

Fifth lecture (44 slides)

Numerical solution of fractional PDEs

- Introduction to fractional calculus
- Numerical methods for fractional integral and derivatives
- Numerical methods for fractional ODEs
- FDM for fractional PDEs
- **FEM for fractional PDEs**

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

$$(u,v)_{L^2(O)} = \int_O uv\,\mathrm{d}O, \qquad \forall u,v \in L^2(O)$$

$$||u||_{L^2(O)} = \sqrt{(u,u)_{L^2(O)}}, \qquad \forall u \in L^2(O)$$

O is a subset of $\mathbb{R}^d$

$$H^r(O) = \{u \in L^2(O) : D^\alpha u \in L^2(O),\ |\alpha| \le k,\ k = 0,1,\cdots,r\}$$

$r \ge 0$

$$D^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_d}^{\alpha_d},\ \alpha = (\alpha_1,\alpha_2,\cdots,\alpha_d)$$

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$$

$$|u|_{H^k(O)} = \left(\sum_{|\alpha|=k} ||D^\alpha u||_{L^2(O)}\right)^{1/2}, \qquad ||u||_{H^k(O)} = \left(\sum_{j=0}^{k} |u|^2_{H^j(O)}\right)^{1/2}$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

Definition 16 ([49]) *Let $\mu > 0, \mu \neq n - 1/2, n \in \mathbb{N}$. Define the semi-norm*

$$|u|_{J_S^\mu(\Omega)} = |({}_{RL}D_{a,x}^\mu u(x), {}_{RL}D_{x,b}^\mu u(x))|^{\frac{1}{2}}$$

and the norm

$$\|u\|_{J_S^\mu(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + |u|_{J_S^\mu(\Omega)}^2\right)^{1/2},$$

and let $J_S^\mu(\Omega)$ (or $J_{S,0}^\mu(\Omega)$) denote the closure of $C^\infty(\Omega)$ (or $C_0^\infty(\Omega)$) with respect to $\|\cdot\|_{J_S^\mu(\Omega)}$.

FEM for fractional PDEs (FPDEs)

Definition 17 *The Fourier transform of $u \in L^2(\mathbb{R})$ is defined as*

$$\hat{u}(\omega) = \mathcal{F}(u(x)) = \int_{\mathbb{R}} e^{-i\omega x} u(x)\, \mathrm{d}x.$$

Definition 18 ([116, 131]) *Let $\mu > 0$. Define the semi-norm*

$$|u|_{H^\mu(\mathbb{R})} = \| |\omega|^\mu \hat{u} \|_{L^2(\mathbb{R})}$$

and the norm

$$\|u\|_{H^\mu(\mathbb{R})} = \left(\|u\|^2_{L^2(\mathbb{R})} + |u|^2_{H^\mu(\mathbb{R})} \right)^{1/2},$$

where $\hat{u}(\mathbb{R})$ is the Fourier transform of function u. And let $H^\mu(\mathbb{R})$ (or $H^\mu_0(\mathbb{R})$) be the closure of $C^\infty(\mathbb{R})$ (or $C^\infty_0(\mathbb{R})$) with respect to $\|\cdot\|_{H^\mu(\mathbb{R})}$.

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

Lemma 5.1.1 ([49, 50]) *Let $\mu > 0$, $u \in L^p(\mathbb{R})$, $p \geq 1$. The Fourier transform of the left and right Riemann–Liouville fractional integral and derivatives satisfy:*

$$\mathcal{F}({}_{RL}\mathrm{D}^{-\mu}_{-\infty,x}u(x)) = (i\omega)^{-\mu}\hat{u}, \qquad \mathcal{F}({}_{RL}\mathrm{D}^{-\mu}_{x,\infty}u(x)) = (-i\omega)^{-\mu}\hat{u}, \tag{5.1}$$

and

$$\mathcal{F}({}_{RL}\mathrm{D}^{\mu}_{-\infty,x}u(x)) = (i\omega)^{\mu}\hat{u}, \qquad \mathcal{F}({}_{RL}\mathrm{D}^{\mu}_{x,\infty}u(x)) = (-i\omega)^{\mu}\hat{u}, \tag{5.2}$$

Lemma 5.1.4 *For $\mu > 0, \mu \neq n - 1/2, n \in \mathbb{N}$, the spaces $J^{\mu}_{L}(\mathbb{R})$ and $J^{\mu}_{S}(\mathbb{R})$ are equal, with equivalent semi-norms and norms.*

Lemma 5.1.5 *Let $\mu > 0$, $\mu \neq n - 1/2, n \in \mathbb{N}$. Then the spaces $J^{\mu}_{S,0}(\Omega)$ and $H^{\mu}_{0}(\Omega)$ are equal, with equivalent semi-norms and norms.*

Lemma 5.1.6 *Let $\mu > 0$. Then the spaces $J^{\mu}_{L,0}(\Omega)$, $J^{\mu}_{R,0}(\Omega)$, and $H^{\mu}_{0}(\Omega)$ are equal. Also, if $\mu \neq n - 1/2, n \in \mathbb{N}$, the spaces $J^{\mu}_{L,0}(\Omega)$, $J^{\mu}_{R,0}(\Omega)$, $J^{\mu}_{S,0}(\Omega)$, and $H^{\mu}_{0}(\Omega)$ have equivalent semi-norms and norms.*

Lemma 5.1.3 ([49]) *Let $\mu > 0$ be given. The spaces $J^{\mu}_{L}(\mathbb{R})$, $J^{\mu}_{R}(\mathbb{R})$, and $H^{\mu}(\mathbb{R})$ are equal with equivalent semi-norms and norms.*

FEM for fractional PDEs (FPDEs)

Lemma 5.1.7 (*Fractional Poincaré-Friedrichs* [49]) *For $u \in J^{\mu}_{L,0}(\Omega)$, $0 \le s \le \mu$, one has*

$$\|u\|_{J^s_L(\Omega)} \le C|u|_{J^{\mu}_L L(\Omega)}, \tag{5.13}$$

and for $u \in J^{\mu}_{R,0}(\Omega), \mu > 0$, we have

$$\|u\|_{J^s_R(\Omega)} \le C|u|_{J^{\mu}_R L(\Omega)}. \tag{5.14}$$

Lemma 5.1.8 *The left and right Riemann–Liouville fractional integral operators are adjoints w.r.t. the inner product in $L^2(\Omega), \Omega = (a,b)$, i.e.,*

$$(\mathrm{D}^{-\mu}_{a,x}u, v)_{L^2(\Omega)} = (u, \mathrm{D}^{-\mu}_{x,b}v)_{L^2(\Omega)}, \quad u, v \in L^2(\Omega), \quad \mu > 0.$$

Lemma 5.1.9 ([64]) *For $0 < \beta, \gamma < 1$, if $u(x) \in H^1(\Omega), \Omega = (a,b)$, then*

$$_{RL}\mathrm{D}^{\beta}_{a,x}\, {}_{RL}\mathrm{D}^{\gamma}_{a,x}u(x) = {}_{RL}\mathrm{D}^{\beta+\gamma}_{a,x}u(x). \tag{5.16}$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

Lemma 5.1.10 ([94, 171, 175]) *Let* $0<\beta<2$, $\Omega=(a,b)$. *Then for any* $u\in H_0^{\beta}(\Omega)$, $v\in H_0^{\beta/2}(\Omega)$, *we have*

$$({}_{RL}\mathrm{D}_{a,x}^{\beta}u, v)_{L^2(\Omega)} = ({}_{RL}\mathrm{D}_{a,x}^{\beta/2}u, {}_{RL}\mathrm{D}_{x,b}^{\beta/2}v)_{L^2(\Omega)},$$

$$({}_{RL}\mathrm{D}_{x,b}^{\beta}u, v)_{L^2(\Omega)} = ({}_{RL}\mathrm{D}_{x,b}^{\beta/2}u, {}_{RL}\mathrm{D}_{a,x}^{\beta/2}v)_{L^2(\Omega)}.$$

Lemma 5.1.11 ([94, 175]) *Let* $0<\beta<1$, $\Omega=(a,b)$. *Then for any* $u\in H^{\beta}(\Omega)$, $v\in H^{\beta/2}(\Omega)$, $u(a)=u(b)=0$, *we have*

$$({}_{RL}\mathrm{D}_{a,x}^{\beta}u, v)_{L^2(\Omega)} = ({}_{RL}\mathrm{D}_{a,x}^{\beta/2}u, {}_{RL}\mathrm{D}_{x,b}^{\beta/2}v)_{L^2(\Omega)}.$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

Lemma 5.1.12 (*Gronwall's inequality*) *Let $a(t), q(t) \in L[t_0, t_1]$, $u(t), b(t)$, $t \in [t_0, t_1]$ be real valued continuous functions; $b(t)$ and $q(t)$ are nonnegative functions satisfying*

$$u(t) \leq a(t) + q(t)\int_{t_0}^{t} b(s)u(s)\,\mathrm{d}s, \quad \forall t \in [t_0, t_1].$$

Then we have

$$u(t) \leq a(t) + q(t)\int_{t_0}^{t} a(s)b(s)\exp\left(\int_{s}^{t} q(r)b(r)\,\mathrm{d}r\right)\mathrm{d}s, \quad \forall t \in [t_0, t_1].$$

Lemma 5.1.13 (*Discrete Gronwall's inequality*) *Let x_n be real positive numbers, $H, C, \Delta t > 0$, $x_0 \leq H$. x_n satisfies*

$$x_n \leq C\Delta t \sum_{k=0}^{n-1} x_k + H.$$

Then we have

$$x_n \leq H\exp(Cn\Delta t).$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

- 1D time-independent equations — RL advection-dispersion equation
- 1D space-fractional equations — Riesz fractional diffusion equation
- 1D time-fractional equations — RL/Caputo time-fractional diffusion equation
- 1D time-space fractional equations — Time-space fractional diffusion equation with Caputo derivative in time

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

$$Lu = -\mathrm{D}\,a\,(p\mathrm{D}_{0,x}^{-\beta} + q\mathrm{D}_{x,1}^{-\beta})\mathrm{D}u + b(x)\mathrm{D}u + c(x)u = f, \quad x \in \Omega = (x_L, x_R) = (0,1)$$

$b(x) \in C^1(\bar{\Omega})$ $\quad u = 0, \quad x \in \partial\Omega$ $\quad c(x) \in C(\bar{\Omega})$ with $c - \mathrm{D}b/2 \geq 0$

$a > 0$ $\quad 0 \leq \beta < 1$ $\quad p + q = 1,\ 0 \leq p, q \leq 1$

Let $\Omega = (x_L, x_R)$ be a general finite domain, and denote by $(\cdot,\cdot)$ the inner product on the space $L^2(\Omega)$ with the L^2 norm $\|\cdot\|$ and the maximum norm $\|\cdot\|_\infty$. Denote $H^r(\Omega)$ and $H_0^r(\Omega)$ as the commonly used Sobolev spaces with the norm $\|\cdot\|_r$ and semi-norm $|\cdot|_r$, respectively. Define $\mathbb{P}_r(\Omega)$ as the space of polynomials defined on Ω with the degree no greater than r, $r \in Z^+$. Let S_h be a uniform partition of Ω, which is given by

Denote by $h = (x_R - x_L)/N = x_i - x_{i-1}$ and $\Omega_i = [x_{i-1}, x_i]$ for $i = 1, 2, \cdots, N$. We define the *finite element space* X_h^r as the set of piecewise polynomials with degree at most r $(r \geq 1)$ on the mesh S_h, which can be expressed by

$$X_h^r = \{v : v|_{\Omega_i} \in \mathbb{P}_r(\Omega_i), v \in C(\Omega)\}.$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

Introduce the *piecewise interpolation operator* $I_h : C(\bar{\Omega}) \to X_h^r$ as

$$I_h u\big|_{\Omega_i} = \sum_{k=0}^{r} u(x_k^i) F_k^i(x), \quad u \in C(\bar{\Omega}),$$

where $F_k^i(x)$ are *Lagrangian basis functions* defined by

$$F_k^i(x) = \prod_{l=0, l\neq k}^{r} \frac{x - x_l^i}{x_k^i - x_l^i}, \quad i = 1, 2, \cdots, N,$$

and $\{x_k^i, k = 0, 1, \cdots, r\}$ are the interpolation nodes on the interval Ω_i with $x_0^i = x_{i-1}$ and $x_r^i = x_i$.

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

Define $\varphi^i\,(i = 0, 1, \cdots, N)$ and $\varphi_k^i\,(k = 1, 2, \cdots, r-1; i = 1, 2, \cdots, N)$ as

$$\varphi_k^i(x) = \begin{cases} F_k^i(x), & x \in [x_{i-1}, x_i], \quad k = 1, 2, \cdots, r-1, i = 1, \cdots, N, \\ 0, & \text{others}, \end{cases}$$

$$\varphi^i(x) = \begin{cases} F_r^i(x), & x \in [x_{i-1}, x_i], \quad i = 1, \cdots, N-1, \\ F_0^{i+1}(x), & x \in [x_i, x_{i+1}], \quad i = 1, \cdots, N-1, \\ 0, & \text{others}, \end{cases}$$

$$\varphi^0(x) = \begin{cases} F_0^1(x), & x \in [x_0, x_1], \\ 0, & \text{others}, \end{cases}$$

$$\varphi^N(x) = \begin{cases} F_r^N(x), & x \in [x_{N-1}, x_N], \\ 0, & \text{others}. \end{cases}$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

Let $X_{h0}^r = X_h^r \cap H_0^1(\Omega)$. Then the spaces X_{h0}^r and X_h^r can be expressed as

$$X_{h0}^r = \operatorname{span}\left\{\varphi_k^i, k = 1,2,\cdots,r-1, i = 1,2,\cdots,N\right\} \cup \left\{\varphi^i, i = 1,2,\cdots,N-1\right\},$$

$$X_h^r = \operatorname{span}\left\{\varphi_k^i, k = 1,2,\cdots,r-1, i = 1,2,\cdots,N\right\} \cup \left\{\varphi^i, i = 0,1,\cdots,N\right\}.$$

$$\phi_j(x) = \begin{cases} \varphi_k^i(x), & j = (i-1)r+k,\ k = 1,2,\cdots,r-1, i = 1,2,\cdots,N, \\ \varphi^i(x), & j = ir,\ i = 0,1,\cdots,N. \end{cases}$$

$$X_{h0}^r = \operatorname{span}\left\{\phi_j, j = 1,2,\cdots,Nr-1\right\} \qquad X_h^r = \operatorname{span}\left\{\phi_j, j = 0,1,2,\cdots,Nr\right\}$$

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

The *orthogonal projection operator* $\Pi_h^{1,0} : H_0^1(\Omega) \to X_{h0}^r$ is defined as

$$(\partial_x(u - \Pi_h^{1,0}u), \partial_x v) = 0, \quad u \in H_0^1(\Omega), \forall v \in X_{h0}^r.$$

Lemma 5.2.1 ([9]) *Let $m, r \in Z^+$, $r \geq 1$, and $u \in H^m(\Omega) \cap H_0^1(\Omega)$. If $1 \leq m \leq r+1$, then there exists a positive constant C independent of h, such that*

$$\|u - \Pi_h^{1,0}u\|_{H^l(\Omega)} \leq Ch^{m-l}\|u\|_{H^m(\Omega)}, \quad 0 \leq l \leq 1.$$

Lemma 5.2.2 ([3]) *Let m, l be nonnegative numbers, $r \in Z^+$, $r \geq 1$, and $u \in H^m(\Omega)$. If $0 \leq l \leq m \leq r+1$, then there exists a positive constant C independent of h, such that*

$$\|u - I_h u\|_{H^l(\Omega)} \leq Ch^{m-l}\|u\|_{H^m(\Omega)}, \quad 0 \leq l \leq 1.$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

$$-\mathrm{D}\,a\,(p\mathrm{D}_{0,x}^{-\beta}+q\mathrm{D}_{x,1}^{-\beta})\mathrm{D}u+b(x)\mathrm{D}u+c(x)u=f \qquad \times \qquad v\in C_0^\infty(\Omega)$$

$$\int_\Omega\left(-\mathrm{D}(ap\mathrm{D}_{0,x}^{-\beta}+aq\mathrm{D}_{x,1}^{-\beta})\mathrm{D}u+b(x)\mathrm{D}u+c(x)u\right)v\,\mathrm{d}x=\int_\Omega f\,v\,\mathrm{d}x$$

$$\int_\Omega\left[a(p\mathrm{D}_{0,x}^{-\beta}+q\mathrm{D}_{x,1}^{-\beta})\mathrm{D}u\mathrm{D}v+b(x)\mathrm{D}u\,v+c(x)uv\right]\mathrm{d}x=\int_\Omega f\,v\,\mathrm{d}x$$

$$(\cdot,\cdot)=(\cdot,\cdot)_{L^2(\Omega)},\ \|\cdot\|=\|\cdot\|_{L^2(\Omega)},\ |\cdot|_r=|\cdot|_{H^r(\Omega)} \qquad \|\cdot\|_r=\|\cdot\|_{H^r(\Omega)}$$

$$0\le\beta<1 \text{ and } u=0 \text{ on } \partial\Omega$$

$$\mathrm{D}_{0,x}^{-\beta}\mathrm{D}u={}_{RL}\mathrm{D}_{0,x}^{1-\beta}u,\qquad \mathrm{D}\mathrm{D}_{0,x}^{-\beta}\mathrm{D}u={}_{RL}\mathrm{D}_{0,x}^{2-\beta}u$$

$$\mathrm{D}_{x,1}^{-\beta}\mathrm{D}u={}_{RL}\mathrm{D}_{x,1}^{1-\beta}u,\qquad \mathrm{D}\mathrm{D}_{x,1}^{-\beta}\mathrm{D}u={}_{RL}\mathrm{D}_{x,1}^{2-\beta}u$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

$$\alpha = 1 - \beta/2 \qquad 1/2 < \alpha \le 1$$

$$A : H_0^{\alpha}(\Omega) \times H_0^{\alpha}(\Omega) \to \mathbb{R}$$

$$A(u,v) = ap(\mathrm{D}_{0,x}^{-\beta} u, \mathrm{D}v) + aq(\mathrm{D}_{x,1}^{-\beta} \mathrm{D}u, \mathrm{D}v) + (b\mathrm{D}u, v) + (cu, v)$$

$$f \in H^{-\alpha}(\Omega) \qquad F : H_0^{\alpha}(\Omega) \to \mathbb{R} \qquad F(v) = (f, v)$$

Variational (weak) formulation (problem)

$$A(u,v) = F(v), \qquad v \in H_0^{\alpha}(\Omega)$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

Definition 20 ([3]) *Let $(V,(\cdot,\cdot))$ be an inner-product space. If the associated normed linear space $(V, \|\cdot\|_V)$ is complete, then $(V,(\cdot,\cdot))$ is called a Hilbert space.*

Definition 21 ([3]) *A bilinear form $A(\cdot,\cdot)$ on a normed linear space H is said to be **bounded** (or **continuous**) if there exists a positive constant C such that*

$$A(v,w) \le C\|v\|_H\|w\|_H, \qquad v,w \in H,$$

*and **coercive** on $V \subset H$ if there exists a positive constant c_0 such that*

$$A(v,v) \ge c_0\|v\|_H^2, \qquad v \in V.$$

Theorem 34 (*Lax-Milgram Theorem* [3]) *Given a Hilbert space $(V,(\cdot,\cdot))$, a continuous, coercive bilinear form $A(\cdot,\cdot)$ and a continuous linear functional $F \in V'$, V' is the dual space of V, there exists a unique $u \in V$ such that*

$$A(u,v) = F(v), \qquad v \in V. \tag{5.26}$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

Theorem

There exists a unique solution $u \in H_0^{\alpha}(\Omega)$ to the variational problem such that

$$||u||_{H^{\alpha}(\Omega)} \leq C||f||_{H^{-\alpha}(\Omega)}$$

In order to prove the uniqueness of the solution u to the variational problem, we need to prove that the bilinear form $A(\cdot,\cdot)$ is continuous and coercive. We first prove the coercivity. Using $c - \mathrm{D}b/2 \geq 0$, it is easy to obtain

$$\begin{aligned} A(u,u) &= ap(\mathrm{D}_{0,x}^{-\beta}\mathrm{D}u, \mathrm{D}u) + aq(\mathrm{D}_{x,1}^{-\beta}\mathrm{D}u, \mathrm{D}u) + (b\mathrm{D}u, u) + (cu, u) \\ &= ap(\mathrm{D}_{0,x}^{-\beta}\mathrm{D}u, \mathrm{D}u) + aq(\mathrm{D}_{x,1}^{-\beta}\mathrm{D}u, \mathrm{D}u) + ((c - \mathrm{D}b/2)u, u) \\ &\geq ap(\mathrm{D}_{0,x}^{-\beta}u, \mathrm{D}u) + aq(\mathrm{D}_{x,1}^{-\beta}\mathrm{D}u, \mathrm{D}u), \end{aligned}$$

Noticing that $u = 0$ on $\partial\Omega$ and using some previous Lemmas, we have

$$(\mathrm{D}_{0,x}^{-\beta}\mathrm{D}u, \mathrm{D}u) = (\mathrm{D}_{0,x}^{-\beta/2}\mathrm{D}_{0,x}^{\alpha}u, \mathrm{D}u) = -(\mathrm{D}_{0,x}^{\alpha}u, \mathrm{D}_{x,1}^{\alpha}u)$$

Proof

1D time-independent equations : RL advection-dispersion equation

$$(\mathrm{D}_{x,1}^{-\beta}\mathrm{D}u, \mathrm{D}u) = -(\mathrm{D}_{x,1}^{\alpha}u, \mathrm{D}_{0,x}^{\alpha}u).$$

Therefore,

$$A(u,u) \geq -a(\mathrm{D}_{0,x}^{\alpha}u, \mathrm{D}_{x,1}^{\alpha}u) = a|u|_{J_S^{\alpha}(\Omega)}^2.$$

The semi-norm equivalence of $J_{S,0}^{\alpha}(\Omega)$ and $H_0^{\alpha}(\Omega)$ implies that

$$A(u,u) \geq a|u|_{J_S^{\alpha}(\Omega)}^2 \geq C|u|_{H^{\alpha}(\Omega)}^2.$$

Since $u = 0$ on $\partial\Omega$, from fractional Poincaré-Friedrichs inequality, we have

$$||u||_{H^{\alpha}(\Omega)}^2 \leq C|u|_{H^{\alpha}(\Omega)}^2.$$

Hence,

$$A(u,u) \geq C_0||u||_{H^{\alpha}(\Omega)}^2.$$

Next, we prove that the bilinear form $A(\cdot,\cdot)$ is continuous. We get

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

$$
\begin{aligned}
|A(u,v)| &\le ap|(\mathrm{D}_{0,x}^{-\beta}\mathrm{D}u,\mathrm{D}v)| + aq|(\mathrm{D}_{x,1}^{-\beta}\mathrm{D}u,\mathrm{D}v)| + |(b\mathrm{D}u,v)| + |(cu,v)| \\
&= ap|(\mathrm{D}_{0,x}^{\alpha}u,\mathrm{D}_{x,1}^{\alpha}v)| + aq|(\mathrm{D}_{x,1}^{\alpha}u,\mathrm{D}_{0,x}^{\alpha}v)| + |(b\mathrm{D}u,v)| + |(cu,v)| \\
&\le ap||u||_{J_L^{\alpha}(\Omega)}||v||_{J_R^{\alpha}(\Omega)} + aq||u||_{J_R^{\alpha}(\Omega)}||v||_{J_L^{\alpha}(\Omega)} \\
&\quad + C||u||_{H^{\alpha}(\Omega)}||v||_{H^{\alpha}(\Omega)} + ||c||_{L^{\infty}(\Omega)}||u||_{L^2(\Omega)}||v||_{L^2(\Omega)} \\
&\le C||u||_{H^{\alpha}(\Omega)}||v||_{H^{\alpha}(\Omega)},
\end{aligned}
$$

where we have used

$$||bu||_{H^{\alpha}(\Omega)} \le C||u||_{H^{\alpha}(\Omega)} \text{ for } b \in C^1(\bar{\Omega})$$

For the linear functional $F(v)$, we have

$$|F(v)| = |(f,v)| \le ||f||_{H^{-\alpha}(\Omega)}||v||_{H^{\alpha}(\Omega)}.$$

Therefore,

$$C||u||_{H^{\alpha}(\Omega)}^2 \le |A(u,u)| \le C_1||f||_{H^{-\alpha}(\Omega)}||u||_{H^{\alpha}(\Omega)}$$

The proof is completed by using the Lax-Milgram Theorem.

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

Finite element method (FEM)

Find $u_h \in X_{h0}^r$, such that

$$A(u_h, v) = F(v), \qquad v \in X_{h0}^r.$$

Theorem (Cea's Lemma)

If u and u_h are solutions to the variational problem and FEM, respectively then

$$\|u - u_h\|_{H^\alpha(\Omega)} \le C\|u - v\|_{H^\alpha(\Omega)}, \qquad \forall v \in X_{h0}^r.$$

Proof

Obviously, we have

$$A(u - u_h, v) = 0, \qquad \forall v \in X_{h0}^r.$$

Then, we can write

$$\|u - u_h\|_{H^\alpha(\Omega)}^2 \le \frac{1}{C_0} A(u - u_h, u - u_h) = \frac{1}{C_0} A(u - u_h, u - v_h + v_h - u_h)$$

$$u_h, v_h \in X_{h0}^r$$

$$= \frac{1}{C_0} A(u - u_h, u - v_h) \le \frac{C}{C_0} \|u - u_h\|_{H^\alpha(\Omega)} \|u - v_h\|_{H^\alpha(\Omega)}$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

Theorem (H^α error estimate)

If $u \in H_0^\alpha(\Omega) \cap H^r(\Omega)$ $(\alpha \le r)$ and u_h are solutions to the variational problem and FEM, respectively then there exists a positive constant C independent of h such that $\|u-u_h\|_{H^\alpha(\Omega)} \le Ch^{r-\alpha}\|u\|_{H^r(\Omega)}$.

Using the Cea's Lemma, we have

$$\|u-u_h\|_{H^\alpha(\Omega)} \le C\|u-I_h u\|_{H^\alpha(\Omega)} \le Ch^{r-\alpha}\|u\|_{H^r(\Omega)}$$

Proof

Aubin-Nitsche trick

$$\begin{cases} -\mathrm{D}\,a(p\mathrm{D}_{0,x}^{-\beta}+q\mathrm{D}_{x,1}^{-\beta})\mathrm{D}w+b(x)\mathrm{D}w+c(x)w=g, & x\in\Omega, \\ w=0, \quad x\in\partial\Omega. \end{cases}$$

$$g=e=u-u_h \longrightarrow A(w,v)=(e,v), \qquad v\in H_0^\alpha(\Omega)$$

$$\|w\|_{H^{2\alpha}(\Omega)} \le C\|e\|_{L^2(\Omega)}, \text{ for } \alpha \ne \frac{3}{4}$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-independent equations : RL advection-dispersion equation

Theorem (L^2 error estimate)

If $u \in H_0^\alpha(\Omega) \cap H^r(\Omega)$ $(3/4 < \alpha \le r)$ and u_h are solutions to the variational problem and FEM, respectively then there exists a positive constant C independent of h such that

$$||u-u_h||_{L^2(\Omega)} \le Ch^r ||u||_{H^r(\Omega)}.$$

Substituting $v = e = u - u_h$ in the new variational problem, and applying Galerkin orthogonality $A(e,v)=0, \forall v \in X_{h0}^r$, we have

$$
\begin{aligned}
||e||_{L^2(\Omega)}^2 &= A(e,w) = A(e, w - I_h w + I_h w) \\
&= A(e, w - I_h w) \le C ||e||_{H^\alpha(\Omega)} ||w - I_h w||_{H^\alpha(\Omega)} \\
&\le Ch^\alpha ||e||_{H^\alpha(\Omega)} ||w||_{H^{2\alpha}(\Omega)} \\
&\le Ch^\alpha ||e||_{H^\alpha(\Omega)} ||e||_{L^2(\Omega)}.
\end{aligned}
$$

It follows that $||u-u_h||_{L^2(\Omega)} \le Ch^\alpha ||u-u_h||_{H^\alpha(\Omega)} \le Ch^r ||u||_{H^r(\Omega)}$.

Proof

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D space-fractional equations : Riesz fractional diffusion equation

$$\begin{cases} \partial_t u = {}_{RZ}\mathrm{D}_x^{2\alpha} u + f(x,t), & (x,t) \in \Omega \times (0,T], \\ u(x,0) = \phi_0(x), & x \in \Omega, \\ u = 0, & (x,t) \in \partial\Omega \times (0,T], \end{cases}$$

$1/2 < \alpha \le 1, \Omega = (a,b)$

$$\begin{aligned} (\partial_t u, v) &= ({}_{RZ}\mathrm{D}_x^{2\alpha} u, v) + (f, v) \\ &= -c_{2\alpha}({}_{RL}\mathrm{D}_{a,x}^{\alpha} u, {}_{RL}\mathrm{D}_{x,b}^{\alpha} v) + ({}_{RL}\mathrm{D}_{x,b}^{\alpha} u, {}_{RL}\mathrm{D}_{a,x}^{\alpha} v) + (f, v) \end{aligned}$$

$v \in H_0^{\alpha}(\Omega)$

Variational (weak) formulation (problem)

Find $U(t) = U(\cdot,t) \in H_0^{\alpha}(\Omega)$ $U(0) = u(0)$

$$(\partial_t U, v) + A(U, v) = (f, v), \quad \forall v \in H_0^{\alpha}(\Omega)$$

$$A(u,v) = c_{2\alpha}\left[({}_{RL}\mathrm{D}_{a,x}^{\alpha} u, {}_{RL}\mathrm{D}_{x,b}^{\alpha} v) + ({}_{RL}\mathrm{D}_{x,b}^{\alpha} u, {}_{RL}\mathrm{D}_{a,x}^{\alpha} v)\right]$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D space-fractional equations : Riesz fractional diffusion equation

Theorem

Let $1/2 < \alpha < 1$ and $t \in (0, T]$. There exists a unique solution $U(t) \in H_0^\alpha(\Omega)$ to the variational problem such that

$$\|U(t)\|^2 + C\int_0^t \|U(s)\|^2_{H^\alpha(\Omega)}\,\mathrm{d}s \le \|u(0)\|^2 + \frac{1}{C}\int_0^t \|f(s)\|^2\,\mathrm{d}s$$

where C is a positive constant.

In order to prove the uniqueness of the solution u to the variational problem, we need to prove that the bilinear form $A(\cdot,\cdot)$ is continuous and coercive. It is easy to verify that

$$\begin{aligned} |A(u,v)| &\le -c_{2\alpha}\left(|({}_{RL}\mathrm{D}^\alpha_{a,x}u, {}_{RL}\mathrm{D}^\alpha_{x,b}v)| + |({}_{RL}\mathrm{D}^\alpha_{x,b}u, {}_{RL}\mathrm{D}^\alpha_{a,x}v)|\right) \\ &\le -c_{2\alpha}\left(|u|_{J^\alpha_L(\Omega)}|v|_{J^\alpha_R(\Omega)} + |u|_{J^\alpha_R(\Omega)}|v|_{J^\alpha_L(\Omega)}\right) \\ &\le C\|u\|_{H^\alpha(\Omega)}\|v\|_{H^\alpha(\Omega)}, \end{aligned}$$

i.e., the bilinear form $A(\cdot,\cdot)$ is continuous. For coercivity, we have

Proof

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D space-fractional equations : Riesz fractional diffusion equation

$$
\begin{aligned}
A(u,u) &= 2c_{2\alpha}({}_{RL}\mathrm{D}^{\alpha}_{a,x}u, {}_{RL}\mathrm{D}^{\alpha}_{x,b}u) = 2c_{2\alpha}\cos(\alpha\pi)\|{}_{RL}\mathrm{D}^{\alpha}_{a,x}\tilde{u}\|^2_{L^2(\mathbb{R})} \\
&= \|{}_{RL}\mathrm{D}^{\alpha}_{a,x}\tilde{u}\|^2_{L^2(\mathbb{R})} = |u|^2_{H^\alpha(\Omega)} \\
&\geq C\|u\|^2_{H^\alpha(\Omega)},
\end{aligned}
$$

where $\tilde{u}$ is the zero extension of u outside Ω. Letting $v = U$ in the weak formulation yields

$$
(\partial_t U, U) + A(U,U) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\|^2 + A(U,U) = (f,U).
$$

Therefore, we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\|^2 + C_0\|U(t)\|^2_{H^\alpha(\Omega)} \leq \frac{1}{4\epsilon}\|f(t)\|^2 + \epsilon\|U(t)\|^2 \leq \frac{1}{4\epsilon}\|f(t)\|^2 + \epsilon\|U(t)\|^2_{H^\alpha(\Omega)}
$$

where ϵ is a suitable positive constant. By choosing $\epsilon = C_0/2$ and integrating over $(0,t)$

$$
\|U(t)\|^2 + C_0\int_0^t \|U(s)\|^2_{H^\alpha(\Omega)}\,\mathrm{d}s \leq \|u(0)\|^2 + \frac{1}{C_0}\int_0^t \|f(s)\|^2\,\mathrm{d}s.
$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D space-fractional equations : Riesz fractional diffusion equation

Next, we prove the uniqueness. Suppose that u and w are two solutions to the weak problem. If $e = u - w$ with $e(0) = 0$ then $(\partial_t e, v) + A(e, v) = 0$. Hence, we have

$$\|e(t)\|^2 + C_0 \int_0^t \|e(s)\|^2_{H^\alpha(\Omega)}\, \mathrm{d}s \le 0$$

which yields $e(t) = 0$, i.e., $u = w$.

Semi-discrete approximation

Find $u_h = u_h(\cdot, t) \in X^\alpha_{h0}$ such that

$$(\partial_t u_h, v) + A(u_h, v) = (I_h f, v), \quad \forall v \in X^\alpha_{h0}$$

with initial condition $u_h(0) = I_h u(0)$

$$v = \phi_j$$

$$(M)_{i,j} = (\phi_i, \phi_j)$$

$$u_h(t) \in X^\alpha_{h0}$$

$$M \frac{\mathrm{d}\mathbf{c}(t)}{\mathrm{d}t} + S\,\mathbf{c}(t) = \mathbf{F}(t)$$

$$\sum_{j=1}^{r-1} c_j(t)\phi_j(x)$$

$$(S)_{i,j} = A(\phi_i, \phi_j)$$

$$\mathbf{c}(t) = (c_1(t), c_2(t), \cdots, c_{N_r-1}(t))^T$$

$$(\mathbf{F}(t))_j = (I_h f(t), \phi_j)$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D space-fractional equations : Riesz fractional diffusion equation

Theorem

Let $1/2 < \alpha < 1$ and $t \in (0, T]$. There exists a unique solution $u_h(t) \in X_{h0}^{\alpha}(\Omega)$ to the semi-discrete problem such that

$$\|u_h(t)\|^2 + C\int_0^t \|u_h(s)\|^2_{H^\alpha(\Omega)}\,\mathrm{d}s \leq \|u_h(0)\|^2 + \frac{1}{C}\int_0^t \|I_h f(s)\|^2\,\mathrm{d}s$$

where C is a positive constant.

Theorem (Error estimate)

Let $1/2 < \alpha < 1$ and $t \in (0, T]$. If $f(t) \in H^{r+1}(\Omega)$ and $u(t) \in H_0^{\alpha}(\Omega) \cap H^{r+1}(\Omega)$ and u_h are solutions to the variational and semi-discrete problems, respectively then there exists a positive constant C such that

$$\left(\int_0^t \|u(s) - u_h(s)\|^2_{H^\alpha(\Omega)}\,\mathrm{d}s\right)^{1/2} \leq Ch^{r+1-\alpha}$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D space-fractional equations : Riesz fractional diffusion equation

Fully-discrete approximation

Find $u_h^n \in X_{h0}^r, n = 1, 2, \cdots, n_T$, such that

$$\begin{cases} (\delta_t u_h^{n-\frac{1}{2}}, v) + A(u_h^n, v) = (I_h f^n, v), \quad \forall v \in X_{h0}^\alpha, \\ u_h^0 = I_h \phi_0, \end{cases}$$

$$\delta_t u_h^{n-\frac{1}{2}} = \frac{u_h^n - u_h^{n-1}}{\Delta t}$$

CN fully-discrete approximation

Find $u_h^n \in X_{h0}^r, n = 1, 2, \cdots, n_T$, such that

$$u_h^{n-\frac{1}{2}} = \frac{u_h^n + u_h^{n-1}}{2}$$

$$\begin{cases} (\delta_t u_h^{n-\frac{1}{2}}, v) + A(u_h^{n-\frac{1}{2}}, v) = (I_h f(t_{n-\frac{1}{2}}), v), \quad \forall v \in X_{h0}^\alpha, \\ u_h^0 = I_h \phi_0, \end{cases}$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D space-fractional equations : Riesz fractional diffusion equation

Theorem

Let $1/2 < \alpha < 1$. If $f \in C([0,T], L^2(\Omega))$ and $u_h^k, k = 1, \dots, n_T$ is a solution to the fully-discrete problem then

$$||u_h^k||^2_{H^\alpha(\Omega)} \leq C||u_h^0||^2_{H^\alpha(\Omega)} + C\Delta t \sum_{n=1}^{k} ||f^n||^2$$

where C is a positive constant independent of n, h and Δt.

Theorem (Error estimate)

Let $1/2 < \alpha < 1$ and $m \geq r + 1$. If $\phi_0 \in H^m(\Omega)$, $f \in C([0,T], H^m(\Omega))$, $u_h^k, k = 1, \dots, n_T$ is the solution to the fully-discrete problem and $u \in H^2([0,T], H_0^1(\Omega) \cap H^m(\Omega))$ is the solution to the variational problem then there exists a positive constant C independent of k, h and Δt such that

$$||u(t_k) - u_h^k||_{H^\alpha(\Omega)} \leq C(\Delta t + h^{r+1-\alpha})$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D space-fractional equations : Riesz fractional diffusion equation

Theorem

Let $1/2 < \alpha < 1$. **If** $f \in C([0,T], L^2(\Omega))$ **and** $u_h^k, k = 1, \dots, n_T$ **is a solution to the CN fully-discrete problem then**

$$\|u_h^k\|_{H^\alpha(\Omega)}^2 \le C\|u_h^0\|_{H^\alpha(\Omega)}^2 + C\Delta t \sum_{n=1}^{k} \|f(t_{n-\frac{1}{2}})\|^2$$

where C **is a positive constant independent of** n, h **and** Δt.

Theorem (Error estimate)

Let $1/2 < \alpha < 1$ **and** $m \ge r + 1$. **If** $\phi_0 \in H^m(\Omega)$, $f \in C([0,T], H^m(\Omega))$, $u_h^k, k = 1, \dots, n_T$ **is the solution to the CN fully-discrete problem and** $u \in H^3([0,T], H_0^1(\Omega) \cap H^m(\Omega))$ **is the solution to the variational problem then there exists a positive constant** C **independent of** k, h **and** Δt **such that**

$$\|u(t_k) - u_h^k\|_{H^\alpha(\Omega)} \le C(\Delta t^2 + h^{r+1-\alpha})$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-fractional equations : RL/Caputo fractional diffusion equation

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\gamma} u = K_\gamma \partial_x^2 u + g(x,t), & (x,t) \in \Omega \times (0,T] \\ u(x,0) = \phi_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \end{cases}$$

$$\Omega = (a,b),\ K_\gamma > 0 \text{ and } 0 < \gamma < 1$$

Semi-discrete approximation

Find $u_h(t) \in X_{h0}^r$, such that

$$\begin{cases} (\partial_t u_h, v) + K_\gamma({}_{RL}\mathrm{D}_{0,t}^{1-\gamma} \partial_x u_h, \partial_x v) = (I_h f, v), & v \in X_{h0}^r, \\ \begin{cases} ({}_C\mathrm{D}_{0,t}^{\gamma} u_h, v) + K_\gamma(\partial_x u_h, \partial_x v) = (I_h g, v), & v \in X_{h0}^r, \\ u_h(0) = \Pi_h^{1,0} \phi_0(x). \end{cases} \end{cases}$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-fractional equations : RL/Caputo fractional diffusion equation

$$M\frac{\mathrm{d}\mathbf{c}(t)}{\mathrm{d}t} + K_\gamma S\,{}_{RL}D_{0,t}^{1-\gamma}\mathbf{c}(t) = \mathbf{F}(t)$$

$$M\,{}_C\mathrm{D}_{0,t}^{\gamma}\mathbf{c}(t) + K_\gamma S\,\mathbf{c}(t) = \mathbf{G}(t)$$

$$\mathbf{c}(t) = (c_1(t), c_2(t), \cdots, c_{Nr-1}(t))^T$$

$$(S)_{i,j} = (\partial_x\phi_i, \partial_x\phi_j)$$

$$(M)_{i,j} = (\phi_i, \phi_j)$$

$$(\mathbf{F}(t))_j = (f(t), \phi_j)$$

$$(\mathbf{G}(t))_j = (g(t), \phi_j)$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-fractional equations : RL/Caputo fractional diffusion equation

Theorem (Error estimate)

Let $1/2 < \gamma < 1$ **and** $t \in (0, T]$. **If** $f \in L^2([0,T], H^{r+1}(\Omega))$, $u_h(t)$ **is the solution to the semi-discrete problem and** $u \in C^1([0,T], H_0^1(\Omega) \cap H^{r+1}(\Omega))$ **is the solution to the RL problem then there exists a positive constant** C **such that**

$$\|u_h(t) - u(t)\|^2 \leq Ch^{2r+2}\|u(t)\|^2_{H^{r+1}(\Omega)} + Ch^{2r+2}\int_0^t \left(\|\partial_s u(s)\|^2_{H^{r+1}(\Omega)} + \|f(s)\|^2_{H^{r+1}(\Omega)}\right) \mathrm{d}s$$

Theorem (Error estimate)

Let $1/2 < \gamma < 1$ **and** $t \in (0, T]$. **If** $g \in L^2([0,T], H^{r+1}(\Omega))$, $u_h(t)$ **is the solution to the semi-discrete problem and** $u \in C^1([0,T], H_0^1(\Omega) \cap H^{r+1}(\Omega))$ **is the solution to the Caputo problem then there exists a positive constant** C **such that**

$$\int_0^t \|{}_C\mathrm{D}_{0,s}^{\gamma/2}(u_h(s) - u(s))\|^2 \mathrm{d}s \leq Ch^{2r+2}$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-fractional equations : RL/Caputo fractional diffusion equation

CN fully-discrete approximation for the RL problem

Find $u_h^n \in X_{h0}^r$ for $n = 0, 1, \cdots, n_T - 1$, such that

$$\begin{cases} (\delta_t u_h^{n+\frac{1}{2}}, v) = \dfrac{\mu}{\tau^{1-\beta}}\Big[-b_0(\partial_x u_h^{n+\frac{1}{2}}, \partial_x v) + \displaystyle\sum_{k=1}^{n}(b_{n-k} - b_{n-k+1})(\partial_x u_h^{k-\frac{1}{2}}, \partial_x v) \\ \qquad + (b_n - B_n)(\partial_x u_h^{\frac{1}{2}}, \partial_x v) + A_n(\partial_x u_h^0, \partial_x v)\Big] + (I_h f(t_{n+\frac{1}{2}}), v), \quad \forall v \in X_{h0}^r, \\ u_h^0 = \Pi_h^{1,0}\phi_0, \end{cases}$$

$$A_n = B_n - \frac{\gamma(n+1/2)^{\gamma-1}}{\Gamma(1+\gamma)}$$

$$b_n = \frac{1}{\Gamma(1+\gamma)}\Big[(n+1)^\gamma - n^\gamma\Big], \quad B_n = \frac{2}{\Gamma(1+\gamma)}\Big[(n+1/2)^\gamma - n^\gamma\Big]$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-fractional equations : RL/Caputo fractional diffusion equation

Theorem

Let $1/2 < \gamma < 1$. If $f \in C([0,T], L^2(\Omega))$ and u_h^n is the solution to the (CN) fully-discrete problem then there is a positive constant C independent of n, h and Δt such that

$$\|u_h^n\|^2 \leq \|u_h^0\|^2 + \Delta t^\gamma K_\gamma \|\partial_x u_h^0\|^2 + C \max_{0\leq k\leq n_T} \|f^k\|^2.$$

Theorem (Error estimate)

If $1/2 < \gamma < 1$, $f \in C([0,T], H^{r+1}(\Omega))$, u_h^n is the solution to the fully-discrete problem and $u \in C^3([0,T], H_0^1(\Omega) \cap H^{r+1}(\Omega))$ is the solution to the RL problem then there exists a positive constant C such that

$$\|u_h^n - u(t_n)\| \leq C(\Delta t + h^{r+1}).$$

$$O(\Delta t^{1+\gamma} + h^{r+1})$$

FEM for fractional PDEs (FPDEs)

1D time-fractional equations : RL/Caputo fractional diffusion equation

L1 method for the Caputo problem

Find $u_h^n \in X_{h0}^r$ for $n = 0, 1, \cdots, n_T - 1$, such that

$$\begin{cases} (\delta_t^{(\gamma)} u_h^n, v) + K_\gamma(\partial_x u_h^n, \partial_x v) = (I_h g^n, v) \\ u_h^0 = \Pi_h^{1,0} \phi_0, \end{cases}$$

$$\delta_t^{(\gamma)} u_h^n = \frac{1}{\Delta t^\gamma} \sum_{k=0}^{n-1} b_{n-k}^{(\gamma)} (u_h^{k+1} - u_h^k)$$

$$b_k^{(\gamma)} = \frac{1}{\Gamma(2-\gamma)} \left[(k+1)^{1-\gamma} - k^{1-\gamma} \right]$$

FEM for fractional PDEs (FPDEs)

1D time-fractional equations : RL/Caputo fractional diffusion equation

Theorem

Let $1/2 < \gamma < 1$**. If** $f \in C([0,T], L^2(\Omega))$ **and** u_h^n **is the solution to the L1 method then there is a positive constant** C **independent of** n, h **and** Δt **such that**

$$\|u_h^n\|^2 \leq 2\|u_h^0\|^2 + 2C \max_{0 \leq k \leq n_T} \|f^k\|^2.$$

Theorem (Error estimate)

If $1/2 < \gamma < 1$**,** $f \in C([0,T], H^{r+1}(\Omega))$**,** u_h^n **is the solution to the fully-discrete problem and** $u \in C^3([0,T], H_0^1(\Omega) \cap H^{r+1}(\Omega))$ **is the solution to the Caputo problem then there exists a positive constant** C **such that**

$$\|u_h^n - u(t_n)\| \leq C(\Delta t^{2-\gamma} + h^{r+1}).$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-fractional equations : RL/Caputo fractional diffusion equation

GL method for the Caputo problem

Find $u_h^n \in X_{h0}^r$ for $n = 0, 1, \cdots, n_T - 1$, such that

$$\begin{cases} (\delta_t^{(\gamma)}(u_h^n - u_h^0), v) = K_\gamma(\partial_x u_h^n, \partial_x v) + (I_h g^n, v), & v \in X_{h0}^r, \\ u_h^0 = \Pi_h^{1,0}\phi_0, & O(\Delta t + h^{r+1}) \end{cases}$$

$$\delta_t^{(\gamma)}(u_h^n - u_h^0) = \frac{1}{\Delta t^\gamma}\sum_{k=0}^{n} \omega_{n-k}^{(\gamma)}(u_h^n - u_h^0), \quad \omega_k^{(\gamma)} = (-1)^k \binom{\gamma}{k}.$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-fractional equations : RL/Caputo fractional diffusion equation

FLMM-FEM-III for the Caputo problem

Find $u_h^n \in X_{h0}^r$ for $n = 0, 1, \cdots, n_T - 1$, such that

$$\begin{cases} \dfrac{1}{\Delta t^\gamma} \displaystyle\sum_{k=0}^{n} \omega_k^{(\gamma)} (u_h^{n-k} - u_h^0, v) = -\dfrac{K_\gamma}{2^\beta} \displaystyle\sum_{k=0}^{n} (-1)^k \omega_k^{(\gamma)} (\partial_x u_h^{n-k}, \partial_x v) - K_\gamma B_n^{(1)} (\partial_x u_h^0, \partial_x v) \\ \quad - K_\gamma C_n^{(1)} (\partial_x (u_h^1 - u_h^0), \partial_x v) + \dfrac{1}{\Delta t^\gamma} \displaystyle\sum_{k=0}^{n} \omega_{n-k}^{(\gamma)} (G^{n-k}, v), \quad v \in X_{h0}^r, \\ u_h^0 = \Pi_h^{1,0} \phi_0, \end{cases}$$

$$\omega_k^{(\gamma)} = (-1)^k \binom{\gamma}{k}, \; G^n = \left[\mathrm{D}_{0,t}^{-\gamma} I_h g(t) \right]_{t=t_n}$$

$$\|u_h^n\|^2 + \frac{1}{2} \Delta t^\gamma \|\partial_x u_h^n\|^2 \le C_0 \left(\|u_h^0\|^2 + \Delta t^\gamma \|\partial_x u_h^0\|^2 \right) + C_1 \max_{0 \le k \le n_T} \|g^k\|^2$$

$$\|u_h^n - u(t_n)\| \le C(\Delta t^2 + h^{r+1})$$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D ... ion

Example

Errors at $t = 1$ for the CN fully-discrete approximation with $N = \frac{1}{h} = 1000$

β	$1/\Delta t$	L^∞-error	order	L^2-error	order
0.25	16	4.1782e−3		3.0566e−3	
	32	1.8315e−3	1.1898	1.3386e−3	1.1912
	64	7.8822e−4	1.2164	5.7592e−4	1.2168
	128	3.3579e−4	1.2310	2.4535e−4	1.2311
	256	1.4227e−4	1.2389	1.0395e−4	1.2389
0.5	16	1.2444e−3		9.1043e−4	
	32	4.7937e−4	1.3762	3.5020e−4	1.3784
	64	1.7888e−4	1.4222	1.3067e−4	1.4222
	128	6.5551e−5	1.4483	4.7884e−5	1.4483
	256	2.3765e−5	1.4638	1.7359e−5	1.4638
0.75	16	1.6258e−4		1.1881e−4	
	32	6.4940e−5	1.3239	4.7437e−5	1.3246
	64	2.3415e−5	1.4717	1.7099e−5	1.4721
	128	8.0019e−6	1.5490	5.8424e−6	1.5493
	256	2.6623e−6	1.5877	1.9436e−6	1.5878

$O(\Delta t^{1+\gamma} + h^{r+1})$

$\|\varepsilon^n\|_\infty = \ldots x_i))^2\Big)^{1/2}$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D t... tion

Errors at $t = 1$ for the CN fully-discrete approximation with $\Delta t = 0.0001$

$r = 1$

$O(\Delta t^{1+\gamma} + h^{r+1})$

β	$N = 1/h$	L^∞-error	order	L^2-error	order
0.25	4	2.9246e−3		2.1485e−3	
	8	7.2782e−4	2.0066	5.3600e−4	2.0031
	16	1.8404e−4	1.9835	1.3462e−4	1.9933
	32	4.7114e−5	1.9658	3.4450e−5	1.9664
	64	1.2889e−5	1.8700	9.4175e−6	1.8711
0.5	4	2.9765e−3		2.1870e−3	
	8	7.3959e−4	2.0088	5.4475e−4	2.0053
	16	1.8598e−4	1.9915	1.3605e−4	2.0014
	32	4.6570e−5	1.9977	3.4057e−5	1.9982
	64	1.1727e−5	1.9896	8.5690e−6	1.9907
0.75	4	2.9801e−3		2.1897e−3	
	8	7.4041e−4	2.0090	5.4536e−4	2.0055
	16	1.8612e−4	1.9921	1.3615e−4	2.0020
	32	4.6532e−5	1.9999	3.4028e−5	2.0004
	64	1.1645e−5	1.9985	8.5093e−6	1.9996

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-fractional equations : RL/Caputo fractional diffusion equation

$0 < \beta \leq 2/3$

L^∞ **errors at** $t = 1$ **with** $h = 0.001$

method	$1/\Delta t$	$\beta = 0.4$	$\beta = 0.5$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1$
CNFDM	16	1.9912e−3	1.9437e−3	3.6981e−3	4.3373e−3	4.9609e−3
	32	8.1666e−4	6.3689e−4	1.1981e−3	1.3732e−3	1.5280e−3
	64	3.2417e−4	2.0120e−4	3.6055e−4	4.0218e−4	4.3742e−4
	128	1.2651e−4	6.3193e−5	1.0259e−4	1.1219e−4	1.2048e−4
	256	4.8861e−5	2.3015e−5	2.8182e−5	3.0469e−5	3.2450e−5
CNFEM	16	2.0946e−3	1.2444e−3	1.4399e−4	1.3778e−4	2.0685e−4
	32	8.4560e−4	4.7937e−4	2.8191e−5	3.0204e−5	5.1681e−5
	64	3.3272e−4	1.7888e−4	1.1328e−5	6.6085e−6	1.2893e−5
	128	1.2907e−4	6.5551e−5	4.0791e−6	1.4141e−6	3.1999e−6
	256	4.9656e−5	2.3765e−5	1.4023e−6	2.8688e−7	7.8039e−7

$2/3 < \beta \leq 1$

Numerical solution of fractional PDEs

FEM for fractional PDEs (FPDEs)

1D time-fractional equations : RL/Caputo fractional diffusion equation

Example

$$ {}_C D_{0,t}^{\gamma} u = K_{\gamma} \partial_x^2 u + g(x,t), \quad (x,t) \in \Omega \times (0,T] $$

$$ u = (t^{2+\beta} + t + 2) \sin(2\pi x) $$

L^2 **errors at** $t = 1$ **for the FLMM-FEM-III with** $\mathrm{r} = 3, N = 1000$

$1/\Delta t$	$\beta = 0.1$	order	$\beta = 0.5$	order	$\beta = 0.9$	order
32	1.8316e−5		8.4548e−5		4.9286e−5	
64	4.5797e−6	1.999	2.0821e−5	2.021	1.2378e−5	1.993
128	1.1438e−6	2.001	5.1989e−6	2.001	3.1014e−6	1.996
256	2.8575e−7	2.001	1.3003e−6	1.999	7.7612e−7	1.998
512	7.1316e−8	2.002	3.2515e−7	1.999	1.9403e−7	2.000

Thanks for your attention

Any question?