Numerical solution of fractional PDEs

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Forth lecture (108 slides)

Numerical solution of fractional PDEs

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- Numerical methods for fractional integral and derivatives
- Numerical methods for fractional ODEs
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Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

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- 1D space-fractional equations
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Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

- 2D time-fractional equations
  - Time-fractional diffusion equation with RL derivative in time
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- 2D time-space fractional equations
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  - Time-space fractional diffusion equation with RL derivative in time

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

$$\begin{cases} \partial_t U = {}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\left(K_\gamma \partial_x^2 U\right) + f(x,t), & (x,t) \in (a,b)\times(0,T], \\ U(x,0) = \phi_0(x), & x \in (a,b), \\ U(a,t) = U_a(t),\ U(b,t) = U_b(t), & t \in (0,T], \end{cases}$$

$0 < \gamma < 1$

$K_\gamma > 0$

$$\Delta t = \frac{T}{n_T} \qquad t_n = n\Delta t, n = 0,1,\dots,n_T \qquad t_{n+\frac{1}{2}} = \frac{t_n +, t_{n+1}}{2}, n = 0,1,\dots,n_T - 1$$

$$\Delta x = \frac{b-a}{N} \qquad x_i = a + i\Delta x, i = 0,1,\dots,N \qquad x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}, i = 0,1,\dots,N-1$$

$$I = (a,b) \qquad U \in \mathrm{C}(\mathrm{I}\times[0,T]) \qquad U^n = U^n(\cdot) = U(\cdot,t_n) \qquad U_i^n = U(x_i,t_n)$$

$$\delta_x U_{i+\frac{1}{2}}^n = \frac{U_{i+1}^n - U_i^n}{\Delta x} \qquad \delta_x^2 U_i^n = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2} \qquad \delta_t U_i^{n+\frac{1}{2}} = \frac{U_i^{n+1} - U_i^n}{\Delta t}$$

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1D time-fractional equations : RL subdiffusion equations

$$\partial_t U(x_i,t_n) = K_\gamma \left({}_{RL}D_{0,t}^{1-\gamma}\partial_x^2 U\right)(x_i,t_n) + f(x_i,t_n)$$

$$\partial_t U(x_i,t_n) = \frac{U(x_i,t_{n+1}) - U(x_i,t_n)}{\Delta t} + O(\Delta t) = \delta_t U_i^{n+\frac{1}{2}} + O(\Delta t)$$

$$\left({}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\partial_x^2 U\right)(x_i,t_n) = {}^{GL}\delta_t^{(1-\gamma)}(\partial_x^2 U^n(x_i)) + O(\Delta t)$$

$${}^{GL}\delta_t^{(\gamma)} U^n = \frac{1}{\Delta t^\gamma}\sum_{k=0}^{n}\omega_{n-k}^{(\gamma)} U^k, \qquad \omega_k^{(\gamma)} = (-1)^k\binom{\gamma}{k}$$

$$\partial_x^2 U(x_i,t_n) = \partial_x^2 U^n(x_i) = \delta_x^2 U_i^n + O(\Delta x^2)$$

$$\delta_t U_i^{n+\frac{1}{2}} = K_\gamma\, {}^{GL}\delta_t^{(1-\gamma)}\delta_x^2 U_i^n + f_i^n + O(\Delta t + \Delta x^2)$$

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1D time-fractional equations : RL subdiffusion equations

Explicit (forward) Euler method

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = K_\gamma \, {}^{GL}\delta_t^{(1-\gamma)}(\delta_x^2 u_i^n) + f_i^n, \quad i = 1,2,\cdots,N-1, \quad n = 0,1,\cdots,n_T-1, \\ u_i^0 = \phi_0(x_i), \quad i = 0,1,2,\cdots,N, \\ u_0^n = U_a(t_n),\ u_N^n = U_b(t_n), \end{cases}$$

$$ {}^{GL}\delta_t^{(\gamma)} U^n = \frac{1}{\Delta t^\gamma} \sum_{k=0}^{n} \omega_{n-k}^{(\gamma)} U^k, \qquad \omega_k^{(\gamma)} = (-1)^k \binom{\gamma}{k}$$

Remark

To construct similar sch ulae for approximating the RL fractiona

Explicit (forward) Euler method

$$u_i^{n+1} = u_i^n + \mu \sum_{k=0}^{n} \omega_{n-k}^{(1-\gamma)} (u_{i+1}^k - 2u_i^k + u_{i-1}^k) + \Delta t f_i^n$$

$$\mu = \frac{K_\gamma \Delta t^\gamma}{\Delta x^2}$$

$$ {}^{L1}_{RL}\delta_t^{(\gamma)} U^n = \frac{1}{\Delta t^\gamma} \left(\sum_{k=0} b_{n-k-1}^{(\gamma)} (U^{k+1} - U^k) + \frac{1}{\Gamma(1-\gamma)} U^0 \right), \quad b_k^{(\gamma)} = \frac{\ldots}{\Gamma(2-\gamma)}$$

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1D time-fractional equations : RL subdiffusion equations

Theorem

Explicit Euler method is stable if $\mu \leq \frac{1}{2^{2-\gamma}}$.

Proof

$f_i^n = 0$ leads to $u_i^{n+1} = u_i^n + \mu \sum_{k=0}^{n} \omega_{n-k}^{(1-\gamma)} (u_{i+1}^k - 2u_i^k + u_{i-1}^k)$. Setting $u_i^k = \rho_k e^{j\sigma i \Delta x}$ where $j^2 = -1$ deduces that $\rho_{n+1} = \rho_n - 4\mu \sin^2\left(\frac{\sigma \Delta x}{2}\right) \sum_{k=0}^{n} \omega_{n-k}^{(1-\gamma)} \rho_k$. Assuming that $\rho_{n+1} = \xi(\sigma)\rho_n$ where ξ is independent of time, we have

$$\xi = 1 - 4\mu \sin^2\left(\frac{\sigma \Delta x}{2}\right) \sum_{k=0}^{n} \omega_k^{(1-\gamma)} \xi^{-k}.$$

If $|\xi| > 1$ for some σ then ρ_n grows to infinity and the method is unstable. Considering the extreme value $\xi = -1$, we obtain the following stability bound on μ

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1D time-fractional equations : RL subdiffusion equations

$$\mu \sin^2\left(\frac{\sigma \Delta x}{2}\right) \le \frac{1}{2\sum_{k=0}^{n} \omega_k^{(1-\gamma)}(-1)^{-k}} \equiv S_{\gamma,n}.$$

This bound depends on n, *i.e.* the number of iterations. Nevertheless, this dependence is weak, i.e. $S_{\gamma,n}$ approaches $S_\gamma = \lim_{n\to\infty} S_{\gamma,n}$ in the form of oscillations with small decaying amplitudes. Since

$$\sum_{k=0}^{\infty} \omega_k^{(1-\gamma)} z^{-k} = (1-z^{-1})^{1-\gamma} = w_1^{(1-\gamma)}(z^{-1})$$

then the explicit Euler method is stable as long as

$$\mu \sin^2\left(\frac{\sigma \Delta x}{2}\right) \le S_\gamma = \frac{1}{2w_1^{(1-\gamma)}(-1)}.$$

Fractional von Neumann stability analysis

Since $\sin^2(\frac{\sigma \Delta x}{2}) \le 1$, we can give a more conservative but simple bound

$$\mu = \frac{K_\gamma \Delta t^\gamma}{\Delta x^2} \le S_\gamma = \frac{1}{2w_1^{(1-\gamma)}(-1)} = \frac{1}{2^{2-\gamma}}.$$

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1D time-fractional equations : RL subdiffusion equations

Remark

If the RL fractional derivative is discretized by the FBDF formula, someone can obtain a series of explicit methods. For example, for $p = 2$ with $w_2^{(1-\gamma)}(z) = (\frac{3}{2} - 2z + \frac{z^2}{2})^{1-\gamma}$, it can be deduced that the explicit Euler method is stable when

$$\mu = \frac{K_\gamma \Delta t^\gamma}{\Delta x^2} \leq S_\gamma = \frac{1}{2w_2^{(1-\gamma)}(-1)} = \frac{1}{4^{3/2-\gamma}}.$$

Remark

Using simple manipulations, someone can obtain that the local truncation error of the explicit Euler method is $O(\Delta t(\Delta t + \Delta x^2))$. It is a little difficult to get the global truncation error.

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1D time-fractional equations : RL subdiffusion equations

Example

$$\begin{cases} \partial_t u = {}_{RL}\mathrm{D}_{0,t}^{1-\gamma} u + f(x,t), & (x,t) \in (0,\pi) \times (0,1], \\ u(x,0) = 0, & x \in [0,\pi], \\ u(0,t) = 0, \quad u(\pi,t) = 0, & t \in [0,1], \end{cases}$$

$$f = \left(2.5t^{1.5} + \frac{\Gamma(3.5)}{\Gamma(3.5-\alpha)} t^{2.5-\alpha}\right)\sin(x)$$

$$u(x,t) = t^{2.5}\sin(x)$$

Maximum L^2 error

$$\max_{0 \le n \le n_T} \|\mathbf{u} - \mathbf{U}^n\|_N = \max_{0 \le n \le n_T} \sqrt{\Delta x \sum_{j=0}^{N-1} (U(x_j,t_n) - u_j^n)^2}$$

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Maximum L^2 error of the explicit Euler method with $\Delta t = 1/40000$

N	$\gamma = 0.2$	$\frac{\Delta t^\gamma}{\Delta x^2}$	$2^{\gamma-2}$	$\gamma = 0.5$	$\frac{\Delta t^\gamma}{\Delta x^2}$	$2^{\gamma-2}$	$\gamma = 0.8$	$\gamma = 0.95$
10	NaN	1.22		3.7919e−3	0.05		2.9621e−3	2.5676e−3
20	NaN	4.87		9.8971e−4	0.20		7.8085e−4	6.8137e−4
30	NaN	10.95	0.29	NaN	0.46	0.35	3.7647e−4	3.3157e−4
40	NaN	19.47		NaN	0.81		2.3491e−4	2.0911e−4
50	NaN	30.42		NaN	1.27		1.6938e−4	1.5242e−4

$$\frac{\Delta t^\gamma}{\Delta x^2} \leq 2^{\gamma-2}$$

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$$\left({}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\partial_x^2 U\right)(x_i,t_n) = {}^{GL}\delta_t^{(1-\gamma)}(\partial_x^2 U^n(x_i)) + O(\Delta t),$$

$$\partial_x^2 U(x_i,t_n) = \delta_x^2 U_i^n + O(\Delta x^2),$$

$$\delta_t U_i^{n-\frac{1}{2}} = K_\gamma\, {}^{GL}\delta_t^{(1-\gamma)}(\delta_x^2 U_i^n) + f_i^n + O(\Delta t + \Delta x^2).$$

Implicit (backward) Euler method

$$\begin{cases} \delta_t u_i^{n-\frac{1}{2}} = K_\gamma\, {}^{GL}\delta_t^{(1-\gamma)}\delta_x^2 u_i^n + f_i^n, \quad i = 1,2,\cdots,N-1, \\ u_i^0 = \phi_0(x_i), \quad i = 0,1,2,\cdots,N, \\ u_0^n = U_a(t_n),\ u_N^n = U_b(t_n), \end{cases}$$

$${}^{GL}\delta_t^{(\gamma)} U^n = \frac{1}{\Delta t^\gamma}\sum_{k=0}^{n}\omega_{n-k}^{(\gamma)} U^k, \qquad \omega_k^{(\gamma)} = (-1)^k\binom{\gamma}{k}$$

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Implicit (backward) Euler method

$$u_i^n = u_i^{n-1} + \mu \sum_{k=0}^{n} \omega_{n-k}^{(1-\gamma)} (u_{i+1}^k - 2u_i^k + u_{i-1}^k) + \Delta t f_i^n$$

$$\mu = K_\gamma \Delta t^\gamma / \Delta x^2$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix}_{(N-1)\times 2}$$

$$S_{N-1} = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix}_{(N-1)\times(N-1)}$$

$$\underline{\mathbf{u}}^n = (u_1^n, \cdots, u_{N-1}^n)^T$$

$$\underline{\mathbf{f}}^n = (f_1^n, \cdots, f_{N-1}^n)^T$$

$$\mathbf{u}_b^n = (u_0^n, u_N^n)^T = (U_a(t_n), U_b(t_n))^T$$

$$(E_{N-1} - \mu S_{N-1})\underline{\mathbf{u}}^n = \underline{\mathbf{u}}^{n-1} + \mu \sum_{k=0}^{n-1} \omega_{n-k}^{(1-\gamma)} (S_{N-1} \underline{\mathbf{u}}^k) + \mu \sum_{k=0}^{n} \omega_{n-k}^{(1-\gamma)} (B\mathbf{u}_b^k) + \underline{\mathbf{f}}^n$$

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Lemma

If $0 < \gamma < 1$ **and** $\omega_k^{(1-\gamma)} = (-1)^k \binom{1-\gamma}{k}$ **then**

$$\omega_0^{(1-\gamma)} = 1, \quad \omega_k^{(1-\gamma)} < 0,\ k > 0;$$

$$\sum_{k=0}^{\infty} \omega_k^{(1-\gamma)} = 0, \quad -\sum_{k=1}^{n} \omega_k^{(1-\gamma)} < 1,\ n \in \mathbb{N}.$$

Theorem

Implicit Euler method is unconditionally stable.

Proof

Similar to the previous Theorem, we obtain the following relation

$$\left(1 + 4\mu \sin^2\left(\frac{\sigma \Delta x}{2}\right)\right)\rho_n = \rho_{n-1} - 4\mu \sin^2\left(\frac{\sigma \Delta x}{2}\right) \sum_{k=1}^{n} \omega_k^{(1-\gamma)} \rho_{n-k}.$$

If $\mu^* = 4\mu \sin^2(\frac{\sigma \Delta x}{2})$ then we have

$$\rho_n = \frac{1}{1+\mu^*}\rho_{n-1} - \frac{\mu^*}{1+\mu^*} \sum_{k=1}^{n} \omega_k^{(1-\gamma)} \rho_{n-k}.$$

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For $n = 1$, it follows using the Lemma that

$$|\rho_1| = \frac{|1-\mu^*\omega_1^{(1-\gamma)}|}{1+\mu^*}|\rho_0| = \frac{1+\mu^*(1-\gamma)}{1+\mu^*}|\rho_0| \le |\rho_0|.$$

Suppose that $|\rho_k| \le |\rho_0|$ for $0 \le k \le n-1$. For $k = n$, we get from the Lemma

$$\begin{aligned}
|\rho_n| &\le \frac{1}{1+\mu^*}|\rho_{n-1}| + \frac{\mu^*}{1+\mu^*}\sum_{k=1}^{n}|\omega_{n-k}^{(1-\gamma)}||\rho_{n-k}| \\
&\le \frac{1}{1+\mu^*}|\rho_0| + \frac{\mu^*}{1+\mu^*}\sum_{k=1}^{n}|\omega_{n-k}^{(1-\gamma)}||\rho_0| \\
&= \frac{1}{1+\mu^*}|\rho_0| + \frac{\mu^*}{1+\mu^*}\left(-\sum_{k=1}^{n}\omega_{n-k}^{(1-\gamma)}\right)|\rho_0| \\
&\le \frac{1}{1+\mu^*}|\rho_0| + \frac{\mu^*}{1+\mu^*}|\rho_0| = |\rho_0|.
\end{aligned}$$

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$$\mathbf{u} = (u_0, u_1, \cdots, u_N)^T$$

$$\mathbf{u}^n = (u_0^n, u_1^n, \cdots, u_N^n)^T$$

$$(\mathbf{u}, \mathbf{v})_N = \Delta x \sum_{i=0}^{N-1} u_i v_i, \quad \mathbf{u}, \mathbf{v} \in R^{(N+1)\times 1}$$

$$||\mathbf{u}||_N = \sqrt{(\mathbf{u}, \mathbf{u})_N}$$

$$(\delta_x \mathbf{u}, \delta_x \mathbf{v})_N = \Delta x \sum_{i=0}^{N-1} \delta_x u_{i+\frac{1}{2}} \delta_x v_{i+\frac{1}{2}}$$

$$(\delta_x^2 \mathbf{u}, \mathbf{v})_N = \Delta x \sum_{i=1}^{N-1} (\delta_x^2 u_i) v_i$$

$$|\mathbf{u}|_{1,N} = \sqrt{(\delta_x \mathbf{u}, \delta_x \mathbf{u})_N}$$

Lemma

If $\boldsymbol{u} = (u_0, \dots, u_N)^T, \boldsymbol{v} = (v_0, \dots, v_N)^T$ **and** $u_0 = u_N = v_0 = v_N = 0$ **then**

$$(\delta_x^2 \mathbf{u}, \mathbf{v})_N = (\mathbf{u}, \delta_x^2 \mathbf{v})_N = -(\delta_x \mathbf{u}, \delta_x \mathbf{v})_N.$$

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Proof

It is easy to see that

$$(\delta_x^2\mathbf{u},\mathbf{v})_N = \Delta x\sum_{i=1}^{N-1} v_i\delta_x^2 u_i = \sum_{i=1}^{N-1} v_i(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}})$$

$$= \sum_{i=1}^{N-1} v_i\delta_x u_{i+\frac{1}{2}} - \sum_{i=0}^{N-2} v_{i+1}\delta_x u_{i+\frac{1}{2}}$$

$$= \sum_{i=0}^{N-1} v_i\delta_x u_{i+\frac{1}{2}} - \sum_{i=0}^{N-1} v_{i+1}\delta_x u_{i+\frac{1}{2}}$$

$$= -\sum_{i=0}^{N-1}(v_{i+1}-v_i)\delta_x u_{i+\frac{1}{2}} = -(\delta_x\mathbf{u},\delta_x\mathbf{v})_N.$$

One can similarly get $\left(\boldsymbol{u},\delta_x^2\boldsymbol{v}\right)_N = -(\delta_x\boldsymbol{u},\delta_x\boldsymbol{v})_N.$

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Lemma

If $\boldsymbol{u} = (u_0, \dots, u_N)^T$ and $u_0 = u_N = 0$ then there exists a positive constant C such that $\|\boldsymbol{u}\|_N \le C|\boldsymbol{u}|_{1,N}$.

Theorem (Energy method)

If $\boldsymbol{u}^n = (u_0^n, \dots, u_N^n)^T$ is the solution of the implicit Euler method, $u_0^n = u_N^n = 0$ and $\boldsymbol{f}^n = (f_0^n, \dots, f_N^n)^T$ then there exists a positive constant C independent of n, Δt and Δx such that

$$\|\mathbf{u}^n\|_N^2 \le \|\mathbf{u}^0\|_N^2 + \Delta t^\gamma K_\gamma |\mathbf{u}^0|_{1,N}^2 + C \max_{0\le k\le n_T} \|\mathbf{f}^k\|_N^2.$$

Proof

We rewrite the implicit Euler scheme in the following form

$$u_i^n = u_i^{n-1} + \Delta t^\gamma K_\gamma \sum_{k=0}^{n} \omega_{n-k}^{(1-\gamma)} \delta_x^2 u_i^k + \Delta t f_i^n.$$

From this relation, one can immediately get

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$$(\mathbf{u}^n,\mathbf{u}^n)_N=(\mathbf{u}^{n-1},\mathbf{u}^n)_N+\Delta t^\gamma K_\gamma\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma)}(\delta_x^2\mathbf{u}^k,\mathbf{u}^n)_N+\Delta t(\mathbf{f}^n,\mathbf{u}^n)_N$$

$$=(\mathbf{u}^{n-1},\mathbf{u}^n)_N-\Delta t^\gamma K_\gamma\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma)}(\delta_x\mathbf{u}^k,\delta_x\mathbf{u}^n)_N+\Delta t(\mathbf{f}^n,\mathbf{u}^n)_N,$$

where previous Lemma is used. Let

$$b_n=\sum_{k=0}^{n}\omega_k^{(1-\gamma)}=\frac{\Gamma(n+\gamma)}{\Gamma(\gamma)\Gamma(n+1)}=\frac{(n+1)^{\gamma-1}}{\Gamma(\gamma)}+O((n+1)^{-2+\gamma}).$$

Obviously $b_n-b_{n-1}=\omega_n^{(1-\gamma)}$ and it can be shown that $C_1 b_n\Delta t^\gamma\leq\Delta t\leq C_2 b_n\Delta t^\gamma$ where C_1 and C_2 positive constants independent of n. Using the Cauchy-Schwarz inequality we obtain

$$\|\mathbf{u}^n\|_N^2+\Delta t^\gamma K_\gamma|\mathbf{u}^n|_{1,N}\leq\frac{1}{2}(\|\mathbf{u}^{n-1}\|_N^2+\|\mathbf{u}^n\|_N^2)$$

$$+\frac{\Delta t^\gamma K_\gamma}{2}\sum_{k=0}^{n-1}(b_{n-k-1}-b_{n-k})(|\mathbf{u}^k|_{1,N}^2+|\mathbf{u}^n|_{1,N}^2)+\Delta t(\epsilon\|\mathbf{u}^n\|_N^2+\frac{1}{4\epsilon}\|\mathbf{f}^n\|_N^2)$$

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where ϵ is a suitable positive constant. Let

$$E^n = \|\mathbf{u}^n\|_N^2 + \Delta t^\gamma K_\gamma \sum_{k=0}^{n} b_{n-k} |\mathbf{u}^k|_{1,N}^2.$$

Then one gets

$$E^n + \Delta t^\gamma K_\gamma b_n |\mathbf{u}^n|_{1,N} \le E^{n-1} + \Delta t \left(\frac{1}{2\epsilon} \|\mathbf{f}^n\|_N^2 + 2\epsilon \|\mathbf{u}^n\|_N^2 \right)$$

$$\le E^{n-1} + \Delta t \left(\frac{1}{2\epsilon} \|\mathbf{f}^n\|_N^2 + 2C_2\epsilon \|\mathbf{u}^n\|_N^2 \right)$$

where the previous Lemma is utilized. Choosing suitable $\epsilon = \frac{K_\gamma}{2C_1C_2}$ satisfies

$$2C_2\epsilon\Delta t \le 2C_2\epsilon C_1 b_n \Delta t^\gamma \le K_\gamma b_n \Delta t^\gamma.$$

Therefore, one obtains

$$E^n \le E^{n-1} + C\Delta t \|\mathbf{f}^n\|_N^2 \le E^0 + C\Delta t \sum_{k=1}^{n} \|\mathbf{u}^k\|_N^2 = \|\mathbf{u}^0\|_N^2 + \Delta t^\gamma K_\gamma |\mathbf{u}^0|_{1,N}^2 + C\Delta t \sum_{k=1}^{n} \|\mathbf{f}^k\|_N^2.$$

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By the definition of E^n, we have $\|\mathbf{u}^n\|_N^2 \le \|\mathbf{u}^0\|_N^2 + \Delta t^\gamma K_\gamma |\mathbf{u}^0|_{1,N}^2 + C \max_{0\le k\le n_T} \|\mathbf{f}^k\|_N^2.$

Theorem

If U is the exact solution of the problem, u is the solution of the implicit Euler method, $e_i^n = u_i^n - U(x_i, t_n), i = 0, \dots, N, n = 1, \dots, n_T$ and $\boldsymbol{e}^n = (e_0^n, \dots, e_N^n)^T$ then there exists a positive constant C independent of n, Δt and Δx such that

$$\|\mathbf{e}^n\|_N \le C(\Delta t + \Delta x^2).$$

Proof

One can get the error equation as follows

$$e_i^n = e_i^{n-1} + \Delta t^\gamma K_\gamma \sum_{k=0}^{n} \omega_{n-k}^{(1-\gamma)} \delta_x^2 e_i^k + \Delta t R_i^n$$

in which $R_i^n = O(\Delta t + \Delta x^2)$ is the truncation error which obtained previously. Based on previous Theorem, to get the error bound, we only need to estimate

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$$\|\mathbf{e}^0\|_N^2 + \Delta t^\gamma K_\gamma |\mathbf{e}^0|_{1,N}^2 + C \max_{0\le k\le n_T} \|\mathbf{R}^k\|_N^2$$

where $\boldsymbol{R}^n = (R_0^n, \dots, R_N^n)^T$ with $R_i^n = O(\Delta t + \Delta x^2)$. Obviously, $\|\boldsymbol{e}^0\|_N = |\boldsymbol{e}^0|_{1,N} = 0$ and $\|\boldsymbol{R}^k\|_N \le C(\Delta t + \Delta x^2)$. Hence $\|\boldsymbol{e}^n\|_N \le C(\Delta t + \Delta x^2)$.

Implicit (backward) Euler method using L1 formula

$$\begin{cases} \delta_t u_i^{n-\frac{1}{2}} = K_\gamma {}_{RL}^{L1}\delta_t^{(1-\gamma)} \delta_x^2 u_i^n + f_i^n, \quad i = 1, 2, \cdots, N-1, \\ u_i^0 = \phi_0(x_i), \quad i = 0, 1, 2, \cdots, N, \\ u_0^n = U_a(t_n),\ u_N^n = U_b(t_n), \end{cases}$$

$${}_{RL}^{L1}\delta_t^{(\gamma)} U^n = \frac{1}{\Delta t^\gamma}\left(\sum_{k=0}^{n-1} b_{n-k-1}^{(\gamma)}(U^{k+1} - U^k) + \frac{n^{-\gamma}}{\Gamma(1-\gamma)} U^0\right), \quad b_k^{(\gamma)} = \frac{(k+1)^{1-\gamma} - k^{1-\gamma}}{\Gamma(2-\gamma)}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

Maximum L^2 error of the implicit Euler method with $N = 1000$

$1/\Delta t$	$\gamma = 0.2$	order	$\gamma = 0.5$	order	$\gamma = 0.8$	order
8	1.8195e−1		1.5777e−1		1.4051e−1	
16	8.9917e−2	1.0169	7.8387e−2	1.0092	7.0388e−2	0.9973
32	4.4687e−2	1.0087	3.9064e−2	1.0048	3.5224e−2	0.9988
64	2.2275e−2	1.0045	1.9499e−2	1.0025	1.7619e−2	0.9994
128	1.1120e−2	1.0022	9.7410e−3	1.0012	8.8112e−3	0.9997

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

Remark

Using the Fourier method or the energy method, we can similarly prove that the Implicit (backward) Euler method using L1 formula is unconditionally stable and convergent of order $O(\Delta t + \Delta x^2)$.

Remark

Cui proposed a compact finite difference scheme to solve the RL subdiffusion equation, in which the time discretization is the same as the implicit Euler method, while the space was discretized by a fourth-order compact finite difference scheme. An implicit method similar to the implicit Euler method which is unconditionally stable and convergent of order $O(\Delta t + \Delta x^2)$ has been presented by Chen for solving a fractional reaction-subdiffusion equation. Chen also constructed an explicit method in which the integer-order time derivative was discretized by the forward Euler method, i.e., $\delta_t u_i^{n-1/2}$ is replaced by $\delta_t u_i^{n+1/2}$.

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

$$\partial_t U(x_i, t_{n-\frac{1}{2}}) = K_\gamma \left({}_{RL}\mathrm{D}_{0,t}^{1-\gamma} \partial_x^2 U \right)(x_i, t_{n-\frac{1}{2}}) + f(x_i, t_{n-\frac{1}{2}})$$

$$\delta_t U_i^{n-\frac{1}{2}} = K_\gamma \delta_t^{(1-\gamma)} \delta_x^2 U_i^{n-\frac{1}{2}} + f(x_i, t_{n-\frac{1}{2}}) + (\Delta t^{1+\gamma} + \Delta x^2)$$

$$\delta_t^{(1-\gamma)} U_i^{n-\frac{1}{2}} = \frac{1}{\Delta t^{1-\gamma}} \left[b_0 U_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (b_{n-1-k} - b_{n-k}) U_i^{k-\frac{1}{2}} - (b_{n-1} - B_{n-1}) U_i^{\frac{1}{2}} - A_{n-1} U_i^0 \right]$$

$$b_n = \frac{1}{\Gamma(1+\gamma)} \left[(n+1)^\gamma - n^\gamma \right]$$

$$B_n = \frac{2}{\Gamma(1+\gamma)} \left[(n+1/2)^\gamma - n^\gamma \right]$$

$$A_n = B_n - \frac{\gamma (n+1/2)^{\gamma-1}}{\Gamma(1+\gamma)}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

First Crank-Nicolson (CN) method

$$\begin{cases} \delta_t u_i^{n-\frac{1}{2}} = K_\gamma \delta_t^{(1-\gamma)} \delta_x^2 u_i^{n-\frac{1}{2}} + f(x_i, t_{n-\frac{1}{2}}), \quad i = 1, 2, \cdots, N-1, \quad n = 1, 2, \cdots, n_T, \\ \delta_t^{(1-\gamma)} U_i^{n-\frac{1}{2}} = \dfrac{1}{\Delta t^{1-\gamma}} \left[b_0 U_i^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (b_{n-1-k} - b_{n-k}) U_i^{k-\frac{1}{2}} - (b_{n-1} - B_{n-1}) U_i^{\frac{1}{2}} - A_{n-1} U_i^0 \right] \end{cases}$$

Second Crank-Nicolson (CN) method

$$\begin{cases} \delta_t u_i^{n-\frac{1}{2}} = \dfrac{K_\gamma}{2} \left[\delta_t^{(1-\gamma)} \delta_x^2 u_i^{n-1} + \delta_t^{(1-\gamma)} \delta_x^2 u_i^n \right] + f(x_i, t_{n+\frac{1}{2}}), \\ \qquad i = 1, 2, \cdots, N-1, \quad n = 1, 2, \cdots, n_T, \\ u_i^0 = \phi_0(x_i), \quad i = 0, 1, 2, \cdots, N, \\ u_0^n = U_a(t_n), \; u_N^n = U_b(t_n), \end{cases}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

$$\partial_t U(x_i,t_n) = K_\gamma \left({}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\partial_x^2 U\right)(x_i,t_n) + f(x_i,t_n),$$

$$\partial_t U(x_i,t_{n-1}) = K_\gamma \left({}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\partial_x^2 U\right)(x_i,t_{n-1}) + f(x_i,t_{n-1})$$

$$\partial_t U(x_i,t_n) + \partial_t U(x_i,t_{n-1})$$
$$= K_\gamma\left[\left({}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\partial_x^2 U\right)(x_i,t_n) + \left({}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\partial_x^2 U\right)(x_i,t_{n-1})\right] + f(x_i,t_n) + f(x_i,t_{n-1})$$

$$\delta_t U_i^{n-\frac{1}{2}} = \frac{K_\gamma}{2}\left[{}_{RL}^{L1}\delta_t^{(1-\gamma)}\delta_x^2 U_i^n + {}_{RL}^{L1}\delta_t^{(1-\gamma)}\delta_x^2 U_i^{n-1}\right] + \frac{1}{2}(f_i^n + f_i^{n-1}) + O(\Delta t^{1+\gamma} + \Delta x^2), \qquad n > 1,$$

$$\delta_t U_i^{\frac{1}{2}} = K_\gamma {}_{RL}^{L1}\delta_t^{(1-\gamma)}\delta_x^2 U_i^1 + f_i^1 + O(\Delta t + \Delta x^2)$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

Third Crank-Nicolson (CN) method (using L1 formula)

$$\begin{cases} \delta_t u_i^{\frac{1}{2}} = K_\gamma \delta_t^{(1-\gamma)} \delta_x^2 u_i^1 + f_i^1, \quad i = 1, 2, \cdots, N-1, \\ \delta_t u_i^{n-\frac{1}{2}} = \dfrac{K_\gamma}{2} \left({}_{RL}^{L1}\delta_t^{(1-\gamma)} \delta_x^2 u_i^n + {}_{RL}^{L1}\delta_t^{(1-\gamma)} \delta_x^2 u_i^{n-1} \right) + \dfrac{1}{2}(f_i^n + f_i^{n-1}), \\ \qquad i = 1, 2, \cdots, N-1, \quad n = 2, 3, \cdots, n_T, \\ u_i^0 = \phi_0(x_i), \quad i = 0, 1, 2, \cdots, N, \\ u_0^n = U_a(t_n), \; u_N^n = U_b(t_n), \end{cases}$$

$${}_{RL}^{L1}\delta_t^{(\gamma)} U^n = \frac{1}{\Delta t^\gamma} \left(\sum_{k=0}^{n-1} b_{n-k-1}^{(\gamma)} (U^{k+1} - U^k) + \frac{n^{-\gamma}}{\Gamma(1-\gamma)} U^0 \right), \quad b_k^{(\gamma)} = \frac{(k+1)^{1-\gamma} - k^{1-\gamma}}{\Gamma(2-\gamma)}$$

Numerical solution of fractional PDEs

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1D time-fractional equations : RL subdiffusion equations

Lemma

If $0 < \gamma \leq 1$, $A_n = B_n - \frac{\gamma(n+1/2)^{\gamma-1}}{\Gamma(1+\gamma)\Delta t^{1-\gamma}}$ and $\boldsymbol{B}_n$ and b_n defined as before then

$$0 \leq b_n \leq b_{n-1}, \quad B_n \leq b_{n-1}, \quad n \geq 1;$$
$$A_n \leq B_n, \quad b_n \leq B_n, \quad n \geq 0,$$

$$\sum_{j=1}^{n} A_j \leq \frac{2^{\gamma}\Delta t^{\gamma-1}}{\Gamma(1-\gamma)}.$$

Theorem

If $\boldsymbol{u}^n = (u_0^n, \dots, u_N^n)^T$ is the solution of the first CN method, $u_0^n = u_N^n = 0$ and $\boldsymbol{f}^{n-1/2} = (0, f_1^{n-\frac{1}{2}}, \dots, f_{N-1}^{n-\frac{1}{2}}, 0)^T$ then

$$\|\mathbf{u}^{n+1}\|_N^2 \leq 2\|\mathbf{u}^0\|_N^2 + C_1 \Delta t^{\gamma} |\mathbf{u}^0|_{1,N}^2 + C_2 \Delta t \sum_{j=0}^{n} \|\mathbf{f}^{k+\frac{1}{2}}\|_N^2$$

where C_1 is a positive constant independent of $n, T, \Delta t$ and Δx and C_2 is a positive constant independent of $n, \Delta t$ and Δx.

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

If $\delta_t \boldsymbol{u}^{n+1/2} = (\delta_t u_0^{n+1/2}, \dots, \delta_t u_N^{n+1/2})^T$ and $\boldsymbol{u}^{n+1/2} = (u_0^{n+1/2}, \dots, u_N^{n+1/2})^T$ then from one of the previous Lemmas, we have

$$(\delta_t \mathbf{u}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N = \mu\Big[-b_0(\delta_x \mathbf{u}^{n+\frac{1}{2}}, \delta_x \mathbf{u}^{n+\frac{1}{2}})_N + \sum_{j=2}^{n-1}(b_{n-j} - b_{n+1-j})(\delta_x \mathbf{u}^{j-\frac{1}{2}}, \delta_x \mathbf{u}^{n+\frac{1}{2}})_N$$

$$+ (b_{n-1} - B_n)(\delta_x \mathbf{u}^{\frac{1}{2}}, \delta_x \mathbf{u}^{n+\frac{1}{2}})_N + A_n(\delta_x \mathbf{u}^0, \delta_x \mathbf{u}^{n+\frac{1}{2}})_N \Big] + (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N.$$

Using Cauchy–Schwarz inequality and previous Lemma, we get

Proof

$$(\delta_t \mathbf{u}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N$$

$$\leq \frac{\mu}{2}\Big[-2b_0|\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2 + \sum_{j=2}^{n-1}(b_{n-j} - b_{n+1-j})(|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2 + |\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2)$$

$$+ (b_{n-1} - B_n)(|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2 + |\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2) + A_n(|\mathbf{u}^0|_{1,N}^2 + |\mathbf{u}^{n-\frac{1}{2}}|_{1,N}^2)\Big] + (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

$$=\frac{\mu}{2}\Big[(-b_0-B_n+A_n)|\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2+\sum_{j=2}^{n-1}(b_{n-j}-b_{n+1-j})|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2$$

$$+(b_{n-1}-B_n)|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2+A_n|\mathbf{u}^0|_{1,N}^2\Big]+(\mathbf{f}^{n+\frac{1}{2}},\mathbf{u}^{n+\frac{1}{2}})_N$$

$$\leq\frac{\mu}{2}\Big[-b_0|\mathbf{u}^{n+\frac{1}{2}}|_{1,N}^2+\sum_{j=1}^{n-1}(b_{n-j}-b_{n+1-j})|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2+A_n|\mathbf{u}^0|_{1,N}^2\Big]+(\mathbf{f}^{n+\frac{1}{2}},\mathbf{u}^{n+\frac{1}{2}})_N.$$

Or

$$||\mathbf{u}^{n+1}||_N^2+\mu\Delta t\sum_{j=1}^{n+1}b_{n+1-j}|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2$$

$$\leq||\mathbf{u}^n||_N^2+\mu\Delta t\sum_{j=1}^{n}b_{n-j}|\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2+\mu\Delta tA_n|\mathbf{u}^0|_{1,N}^2+2\Delta t(\mathbf{f}^{n+\frac{1}{2}},\mathbf{u}^{n+\frac{1}{2}})_N.$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

If $E^{n+1} = \|\mathbf{u}^{n+1}\|_N^2 + \mu\Delta t \sum_{j=1}^{n+1} b_{n+1-j} |\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2$ then

$$
\begin{aligned}
E^{n+1} &\le E^n + \mu\Delta t |\mathbf{u}^0|_{1,N}^2 A_n + 2\Delta t (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N \\
&\le E^{n-1} + \mu\Delta t |\mathbf{u}^0|_{1,N}^2 (A_n + A_{n-1}) + 2\Delta t \left[(\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}})_N + (\mathbf{f}^{n-\frac{1}{2}}, \mathbf{u}^{n-\frac{1}{2}})_N \right] \\
&\le E^1 + \mu\Delta t |\mathbf{u}^0|_{1,N}^2 \sum_{j=1}^{n} A_n + 2\Delta t \sum_{k=1}^{n} (\mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{k+\frac{1}{2}})_N \\
&\le E^1 + \mu\Delta t |\mathbf{u}^0|_{1,N}^2 \sum_{j=1}^{n} A_n + \epsilon\mu\Delta t \sum_{j=1}^{n+1} b_{n+1-j} |\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2 + \sum_{j=1}^{n+1} \frac{\Delta t}{\epsilon\mu b_{n+1-j}} \|\mathbf{f}^{j-\frac{1}{2}}\|_N^2.
\end{aligned}
$$

Here we have used the Cauchy–Schwarz inequality and $\epsilon\|\mathbf{u}^{j-\frac{1}{2}}\|_N^2 \le |\mathbf{u}^{j-\frac{1}{2}}|_{1,N}^2$ using one of the previous Lemmas. ϵ is a suitable positive constant independent of $j, \Delta t$ and Δx. Therefore, we have

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

$$||\mathbf{u}^{n+\frac{1}{2}}||_N^2 \leq E^1 + \mu\Delta t^{\gamma}||\mathbf{u}^0||^2 \sum_{j=1}^{n} A_n + C\Delta t \sum_{j=1}^{n+1} ||\mathbf{f}^{j-\frac{1}{2}}||_N^2$$

where $1/b_n \leq C_\gamma n^{1-\gamma}\Delta t^{1-\gamma} \leq C_\gamma T^{1-\gamma}$ in which C_γ only depends on γ. Letting $n = 0$ and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&||\mathbf{u}^1||_N^2 + \mu\Delta t B_0|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2 \\
=&||\mathbf{u}^0||_N^2 + \mu\Delta t A_0(\delta_x\mathbf{u}^{\frac{1}{2}}, \delta_x\mathbf{u}^0)_N + 2\Delta t(\mathbf{f}^{\frac{1}{2}}, \mathbf{u}^{\frac{1}{2}})_N \\
\leq&||\mathbf{u}^0||_N^2 + \mu\Delta t A_0\left(\epsilon_1|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2 + \frac{1}{4\epsilon_1}|\mathbf{u}^0|_{1,N}^2\right) + 2\Delta t\left(\frac{1}{4\epsilon_2}||\mathbf{f}^{\frac{1}{2}}||_N^2 + \epsilon_2||\mathbf{u}^{\frac{1}{2}}||_N^2\right)
\end{aligned}$$

where $\epsilon_1, \epsilon_2 > 0$ are suitable constants such that

$$\epsilon_1\mu A_0|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2 + 2\epsilon_2||\mathbf{u}^{\frac{1}{2}}||_N^2 \leq \frac{1}{2}\mu B_0|\mathbf{u}^{\frac{1}{2}}|_{1,N}^2.$$

Therefore, we obtain

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

$$E^1 = \|\mathbf{u}^1\|_N^2 + \mu\Delta t B_0 |\mathbf{u}^{\frac{1}{2}}|_{1,N}^2 \le 2\|\mathbf{u}^1\|_N^2 + \mu\Delta t B_0 |\mathbf{u}^{\frac{1}{2}}|_{1,N}^2$$

$$\le 2\|\mathbf{u}^0\|_N^2 + \frac{\mu\Delta t A_0}{2\epsilon_1}|\mathbf{u}^0|_{1,N}^2 + \frac{\Delta t}{\epsilon_2}\|\mathbf{f}^{\frac{1}{2}}\|_N^2.$$

Hence

$$\|\mathbf{u}^{n+1}\|_N^2 \le 2\|\mathbf{u}^0\|_N^2 + C_1\Delta t^\gamma |\mathbf{u}^0|_{1,N}^2 + C_2\Delta t\sum_{j=0}^{n}\|\mathbf{f}^{k+\frac{1}{2}}\|_N^2$$

in which Δt and Δx and the

$$\|\mathbf{e}^{n+1}\|_N^2 \le 2\|\mathbf{e}^0\|_N^2 + C_1\Delta t^\gamma |\mathbf{e}^0|_{1,N}^2 + C_2\Delta t\sum_{k=0}^{n}\|\mathbf{R}^k\|_N^2 \le C(\Delta t^{1+\gamma} + \Delta x^2)$$

$$e_i^n = U(x_i, t_n) - u_i^n$$

$$\delta_t e_i^{n-\frac{1}{2}} = K_\gamma \delta_t^{(1-\gamma)}\delta_x^2 e_i^{n-\frac{1}{2}} + R_i^n$$

$$R_i^n = O(\Delta t^{1+\gamma} + \Delta x^2)$$

$$e_0^n = e_N^n = 0$$

$$e_i^0 = 0, i = 0, 1, \cdots, N$$

$$\mathbf{R}^n = (R_0^n, R_1^n, \cdots, R_N^n)^T$$

$$\boldsymbol{e}^n = (e_0^n, \dots, e_N^n)^T$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

Remark

In a similar manner, we can show that the third CN method is unconditionally stable and convergent of order $O(\Delta t^{\min\{2-\gamma/2+\gamma+1\}} + \Delta x^2)$. For the second CN method, one can prove that it is unconditionally stable if the time fractional operator is discretized by the following Grünwald–Letnikov formula

$$ {}^{GL}\delta_t^{(\gamma)} U^n = \frac{1}{\Delta t^\gamma} \sum_{k=0}^{n} \omega_{n-k}^{(\gamma)} U^k, \qquad \omega_k^{(\gamma)} = (-1)^k \binom{\gamma}{k}. $$

But the convergence rate is not very satisfactory for the smooth enough solutions. We can see that the first CN type method is more natural than the other two CN methods.

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

Maximum L^2 error of the first CN method with $N = 1000$

$1/\Delta t$	$\gamma = 0.2$	order	$\gamma = 0.5$	order	$\gamma = 0.8$	order
8	4.2600e−2		9.6759e−3		2.6338e−3	
16	2.0452e−2	1.0586	4.4424e−3	1.1231	4.8959e−4	2.4275
32	9.3682e−3	1.1264	1.8185e−3	1.2886	9.4388e−5	2.3749
64	4.1927e−3	1.1599	7.0398e−4	1.3692	1.8835e−5	2.3252
128	1.8536e−3	1.1775	2.6416e−4	1.4141	8.4538e−6	1.1558

Numerical solution of fractional PDEs

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1D time-fractional equations : RL subdiffusion equations

θ-method (weighted average method)

$$\begin{cases} \delta_t u_i^{n-\frac{1}{2}} = K_\gamma \left[(1-\theta) \delta_t^{(1-\gamma)} \delta_x^2 u_i^n + \theta \delta_t^{(1-\gamma)} \delta_x^2 u_i^{n-1} \right] + f(x_i, t_{n-\frac{1}{2}}), \\ i = 1, 2, \cdots, N-1, n = 1, 2, \cdots, n_T \end{cases}$$

$${}_{RL}D_{0,t}^{1-\gamma}$$

Remark

If $\theta = 1/2$, θ-method is reduced to the second CN method. The stability was studied by using the *fractional von Neumann analysis* when $\delta_t^{(1-\gamma)} = {}_p^B \delta_t^{(1-\gamma)}$. The θ-method is unconditionally stable for $0 \le \theta \le 1/2$ and stable for $1/2 < \theta \le 1$ under the condition $\frac{K_\gamma \Delta t^\gamma}{\Delta x^2} \le 1/(2(2\theta - 1) w_p^{(\gamma-1)}(-1))$ where

$$w_p^{(-\gamma)}(z) = \left(w_p(z)\right)^\gamma, \quad w_p(z) = \sum_{j=1}^{p} \frac{1}{j}(1-z)^j, \quad p = 1, 2, \cdots, 6.$$

If $\theta = 0$ (or $\theta = 1$), the explicit (or implicit) Euler method is recovered.

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1D time-fractional equations : RL subdiffusion equations

$$\partial_t U = {}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\left(K_\gamma \partial_x^2 U\right) + f(x,t)$$

$$F_i^n = \int_{t_{n-1}}^{t_n} f(x_i, s)\,\mathrm{d}s$$

$$U(x_i,t_n) = U(x_i,t_{n-1}) + K_\gamma\left\{\left[\mathrm{D}_{0,t}^{-(1-\gamma)}\partial_x^2 U(x_i,t)\right]_{t=t_n} - \left[\mathrm{D}_{0,t}^{-(1-\gamma)}\partial_x^2 U(x_i,t)\right]_{t=t_{n-1}}\right\} + F_i^n$$

Using left fractional rectangular formula

$$U_i^n = U_i^{n-1} + \Delta t^\gamma K_\gamma\left[\sum_{k=1}^{n} b_{n-k}^{(1-\gamma)}\delta_x^2 U_i^{k-1} - \sum_{k=1}^{n-1} b_{n-k-1}^{(1-\gamma)}\delta_x^2 U_i^{k-1}\right] + F_i^n + O(\Delta t(\Delta t + \Delta x^2))$$

New explicit method

$$\begin{cases} u_i^n = u_i^{n-1} + \Delta t^\gamma K_\gamma\left[\sum_{k=1}^{n} b_{n-k}^{(1-\gamma)}\delta_x^2 u_i^{k-1} - \sum_{k=1}^{n-1} b_{n-k-1}^{(1-\gamma)}\delta_x^2 u_i^{k-1}\right] + F_i^n, \quad i = 1,2,\cdots,N-1, \\ u_i^0 = \phi_0(x_i), \quad i = 0,1,2,\cdots,N, \\ u_0^n = U_a(t_n), \quad u_N^n = U_b(t_n), \end{cases}$$

$$O(\Delta t + \Delta x^2)$$

$$\frac{K_\gamma \Delta t^\gamma}{\Delta x^2} \le 1/4$$

$$b_k^{(1-\gamma)} = \frac{1}{\Gamma(1+\gamma)}\left[(k+1)^\gamma - k^\gamma\right]$$

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1D time-fractional equations : RL subdiffusion equations

New implicit method (using right fractional rectangular formula)

$$\begin{cases} u_i^n = u_i^{n-1} + \Delta t^\gamma K_\gamma \left[\sum_{k=1}^{n} b_{n-k}^{(1-\gamma)} \delta_x^2 u_i^k - \sum_{k=1}^{n-1} b_{n-k-1}^{(1-\gamma)} \delta_x^2 u_i^k \right] + F_i^n, \quad i = 1, 2, \cdots, N-1, \\ a_{k,n} = \dfrac{1}{\Gamma(\gamma+2)} \begin{cases} (n-1)^{\gamma+1} - (n-1-\gamma) n^\gamma, & k = 0, \\ (n-k+1)^{\gamma+1} + (n-1-k)^{\gamma+1} - 2(n-k)^{\gamma+1}, & 1 \le k \le n-1, \\ 1, & k = n. \end{cases} \end{cases}$$

Unconditionally stable

$$\begin{cases} u_i^n = u_i^{n-1} + \Delta t^\gamma K_\gamma \left(\sum_{k=0}^{n} a_{k,n} \delta_x^2 u_i^k - \sum_{k=0}^{n-1} a_{k,n-1} \delta_x^2 u_i^k \right) + F_i^n, \quad i = 1, 2, \cdots, N-1, \\ u_i^0 = \phi_0(x_i), \quad i = 0, 1, 2, \cdots, N, \\ u_0^n = U_a(t_n), \; u_N^n = U_b(t_n), \end{cases}$$

Unconditionally stable

$O(\Delta t^2 + \Delta x^2)$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

Maximum L^2 error of the new explicit method with $\Delta t = 1/40000$

N	$\gamma = 0.2$	$\gamma = 0.5$	$\gamma = 0.8$	$\gamma = 0.95$
10	NaN	3.7998e−3	2.9649e−3	2.5682e−3
20	NaN	9.9768e−4	7.8359e−4	6.8199e−4
30	NaN	NaN	3.7921e−4	3.3219e−4
40	NaN	NaN	2.3765e−4	2.0973e−4
50	NaN	NaN	1.7212e−4	1.5304e−4

$$\frac{K_\gamma \Delta t^\gamma}{\Delta x^2} \leq 1/4$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

Maximum L^2 error of the new implicit method with $N = 1000$

$1/\Delta t$	$\gamma = 0.2$	order	$\gamma = 0.5$	order	$\gamma = 0.8$	order
8	2.2569e−1		2.4567e−1		2.4628e−1	
16	1.1822e−1	0.9328	1.2830e−1	0.9372	1.2735e−1	0.9515
32	6.1127e−2	0.9516	6.5855e−2	0.9622	6.4805e−2	0.9746
64	3.1352e−2	0.9633	3.3465e−2	0.9767	3.2703e−2	0.9867
128	1.5994e−2	0.9710	1.6904e−2	0.9852	1.6431e−2	0.9930

Maximum L^2 error of the new CN method with $N = 1000$

$1/\Delta t$	$\gamma = 0.2$	order	$\gamma = 0.5$	order	$\gamma = 0.8$	order
8	4.0242e−3		3.1760e−3		3.0590e−3	
16	9.8681e−4	2.0279	7.8837e−4	2.0103	7.7210e−4	1.9862
32	2.4196e−4	2.0280	1.9645e−4	2.0047	1.9479e−4	1.9868
64	5.9575e−5	2.0220	4.9238e−5	1.9963	4.9222e−5	1.9846
128	1.4937e−5	1.9958	1.2563e−5	1.9706	1.2582e−5	1.9680

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : RL subdiffusion equations

Remark

The new explicit, implicit and CN methods known as integration methods can be used to solve other fractional equations, such as the *fractional Fokker–Planck equation*

$$\partial_t U = {}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\left(\partial_x\left(d(x)\partial_x U\right) + f(x,t)\right)$$

the *fractional cable equation*

$$\partial_t U = {}_{RL}\mathrm{D}_{0,t}^{1-\gamma_1}\left(K_1\partial_x^2 U\right) - K_2\,{}_{RL}\mathrm{D}_{0,t}^{1-\gamma_2} U + f(x,t),\ 0<\gamma_1,\gamma_2<1$$

and the *Stokes' first problem for a heated generalized second grade fluid with fractional derivative*

$$\partial_t U = {}_{RL}\mathrm{D}_{0,t}^{1-\gamma_1}\left(K_1\partial_x^2 U\right) + K_2\,{}_{RL}\mathrm{D}_{0,t}^{1-\gamma_2} U + f(x,t),\ 0<\gamma_1,\gamma_2<1.$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

$\mathrm{D}_{0,t}^{-(1-\gamma)}$

$$\begin{cases} \partial_t U = {}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\left(K_\gamma \partial_x^2 U\right) + f(x,t), & (x,t) \in (a,b)\times(0,T], \\ U(x,0) = \phi_0(x), \quad x \in (a,b), \\ U(a,t) = U_a(t),\ U(b,t) = U_b(t), & t \in (0,T], \end{cases}$$

$0 < \gamma < 1$

$K_\gamma > 0$

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\gamma} U = K_\gamma \partial_x^2 U + g(x,t), & (x,t) \in (a,b)\times(0,T], \\ U(x,0) = \phi_0(x), \quad x \in (a,b), \\ U(a,t) = U_a(t),\ U(b,t) = U_b(t), & t \in (0,T], \end{cases}$$

$g(x,t) = D_{0,t}^{\gamma-1} f(x,t)$

Remark

If U is suitably smooth in time then ${}_C\mathrm{D}_{0,t}^{\gamma} U(x,t) = {}_{RL}\mathrm{D}_{0,t}^{\gamma}(U(x,t) - U(x,0))$. Hence, a natural way to discretize the above Caputo derivative is to use the Grünwald–Letnikov approximation, the L1 method, the FLMM and so on. The space derivative is discretized by the classical methods such as the central difference method or any compact difference method.

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

$$ {}_C\mathrm{D}_{0,t}^{\gamma} U(x_i,t_n) = K_\gamma \partial_x^2 U(x_i,t_n) + g_i^n = K_\gamma \partial_x^2 U(x_i,t_{n-1}) + g_i^n + O(\Delta t) $$

$$ \delta_t^{(\gamma)} U_i^n = K_\gamma \delta_x^2 U_i^{n-1} + g_i^n + O(\Delta t + \Delta x^2) $$

Explicit Euler method (using GL formula)

$$ \begin{cases} \delta_t^{(\gamma)}(u_i^n - u_i^0) = K_\gamma \delta_x^2 u_i^{n-1} + g_i^n, \quad i = 1,2,\cdots,N-1, \\ u_i^0 = \phi_0(x_i), \quad i = 0,1,2,\cdots,N, \\ u_0^n = U_a(t_n),\ u_N^n = U_b(t_n), \end{cases} $$

$$ \delta_t^{(\gamma)}(u_i^n - u_i^0) = \frac{1}{\Delta t^\gamma} \sum_{k=0}^{n} \omega_{n-k}^{(\gamma)} (u_i^k - u_i^0), \quad \omega_k^{(\gamma)} = (-1)^k \binom{\gamma}{k} $$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

Explicit Euler method (using L1 formula)

$$\begin{cases} \delta_t^{(\gamma)} u_i^n = K_\gamma \delta_x^2 u_i^{n-1} + g_i^n, \quad i = 1, 2, \cdots, N-1, \\ u_i^0 = \phi_0(x_i), \quad i = 0, 1, 2, \cdots, N, \\ u_0^n = U_a(t_n), \; u_N^n = U_b(t_n), \end{cases}$$

$$\delta_t^{(\gamma)} u_i^n = \frac{1}{\Delta t^\gamma} \sum_{k=0}^{n-1} b_{n-k-1}^{(\gamma)} (u_i^{k+1} - u_i^k) = \frac{1}{\Delta t^\gamma} \left[b_0 u_i^n - \sum_{k=0}^{n-1} (b_{n-k-1}^{(\gamma)} - b_{n-k}^{(\gamma)}) u_i^k - b_n u_i^0 \right],$$

$$b_k^{(\gamma)} = \frac{1}{\Gamma(2-\gamma)} \left[(k+1)^{1-\gamma} - k^{1-\gamma} \right].$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

Theorem

The explicit Euler method (using GL formula) is stable if $\mu \leq \gamma/2$ where $\mu = \frac{K_\gamma \Delta t^\gamma}{\Delta x^2}$.

Proof

Let

$$A = \begin{bmatrix} \omega_1^{(\gamma)} + 2\mu & -\mu & 0 & \cdots & 0 & 0 \\ -\mu & \omega_1^{(\gamma)} + 2\mu & -\mu & \cdots & 0 & 0 \\ 0 & -\mu & \omega_1^{(\gamma)} + 2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_1^{(\gamma)} + 2\mu & -\mu \\ 0 & 0 & 0 & \cdots & -\mu & \omega_1^{(\gamma)} + 2\mu \end{bmatrix}_{(N-1)\times(N-1)}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

$$B = \begin{bmatrix} 1-2\mu & \mu & 0 & \cdots & 0 & 0 \\ \mu & 1-2\mu & \mu & \cdots & 0 & 0 \\ 0 & \mu & 1-2\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-2\mu & \mu \\ 0 & 0 & 0 & \cdots & \mu & 1-2\mu \end{bmatrix}_{(N-1)\times(N-1)}$$

and suppose that for $i = 1, \dots, N-1$, $\tilde{u}_i^n$ and $\tilde{g}_i^n$ are perturbations of u_i^n and g_i^n, respectively and $\tilde{\boldsymbol{u}}^n = (\tilde{u}_1^n, \dots, \tilde{u}_{N-1}^n)^T$ and $\tilde{\boldsymbol{g}}^n = (\tilde{g}_1^n, \dots, \tilde{g}_{N-1}^n)^T$. Then, we have

$$\sum_{k=0}^{n} \omega_{n-k}^{(\gamma)} (\tilde{u}_i^k - \tilde{u}_i^0) = \mu \tilde{u}_i^{n-1} (\tilde{u}_{i+1}^{n-1} - 2\tilde{u}_i^{n-1} + \tilde{u}_{i-1}^{n-1}) + \Delta t^\gamma \tilde{g}_i^n$$

Then the matrix representation of this perturbation equation is as follows

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

$$\begin{cases} \tilde{\mathbf{u}}^1 = B\tilde{\mathbf{u}}^0 + \Delta t^\gamma \tilde{\mathbf{g}}^1, \quad n = 1, \\ \tilde{\mathbf{u}}^n = -A\tilde{\mathbf{u}}^{n-1} - \sum_{k=1}^{n-2} \omega_{n-k}^{(\gamma)} \tilde{\mathbf{u}}^k + \sum_{k=1}^{n} \omega_{n-k}^{(\gamma)} \tilde{\mathbf{u}}^0 + \Delta t^\gamma \tilde{\mathbf{g}}^n, \quad n > 1. \end{cases}$$

Since $2\mu \le \gamma = -\omega_1^\gamma$, it is easy to obtain $\|A\| \le -\omega_1^\gamma$ and $\|B\| \le 1$ by the Greschgorin's theorem. Here $\|A\|$ denotes the spectral norm (or 2-norm) of the matrix A, which is equal to the absolute largest eigenvalue of A when A is symmetric. For convenience, we set

$$b_{n-1} = \sum_{k=0}^{n-1} \omega_k^{(\gamma)} = \frac{\Gamma(n-\gamma)}{\Gamma(1-\gamma)\Gamma(n)} = \frac{n^{-\gamma}}{\Gamma(1-\gamma)} + O(n^{-1-\gamma}).$$

Then one can easily prove that $\Delta t^\gamma \le C b_{n-1}$, where C is a positive constant only dependent on γ and T. Next, we prove that

$$\|\tilde{\mathbf{u}}^n\| \le \|\tilde{\mathbf{u}}^0\| + C \max_{1 \le n \le n_T} \|\tilde{\mathbf{g}}^n\| = E.$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

where $\|\cdot\|$ is the discrete L^2 norm for the vector, which is defined by

$$\|\mathbf{u}\| = \left(\sum_{i=1}^{N-1} u_i^2\right)^{1/2}, \quad \mathbf{u} = (u_1, u_2, \cdots, u_{N-1})^T \in R^{N-1}.$$

Using the mathematical induction for $n = 1$, we get

$$\|\tilde{\mathbf{u}}^1\| = \|B\tilde{\mathbf{u}}^0 + \Delta t^\gamma \tilde{\mathbf{g}}^1\| \le \|B\|\|\tilde{\mathbf{u}}^0\| + Cb_0\|\tilde{\mathbf{g}}^1\| \le \|\tilde{\mathbf{u}}^0\| + C\|\tilde{\mathbf{g}}^1\| \le E.$$

Suppose that $\|\tilde{\boldsymbol{u}}^n\| \le E, n = 1, \dots, m-1$. For $n = m$, we have

$$\|\tilde{\mathbf{u}}^m\| \le -\omega_1^{(\gamma)} E - \sum_{k=1}^{m-2} \omega_{m-k}^{(\gamma)} E + b_{m-1} E$$

$$= b_0 E - \sum_{k=0}^{m-1} \omega_1^{(\gamma)} E + b_{m-1} E = E.$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

Remark

From this Theorem, we can obtain that the explicit Euler method (using GL formula) is convergent with order $O(\Delta t + \Delta x^2)$ if $\frac{K_\gamma \Delta t^\gamma}{\Delta x^2} \leq -\frac{1}{2}\omega_1^{(\gamma)} \leq \frac{\gamma}{2}$. One can simil onally stabl

$$\delta_t^{(\gamma)}(u_i^n - u_i^0) = \frac{1}{\Delta t^\gamma}\sum_{k=0}^{n}\omega_{n-k}^{(\gamma)}(u_i^n - u_i^0), \quad \omega_k^{(\gamma)} = (-1)^k\binom{\gamma}{k}$$

Implicit Euler method (using GL formula)

$$\begin{cases} \delta_t^{(\gamma)}(u_i^n - u_i^0) = K_\gamma \delta_x^2 u_i^n + g_i^n, & i = 1, 2, \cdots, N-1, \\ u_i^0 = \phi_0(x_i), & i = 0, 1, 2, \cdots, N, \\ u_0^n = U_a(t_n),\ u_N^n = U_b(t_n), \end{cases}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

Implicit Euler method (using L1 formula)

$$\begin{cases} \delta_t^{(\gamma)} u_i^n = K_\gamma \delta_x^2 u_i^n + g_i^n, \quad i = 1, 2, \cdots, N-1, \\ u_i^0 = \phi_0(x_i), \quad i = 0, 1, 2, \cdots, N, \\ u_0^n = U_a(t_n),\ u_N^n = U_b(t_n), \end{cases}$$

$$\delta_t^{(\gamma)} u_i^n = \frac{1}{\Delta t^\gamma} \sum_{k=0}^{n-1} b_{n-k}^{(\gamma)} (u_i^{k+1} - u_i^k)$$

$$b_k^{(\gamma)} = \frac{1}{\Gamma(2-\gamma)} \left[(k+1)^{1-\gamma} - k^{1-\gamma} \right]$$

Remark

It is easy to prove that the implicit Euler method (using GL formula) and implicit Euler method (using L1 formula) are unconditionally stable using the Fourier method or the energy method, and are convergent of order $O(\Delta t + \Delta x^2)$ and $O(\Delta t^{2-\gamma} + \Delta x^2)$, respectively.

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

$$ {}_0^C D_t^\alpha u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),\ x \in (0,1), t \in [0,T] $$

$$ \mu = (\Delta t)^\alpha \Gamma(2-\alpha) $$

$$ \gamma_0^{(\alpha)} u(x,t_k) - \mu \frac{\partial^2 u(x,t_k)}{\partial x^2} = \gamma_{k-1}^{(\alpha)} u(x,t_0) + \sum_{i=1}^{k-1} (\gamma_{i-1}^{(\alpha)} - \gamma_i^{(\alpha)}) u(x,t_{k-i}) + \mu f(x,t_k) + R_k $$

$$ |R_k| = |R(u(x,t_k))| \le \frac{\alpha}{\Gamma(1-\alpha)} \left\{ 12\,(t_k - t_1)^{-\alpha-1} \max_{t_0 \le t \le t_1} \left| \frac{\partial^2 u}{\partial t^2}(x,t) \right| \Delta t^3 \right. $$

Implicit Euler method (using L1-2-3 formula)

$$ \gamma_0^{(\alpha)} u^k - \mu \frac{\partial^2 u^k}{\partial x^2} = \gamma_{k-1}^{(\alpha)} u^0 + \sum_{i=1}^{k-1} (\gamma_{i-1}^{(\alpha)} - \gamma_i^{(\alpha)}) u^{k-i} + \mu f^k $$

$$ \left. + \left(\frac{\ }{2} + \frac{\ }{12} \frac{\ }{\prod_{i=1}^{4} (\alpha - i)} \right) \max_{t_0 \le t \le t_k} \left| \frac{\ }{\partial t^4}(x,t) \right| \Delta t^{\ } \right\}. $$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

$$
\begin{aligned}
|T(x,t_k)| \le \frac{\alpha}{\Gamma(1-\alpha)} \Bigg\{ & 12\,(t_k - t_1)^{-\alpha-1} \max_{t_0 \le t \le t_1} \left| \frac{\partial^2 u}{\partial t^2}(x,t) \right| \Delta t^3 \\
& + \frac{1}{8}(t_k - t_2)^{-\alpha-1} \max_{t_0 \le t \le t_2} \left| \frac{\partial^3 u}{\partial t^3}(x,t) \right| \Delta t^4 \\
& + \left(\frac{1}{2} + \frac{1}{12} \frac{27 - 10\alpha + \alpha^2}{\prod_{i=1}^{4} (\alpha - i)} \right) \max_{t_0 \le t \le t_k} \left| \frac{\partial^4 u}{\partial t^4}(x,t) \right| \Delta t^{4-\alpha} \Bigg\} \\
& + \frac{1}{12} \max_{0<x<1} \left| \frac{\partial^4 u}{\partial x^4}(x,t) \right| \Delta x^2, \qquad k \ge 3.
\end{aligned}
$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

Example

$$u(x,t) = e^x t^{4+\alpha} \qquad \Delta x = 1/20\,000$$

$$EU_j^N(\Delta t) = \max_{0\le i\le M}\left|U_i^N - (u_j)_i^N\right| \qquad \text{order}(u_j) = \log_2 \frac{EU_j^N(\Delta t)}{EU_j^N(\Delta t/2)}$$

α	Δt	$EU_1^N(\Delta t)$	order (u_1) $\simeq 2-\alpha$	$EU_2^N(\Delta t)$	order (u_2) $\simeq 3-\alpha$	$EU_3^N(\Delta t)$	order (u_3) $\simeq 4-\alpha$
0.1	1/10	9.668 9E−4	0.00	1.190 6E−4	0.00	1.078 6E−5	0.00
	1/20	3.015 3E−4	1.68	1.844 6E−5	2.69	7.897 7E−7	3.77
	1/40	9.111 2E−5	1.73	2.753 2E−6	2.74	5.707 0E−8	3.79
	1/80	2.694 9E−5	1.76	3.977 3E−7	2.79	4.114 3E−9	3.79
	1/160	7.849 2E−6	1.78	5.203 3E−8	2.93	3.281 4E−10	3.65

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

$$ {}_C\mathrm{D}_{0,t}^{\gamma} y(t) = \lambda y(t) + g(t), \quad y(0) = y_0, \quad 0 < \gamma < 1 $$

$$ \frac{1}{\Delta t^{\gamma}} \sum_{k=0}^{n} \omega_{n-k}(y(t_k) - y_0) = \lambda \sum_{k=0}^{n} \theta_{n-k}^{(m)} y(t_k) + \lambda B_n^{(m)} y_0 + \frac{1}{\Delta t^{\gamma}} \sum_{k=0}^{n} \omega_{n-k} G_k + R_n $$

$$ \begin{cases} {}_C\mathrm{D}_{0,t}^{\gamma} U = K_{\gamma} \partial_x^2 U + g(x,t), & (x,t) \in (a,b) \times (0,T], \\ U(x,0) = \phi_0(x), & x \in (a,b), \\ U(a,t) = U_a(t),\ U(b,t) = U_b(t), & t \in (0,T], \end{cases} \qquad g(x,t) = D_{0,t}^{\gamma-1} f(x,t) $$

$$ x = x_i \qquad t = t_n $$

$$ \frac{1}{\Delta t^{\gamma}} \sum_{k=0}^{n} \omega_{n-k}(U_i^k - U_i^0) = \lambda \sum_{k=0}^{n} \theta_{n-k}^{(m)} \delta_x^2 U_i^k + \lambda B_n^{(m)} \delta_x^2 U_i^0 + \frac{1}{\Delta t^{\gamma}} \sum_{k=0}^{n} \omega_{n-k} G_i^k + R_i^n $$

$$ \theta_n^{(1)} = \frac{1}{2^{\gamma}}(- \qquad B_n^{(m)} = \frac{1}{\Gamma(1+\gamma)} \sum_{k=0}^{n} \omega_{n-k} k^{\gamma} - \sum_{k=0}^{n} \theta_k^{(m)} = O(n^{-1}) \qquad m = 1, 2 $$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

$$\begin{cases} \dfrac{1}{\Delta t^{\gamma}}\left[\sum_{k=0}^{n}\omega_k(u_i^{n-k}-u_i^0)\right] = \dfrac{K_\gamma}{2^\gamma}\sum_{k=0}^{n}(-1)^k\omega_k\delta_x^2u_i^{n-k} + K_\gamma B_n^{(1)}\delta_x^2u_i^0 \\ \qquad + \dfrac{1}{\Delta t^{\gamma}}\sum_{k=0}^{n}\omega_{n-k}G_i^{n-k}, \end{cases}$$

FLMM-FDM I

unconditionally stable and convergent of order $O(\Delta t+\Delta x^2)$

$$\begin{cases} \dfrac{1}{\Delta t^{\gamma}}\left[\sum_{k=0}^{n}\omega_k(u_i^{n-k}-u_i^0)\right] = K_\gamma\left[(1-\frac{\gamma}{2})\delta_x^2u_i^n + \frac{\gamma}{2}\delta_x^2u_i^{n-1}\right] + K_\gamma B_n^{(2)}\delta_x^2u_i^0 \\ \qquad + \dfrac{1}{\Delta t^{\gamma}}\left[(1-\frac{\gamma}{2})G_i^n + \frac{\gamma}{2}G_i^{n-1}\right], \\ u_i^0 = \phi_0(x_i), \quad i=0,1,2,\cdots,N, \\ u_0^n = U_a(t_n),\ u_N^n = U_b(t_n), \end{cases}$$

FLMM-FDM II

$$\omega_k = (-1)^k\binom{\gamma}{k}$$

$$G_i^n = \left[\mathrm{D}_{0,t}^{-\gamma}g(x_i,t)\right]_{t=t_n}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

Improved FLMM-FDM I

$$\frac{1}{\Delta t^{\gamma}}\left[\sum_{k=0}^{n}\omega_k(u_i^{n-k}-u_i^0)\right]=\frac{K_\gamma}{2^\gamma}\sum_{k=0}^{n}(-1)^k\omega_k\delta_x^2u_i^{n-k}+K_\gamma B_n^{(1)}\delta_x^2u_i^0$$

$$+K_\gamma C_n^{(1)}\delta_x^2(u_i^1-u_i^0)+\frac{1}{\Delta t^{\gamma}}\sum_{k=1}^{n}\omega_{n-k}G_i^{n-k}$$

Improved FLMM-FDM II

$$\frac{1}{\Delta t^{\gamma}}\left[\sum_{k=0}^{n}\omega_k(u_i^{n-k}-u_i^0)\right]=K_\gamma\left[(1-\frac{\gamma}{2})\delta_x^2u_i^n+\frac{\gamma}{2}\delta_x^2u_i^{n-1}\right]+K_\gamma B_n^{(2)}\delta_x^2u_i^0$$

$$+K_\gamma C_n^{(2)}\delta_x^2(u_i^1-u_i^0)+\frac{1}{\Delta t^{\gamma}}\left[(1-\frac{\gamma}{2})G_i^n+\frac{\gamma}{2}G_i^{n-1}\right],$$

$$u_i^0=\phi_0(x_i),\quad i=0,1,2,\cdots,N,$$

$$u_0^n=U_a(t_n),\ u_N^n=U_b(t_n),$$

$$G_i^n=\left[\mathrm{D}_{0,t}^{\gamma}g(x_i,t)\right]_{t=t_n}$$

$$\omega_k=(-1)^k\binom{\gamma}{k}$$

unconditional stability and convergence of order $O(\Delta t^2+\Delta x^2)$, 2

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-fractional equations : Caputo subdiffusion equations

Example

Maximum L^2 error of the Improved-FLMM-FDM-I with $N = 5000$

$1/\Delta t$	$\gamma = 0.2$	order	$\gamma = 0.5$	order	$\gamma = 0.8$	order
16	3.8041e−4		6.2221e−4		6.2077e−4	
32	8.1801e−5	2.2174	1.0391e−4	2.5820	9.0134e−5	2.7839
64	1.7541e−5	2.2214	2.2495e−5	2.2077	2.2715e−5	1.9884
128	3.7303e−6	2.2334	5.7013e−6	1.9803	5.7557e−6	1.9806
256	7.6683e−7	2.2823	1.4901e−6	1.9359	1.5035e−6	1.9366

Maximum L^2 error of the Improved-FLMM-FDM-II with $N = 5000$

$1/\Delta t$						
16	3.8041e−4		6.2221e−4		6.2077e−4	
32	8.1801e−5	2.2174	1.0391e−4	2.5820	7.9829e−5	2.9591
64	1.7541e−5	2.2214	1.7223e−5	2.5930	1.0414e−5	2.9384
128	3.7303e−6	2.2334	2.8273e−6	2.6068	1.8163e−6	2.5194
256	7.6682e−7	2.2823	4.5046e−7	2.6500	5.1733e−7	1.8119

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D space-fractional equations

One-sided space-fractional diffusion equation

$$\begin{cases} \partial_t U = d(x)\,{}_{RL}\mathrm{D}^{\alpha}_{a,x} U + g(x,t), & (x,t) \in (a,b)\times(0,T], \\ U(x,0) = \phi_0(x), \quad x \in (a,b), & \boxed{d(x) > 0} \quad \boxed{1 < \alpha \le 2} \\ U(a,t) = U_a(t),\ U(b,t) = U_b(t), & t \in (0,T], \end{cases}$$

$$\left({}_{RL}\mathrm{D}^{\alpha}_{a,x} U\right)(x_i,t) = \frac{1}{\Delta x^{\alpha}} \sum_{j=0}^{i} \omega_j^{(\alpha)} U(t, x_{i-j}) + O(\Delta x)$$

Proposition

The explicit Euler method solution based on the GL approximation is unstable.

Let u_i^n be the approximate solutions. The explicit Euler method on the Günwald-Letnikov approximation is as follows

$$\boxed{\omega_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{d_i}{\Delta x^{\alpha}} \sum_{j=0}^{i} \omega_j^{(\alpha)} u_{i-j}^n + g_i^n, \quad i = 1, 2, \cdots, N-1$$

Proof

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D space-fractional equations
One-sided space-fractional diffusion equation

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x^\alpha} d_i \sum_{j=0}^{i} \omega_j^{(\alpha)} u_{i-j}^n + \Delta t g_i^n.$$

If $\underline{u}_i^0 = u_i^0 + \varepsilon_i^0$ then

$$\underline{u}_i^1 = \mu_i \underline{u}_i^0 + \frac{\Delta t}{\Delta x^\alpha} d_i \sum_{j=1}^{i} \omega_j^{(\alpha)} u_{i-j}^0 + \Delta t g_i^0 = \mu_i \varepsilon_i^0 + u_i^1$$

where

$$\mu_i = 1 + \frac{\Delta t}{\Delta x^\alpha} d_i$$

So, $\varepsilon_i^1 = \mu_i \varepsilon_i^0$ which leads to $\varepsilon_i^n = (\mu_i)^n \varepsilon_i^0$. Obviously, $\mu_i > 1$ (amplification factor) and ε_i^n will amplify when n increases. Hence, the method is unstable.

Numerical solution of fractional PDEs

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Remark

Similarly, one can show that the implicit Euler method (using GL formula) and CN method (using GL formula) are also unstable. The shifted GL formula

$$\left({}_{RL}\mathrm{D}_{a,x}^{\alpha}U\right)(x_i,t)=\frac{1}{\Delta x^{\alpha}}\sum_{j=0}^{i+p}\omega_j^{(\alpha)}U(t,x_{i-j}+p\Delta x)+C(p-\alpha/2)\Delta x+O(\Delta x^2)$$

can be used to overcome this drawback. Here C **is independent of** Δx**. Obviously, the best performance comes from minimizing** $|p-\alpha/2|$**. For** $1<\alpha\leq 2$**, the optimal choice is** $p=1$**.**

Numerical solution of fractional PDEs

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1D space-fractional equations
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Explicit (left) and implicit (right) Euler method (with shifted (1 shift) GL formula)

$$\delta_t u_i^{n+\frac{1}{2}} = d_{iL}\delta_x^{(\alpha)} u_{i+1}^n + g_i^n \quad i = 1, 2, \cdots, N-1 \qquad \delta_t u_i^{n+\frac{1}{2}} = d_{iL}\delta_x^{(\alpha)} u_{i+1}^{n+1} + g_i^{n+1}$$

$$u_i^0 = \phi_0(x_i), \quad i = 0, 1, 2, \cdots, N,$$

$$O(\Delta t + \Delta x) \qquad u_0^n = U_a(t_n),\ u_N^n = U_b(t_n), \qquad O(\Delta t + \Delta x)$$

$${}_L\delta_x^{(\alpha)} u_{i+1}^n = \frac{1}{\Delta x^\alpha} \sum_{j=0}^{i+1} \omega_j^{(\alpha)} u_{i+1-j}^n, \ \omega_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}$$

CN method (with shifted (1 shift) GL formula)

$$\delta_t u_i^{n+\frac{1}{2}} = d_{iL}\delta_x^{(\alpha)} u_{i+1}^{n+\frac{1}{2}} + g(x_i, t_{n+\frac{1}{2}}) \qquad O(\Delta t^2 + \Delta x)$$

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$$S_{N-1}^{(\alpha)} = \begin{pmatrix} \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{N-2}^{(\alpha)} & \omega_{N-3}^{(\alpha)} & \omega_{N-4}^{(\alpha)} & \cdots & \omega_0^{(\alpha)} \\ \omega_{N-1}^{(\alpha)} & \omega_{N-2}^{(\alpha)} & \omega_{N-3}^{(\alpha)} & \cdots & \omega_1^{(\alpha)} \end{pmatrix}_{(N-1)\times(N-1)}$$

$$B_L^{(\alpha)} = \begin{pmatrix} \omega_2^{(\alpha)} \\ \omega_3^{(\alpha)} \\ \vdots \\ \omega_{N-1}^{(\alpha)} \\ \omega_N^{(\alpha)} \end{pmatrix}_{(N-1)\times 1}, \quad B_R^{(\alpha)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \omega_0^{(\alpha)} \end{pmatrix}_{(N-1)\times 1}$$

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$$\mu = \frac{\Delta t}{\Delta x^{\alpha}},\ \underline{\mathbf{u}}^n = (u_1^n, \cdots, u_{N-1}^n)^T,\ \underline{\mathbf{g}}^n = (g_1^n, \cdots, g_{N-1}^n)^T$$

$$S = D_{N-1} S_{N-1}^{(\alpha)}, \quad D_{N-1} = diag(d_1, d_2, \cdots, d_{N-1})$$

$$B_L = B_0^{(\alpha)} .* d, \quad B_R = B_N^{(\alpha)} .* d, \quad d = (d_1, \cdots, d_{N-1})^T$$

Matrix form of the explicit, implicit and CN methods (with shifted GL formula)

$$\underline{\mathbf{u}}^{n+1} = (E + \mu S)\underline{\mathbf{u}}^n + \Delta t \underline{\mathbf{g}}^n + \mu(B_L u_0^n + B_R u_N^n)$$

$$(E - \mu S)\underline{\mathbf{u}}^{n+1} = \underline{\mathbf{u}}^n + \Delta t \underline{\mathbf{g}}^{n+1} + \mu(B_L u_0^{n+1} + B_R u_N^{n+1})$$

$$(E - \frac{\mu}{2} S)\underline{\mathbf{u}}^{n+1} = (E + \frac{\mu}{2} S)\underline{\mathbf{u}}^n + \frac{\Delta t}{2}(\underline{\mathbf{g}}^n + \underline{\mathbf{g}}^{n+1}) + \mu(B_L u_0^{n+\frac{1}{2}} + B_R u_N^{n+\frac{1}{2}})$$

Numerical solution of fractional PDEs

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Remark

Using the *Gerschgorin theorem*, we can obtain that the eigenvalues λ of the matrix S satisfy

$$|\lambda - d_i\omega_1^{(\alpha)}| \le d_i\omega_0^{(\alpha)} + d_i\sum_{j=2}^{i}\omega_j^{(\alpha)} \le -d_i\omega_1^{(\alpha)}$$

which implies

$$-2\alpha d_{\max} = 2d_{\max}\omega_1^{(\alpha)} \le 2d_i\omega_1^{(\alpha)} \le \lambda < 0 \qquad d_{\max} = \max_{0\le i\le N} d_i$$

Therefore, the eigenvalue of the matrix $E + \mu S$ lies in $[1 - 2\mu\alpha d_{\max}, 1]$, and the eigenvalue of the matrix $E - \mu S$ lies in $[1, 1 + 2\mu\alpha d_{\max}]$. Thus, we can obtain that the explicit Euler method is stable if

$$1 - 2\mu\alpha d_{\max} \ge -1 \quad \Longleftrightarrow \quad \frac{\Delta t}{\Delta x^{\alpha}} = \mu \le \frac{1}{\alpha d_{\max}}.$$

Since the eigenvalues of the matrix $E - \mu S$ are all equal to or greater than 1, the eigenvalues of $(E - \mu S)^{-1}$ are not greater than 1. Hence, the implicit Euler method is unconditionally stable.

Numerical solution of fractional PDEs

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1D space-fractional equations

One-sided space-fractional diffusion equation

Remark

For the Crank–Nicolson method, we have

$$\underline{\mathbf{u}}^{n+1} = (E - \frac{\mu}{2}S)^{-1}(E + \frac{\mu}{2}S)\underline{\mathbf{u}}^{n} + \underline{\mathbf{b}}^{n}$$

It is known that if λ is the eigenvalue of the matrix S then $(1-\mu\lambda)^{-1}(1+\mu\lambda)$ is the eigenvalue of $(E-\mu S)^{-1}(E+\mu S)$. Since λ has negative real part then $(1-\mu\lambda)^{-1}(1+\mu\lambda) < 1$. Hence, the CN method is unconditionally stable.

Remark

The explicit, implicit and CN methods are special cases of the following θ-method

If $\theta = 1/2$, θ-method is reduced to the CN method. The θ-method is unconditionally stable for $0 \le \theta \le 1/2$ and stable for $1/2 < \theta \le 1$ under the condition $\frac{\Delta t}{\Delta x^{\alpha}} \le 1/((2\theta-1)\alpha d_{\max})$. If $\theta = 0$ (or $\theta = 1$), the explicit (or implicit) method is recovered.

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D space-fractional equations

Two-sided space-fractional diffusion equation

$$\begin{cases} \partial_t U = c(x,t)\,{}_{RL}\mathrm{D}^{\alpha}_{a,x}U + d(x,t)\,{}_{RL}\mathrm{D}^{\alpha}_{x,b}U + g(x,t), & (x,t)\in(a,b)\times(0,T], \\ U(x,0)=\phi_0(x), \quad x\in(a,b), & \\ U(a,t)=U(b,t)=0, \quad t\in(0,T], & \end{cases}$$

$$1<\alpha<2$$

$$c(x,t),d(x,t)\geq 0$$

θ-method (with shifted GL formula)

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = \left[(1-\theta)c_i^{n+1}\,{}_L\delta_x^{(\alpha)}u_{i+1}^{n+1} + \theta c_i^n\,{}_L\delta_x^{(\alpha)}u_{i+1}^n\right] \\ \qquad + \left[(1-\theta)d_i^{n+1}\,{}_R\delta_x^{(\alpha)}u_{i-1}^{n+1} + \theta d_i^n\,{}_R\delta_x^{(\alpha)}u_{i-1}^n\right] \\ \qquad + (1-\theta)g_i^{n+1} + \theta g_i^n, \quad i=1,2,\cdots,N-1, \\ u_i^0 = \phi_0(x_i), \quad i=0,1,2,\cdots,N, \\ u_0^n = U_a(t_n),\ u_N^n = U_b(t_n), \end{cases}$$

Numerical solution of fractional PDEs

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1D space-fractional equations
Two-sided space-fractional diffusion equation

$$\mu = \frac{\Delta t}{\Delta x^{\alpha}},\ \mu_{c,i}^{n} = \mu c_i^n,\ \text{and}\ \mu_{d,i}^{n} = \mu d_i^n$$

θ-method (with shifted GL formula)

$$u_i^{n+1} - (1-\theta)\left[\mu_{c,i}^{n+1}\,{}_L\delta_x^{(\alpha)} u_i^{n+1} + \mu_{d,i}^{n+1}\,{}_R\delta_x^{(\alpha)} u_i^{n+1}\right]$$
$$= u_i^n + \theta\left[\mu_{c,i}^{n}\,{}_L\delta_x^{(\alpha)} u_i^{n} + \mu_{d,i}^{n}\,{}_R\delta_x^{(\alpha)} u_i^{n}\right] + \Delta t\left[(1-\theta) g_i^{n+1} + \theta g_i^n\right], \quad i = 1, 2, \cdots, N-1.$$

$$\left[E - (1-\theta)\mu S^{n+1}\right]\underline{\mathbf{u}}^{n+1} = (E + \theta\mu S^n)\underline{\mathbf{u}}^n + \Delta t\left[(1-\theta)\underline{\mathbf{g}}^{n+1} + \theta\underline{\mathbf{g}}^n\right]$$

$$S^n = diag(c_1^n, c_2^n, \cdots, c_{N-1}^n) S_{N-1}^{(\alpha)} + diag(d_1^n, d_2^n, \cdots, d_{N-1}^n)(S_{N-1}^{(\alpha)})^T$$

$$c_{\max} = \max_{0 \le i \le N} c(x_i),\ d_{\max} = \max_{0 \le i \le N} d(x_i) \qquad S = S^n$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D space-fractional equations
Two-sided space-fractional diffusion equation

Remark

Using the *Gerschgorin theorem*, we can obtain that the eigenvalues λ of the matrix S satisfy

$$\left|\lambda-\omega_1^{(\alpha)}(c_i+d_i)\right| \le c_i \sum_{j=0,j\neq 1}^{i} |\omega_j^{(\alpha)}| + d_i \sum_{j=0,j\neq 1}^{N-i} |\omega_j^{(\alpha)}|.$$

According to the definition of $\omega_j^{(\alpha)}$, we have $\left|\lambda-\omega_1^{(\alpha)}(c_i+d_i)\right| \le -\omega_1^{(\alpha)}(c_i+d_i).$

So

$$-2\alpha(c_{\max}+d_{\max}) \le 2\omega_1^{(\alpha)}(c_i+d_i) \le \lambda \le 0.$$

On the other hand, if λ is the eigenvalue of the matrix S then $\tilde{\lambda} = (1-\mu(1-\theta)\lambda)^{-1}(1+\mu\theta\lambda)$ is the eigenvalue of $(E-\mu(1-\theta)S)^{-1}(E+\mu\theta S)$. Therefore, the θ-method is unconditionally stable for $0 \le \theta \le 1/2$ since $|\tilde{\lambda}| \le 1$. For $1/2 < \theta \le 1$, we deduce from $|\tilde{\lambda}| \le 1$ that $\mu = \frac{\Delta t}{\Delta x^{\alpha}} \le 1/((2\theta-1)\alpha(c_{\max}+d_{\max}))$ which is the stability condition.

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D space-fractional equations

Riesz space-fractional diffusion equation

$$\begin{cases} \partial_t U = {}_{RZ}\mathrm{D}_x^{\alpha} U + g(x,t), & (x,t) \in (a,b)\times(0,T], \\ U(x,0) = \phi_0(x), & x \in (a,b), \\ U(a,t) = U(b,t) = 0, & t \in (0,T], \end{cases}$$

$d(x) > 0$

$1 < \alpha \leq 2$

$${}_{RZ}\mathrm{D}_x^{\alpha} U = -c_\alpha \left({}_{RL}\mathrm{D}_{a,x}^{\alpha} U + {}_{RL}\mathrm{D}_{x,b}^{\alpha} U\right) \qquad c_\alpha = \frac{1}{2\cos(\alpha\pi/2)}$$

Remark

Obviously, the Riesz space fractional derivative can be seen as the linear combination of the left and right Riemann–Liouville derivatives. Therefore, we can use the previous θ–method. However, the symmetrical fractional central (SFC) difference method is more popular.

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D space-fractional equations

Riesz space-fractional diffusion equation

Fractional central difference operator

$$D^{\alpha}_{\Delta x} f(x) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)} f(x-k\Delta x), \quad \alpha > -1$$

Lemma

If $\alpha > -1$ **and**

$$g_k = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)} \qquad k = 0, \mp 1, \mp 2, \cdots$$

are the coefficients of the central difference approximation then

$$g_0 \geq 0, \quad g_{-k} = g_k \leq 0, \quad |k| > 0$$

and

$$\sum_{k=-\infty}^{\infty} g_k e^{ikz} = |2\sin(z/2)|^{\alpha}, \quad i^2 = -1.$$

Numerical solution of fractional PDEs

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1D space-fractional equations

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Lemma

If $f \in C^5(\mathbb{R})$ and all derivatives up to order 5 of f belong to $L_1(\mathbb{R})$ then

$$-\frac{D^{\alpha}_{\Delta x} f(x)}{\Delta x^{\alpha}} = {}_{RZ}D^{\alpha}_{x} f(x) + O(\Delta x^2)$$

where

$${}_{RZ}D^{\alpha}_{x} f(x) = -c_{\alpha}({}_{RL}D^{\alpha}_{-\infty,x} f(x) + {}_{RL}D^{\alpha}_{x,\infty} f(x)) \text{ and } 1 < \alpha \le 2.$$

$$f(a) = f(b) = 0 \quad \Longrightarrow \quad {}_{RZ}D^{\alpha}_{x} f(x) = -\frac{1}{\Delta x^{\alpha}} \sum_{k=-\frac{b-x}{\Delta x}}^{\frac{x-a}{\Delta x}} g_k f(x - k\Delta x) + O(\Delta x^2)$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D space-fractional equations
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Fractional central difference θ-method

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = (1-\theta)\,{}_{RZ}\delta_x^{(\alpha)} u_i^{n+1} + \theta\,{}_{RZ}\delta_x^{(\alpha)} u_i^{n} \\ \qquad\qquad + (1-\theta) g_i^{n+1} + \theta g_i^{n}, \quad i = 1, 2, \cdots, N-1, \; 0 \le \theta \le 1, \\ u_i^0 = \phi_0(x_i), \quad i = 0, 1, 2, \cdots, N, \\ u_0^n = 0, \; u_N^n = 0, \end{cases}$$

$${}_{RZ}\delta_x^{(\alpha)} u_i^n = -\frac{1}{\Delta x^\alpha} \sum_{k=-N+i}^{i} g_k u_{i-k}^n, \quad g_k = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)}$$

Numerical solution of fractional PDEs

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$$u_0^n = u_N^n = 0 \Rightarrow [E-(1-\theta)\mu S]\underline{\mathbf{u}}^{n+1} = (E+\mu\theta S)\underline{\mathbf{u}}^n + \Delta t\left[(1-\theta)\underline{\mathbf{g}}^{n+1} + \theta\underline{\mathbf{g}}^n\right]$$

$$S = -\begin{pmatrix} g_0 & g_{-1} & g_{-2} & \cdots & g_{-N+1} \\ g_1 & g_0 & g_{-1} & \cdots & g_{-N+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{N-2} & g_{N-3} & g_{N-4} & \cdots & g_1 \\ g_{N-1} & g_{N-2} & g_{N-3} & \cdots & g_0 \end{pmatrix}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D space-fractional equations

Riesz space-fractional diffusion equation

Remark

Using the *Gerschgorin theorem*, we can obtain that the eigenvalues λ of the matrix S satisfy

$$|\lambda + g_0| \le \mu \sum_{k=-N+i, k\neq 0}^{i-1} |g_k| \le g_0 \quad \Longrightarrow \quad -2g_0 \le \lambda \le 0$$

On the other hand, if λ is the eigenvalue of the matrix S then $\tilde{\lambda} = (1-\mu(1-\theta)\lambda)^{-1}(1+\mu\theta\lambda)$ is the eigenvalue of $(E-\mu(1-\theta)S)^{-1}(E+\mu\theta S)$. Therefore, the θ-method is unconditionally stable for $0 \le \theta \le 1/2$ since $|\tilde{\lambda}| \le 1$. For $1/2 < \theta \le 1$, we deduce from $|\tilde{\lambda}| \le 1$ that $\mu = \frac{\Delta t}{\Delta x^{\alpha}} \le 1/((2\theta - 1)g_0)$ which is the stability condition.

Numerical solution of fractional PDEs

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Example

$$\begin{cases} \partial_t U = {}_{RL}\mathrm{D}^{\alpha}_{0,x} U + {}_{RL}\mathrm{D}^{\alpha}_{x,1} U + g(x,t), & (x,t) \in (0,1)\times(0,1], \\ U(x,0) = x^4(1-x)^4, & x \in (0,1), \\ U(0,t) = U(1,t) = 0, & t \in (0,1], \end{cases}$$

$$U(x,t) = \cos(t)x^4(1-x)^4$$

$$1 < \alpha < 2$$

L^2 **error at** $t = 1$ **of the SFC difference method with** $\theta = 0.5$ **and** $\Delta t = 1/1000$

N	$\alpha = 1.2$	order	$\alpha = 1.5$	order	$\alpha = 1.8$	order
8	8.9228e−5		1.5660e−4		1.4640e−4	
16	3.5343e−5	1.3361	5.6465e−5	1.4717	4.1501e−5	1.8186
32	1.5504e−5	1.1888	1.6347e−5	1.7883	1.0880e−5	1.9315
64	4.7751e−6	1.6991	4.3544e−6	1.9085	2.7739e−6	1.9717
128	1.3052e−6	1.8713	1.1218e−6	1.9566	6.9973e−7	1.9870

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-space fractional equations

Time-space fractional diffusion equation with Caputo derivative

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\gamma} U = (L^{(\alpha)}U)(x,t) + g(x,t), & (x,t) \in (a,b)\times(0,T], \\ U(x,0) = \phi_0(x), & x \in (a,b), \\ U(a,t) = 0,\ U(b,t) = 0, & t \in (0,T], \end{cases}$$

$$L^{(\alpha)} = c(x,t)\,{}_{RL}\mathrm{D}_{a,x}^{\alpha} + d(x,t)\,{}_{RL}\mathrm{D}_{x,b}^{\alpha},\ 0 < \gamma \le 1,\ 1 < \alpha < 2, \text{ and } c,d > 0$$

$$L_{\Delta x,q}^{(\alpha,n)} U_i^n = \begin{cases} d_i^n\,{}_L\delta_x^{(\alpha)} U_{i+1}^n + c_i^n\,{}_R\delta_x^{(\alpha)} U_{i-1}^n, & q = 1, \\ d_i^n\,{}_L\delta_x^{(\alpha,1)} U_i^n + c_i^n\,{}_R\delta_x^{(\alpha,1)} U_i^n, & q = 2, \\ d_i^n\,{}_L\delta_x^{(\alpha,2)} U_i^n + c_i^n\,{}_R\delta_x^{(\alpha,2)} U_i^n, & q = 3, \\ {}_{RZ}\delta_x^{(\alpha)} U_i^n, & q = 4, \end{cases}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-space fractional equations

Time-space fractional diffusion equation with Caputo derivative

$$L_{\Delta x,q}^{(\alpha,n)} U_i^n = (L^{(\alpha)} U)(x_i, t_n) + O(\Delta t^p)$$

$$p = \begin{cases} \delta_t^{(\gamma)}(u_i^n - u_i^0) = L_{\Delta x,q}^{(\alpha,n)} u_i^n + g_i^n, & i = 1, 2, \cdots, N-1, \\ u_i^0 = \phi_0(x_i), & i = 0, 1, 2, \cdots, N, \\ u_0^n = u_N^n = 0, \end{cases}$$

$$\|\mathbf{e}^n\|_N^2 \le C(\Delta t + \Delta x^p)$$

$$\|\mathbf{u}^n\|_N^2 \le 2\|\mathbf{u}^0\|_N^2 + C \max_{1 \le n \le n_T} \|\mathbf{g}^n\|_N^2$$

$$\begin{cases} {}_C^{L1}\delta_t^{(\gamma)} u_i^n = L_{\Delta x,q}^{(\alpha,n)} u_i^n + g_i^n, & i = 1, 2, \cdots, N-1, \\ u_i^0 = \phi_0(x_i), & i = 0, 1, 2, \cdots, N, \\ u_0^n = u_N^n = 0, \end{cases}$$

$$\|\mathbf{e}^n\|_N^2 \le C(\Delta t^{2-\gamma} + \Delta x^p)$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-space fractional equations

Time-space fractional diffusion equation with Caputo derivative

$$\begin{cases} \dfrac{1}{\Delta t^{\gamma}} \displaystyle\sum_{k=0}^{n} \omega_k (u_i^{n-k} - u_i^0) = (1-\frac{\gamma}{2}) L_{\Delta x,q}^{(\alpha,n)} u_i^n + \frac{\gamma}{2} L_{\Delta x,q}^{(\alpha,n-1)} u_i^{n-1} \\ \qquad\qquad + B_n^{(2)} L_{\Delta x,q}^{(\alpha,0)} u_i^0 + \dfrac{1}{\Delta t^{\gamma}} \left[(1-\frac{\gamma}{2}) G_i^n + \frac{\gamma}{2} G_i^{n-1} \right], \\ u_i^0 = \phi_0(x_i), \quad i = 0,1,2,\cdots,N, \\ u_0^n = u_N^n = 0, \end{cases}$$

$$\omega_k = (-1)^k \binom{\gamma}{k}, \; G_i^n = \left[\mathrm{D}_{0,t}^{\gamma} g(x_i,t) \right]_{t=t_n}$$

$$\|\mathbf{u}^n\|_N^2 \le C_1 \left(\|\mathbf{u}^0\|_N^2 + \Delta t^{\gamma} \Delta x \|\underline{\mathbf{u}}^0\|_S^2 \right) + C_2 \max_{0 \le k \le n_T} \|\mathbf{g}^n\|_N^2$$

$$O(\Delta t + \Delta x^p)$$

Numerical solution of fractional PDEs

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1D time-space fractional equations

Time-space fractional diffusion equation with Caputo derivative

$$\begin{cases} \frac{1}{\Delta t^{\gamma}} \sum_{k=0}^{n} \omega_k (u_i^{n-k} - u_i^0) = (1 - \frac{\gamma}{2}) L_{\Delta x,q}^{(\alpha,n)} u_i^n + \frac{\gamma}{2} L_{\Delta x,q}^{(\alpha,n-1)} u_i^{n-1} \\ \qquad + B_n^{(2)} L_{\Delta x,q}^{(\alpha,0)} u_i^0 + C_n^{(2)} (L_{\Delta x,q}^{(\alpha,1)} u_i^1 - L_{\Delta x,q}^{(\alpha,0)} u_i^0) \\ \qquad + \frac{1}{\Delta t^{\gamma}} \left[(1 - \frac{\gamma}{2}) G_i^n + \frac{\gamma}{2} G_i^{n-1} \right], \\ u_i^0 = \phi_0(x_i), \quad i = 0, 1, 2, \cdots, N, \\ u_0^n = u_N^n = 0, \end{cases}$$

$$\|\mathbf{u}^n\|_N^2 \le C_1 \left(\|\mathbf{u}^0\|_N^2 + \Delta t^{\gamma} \Delta x \|\underline{\mathbf{u}}^0\|_S^2 \right) + C_2 \max_{0 \le k \le n_T} \|\mathbf{g}^n\|_N^2$$

$$O(\Delta t^2 + \Delta x^p)$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

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Time-space fractional diffusion equation with Caputo derivative

Example

$$\begin{cases} {}_{C}D^{\gamma}_{0,t}U = {}_{RL}\mathrm{D}^{\alpha}_{0,x}U + {}_{RL}\mathrm{D}^{\alpha}_{x,1}U + g(x,t), & (x,t) \in (0,1)\times(0,1], \\ U(x,0) = 2x^4(1-x)^4, & x \in (0,1), \\ U(0,t) = U(1,t) = 0, & t \in (0,1], \end{cases}$$

$0 < \gamma < 1, 1 < \alpha < 2$

$U(x,t) = (t^{2+\gamma} + t + 2)x^4(1-x)^4$

L^2 **error at** $t = 1$ **for the L1 method with** $\gamma = 0.8, \Delta t = 0.001$

q	N	$\alpha = 1.2$	order	$\alpha = 1.5$	order	$\alpha = 1.8$	order
	8	2.0521e−4		2.8394e−4		3.5194e−4	
	16	4.8876e−5	2.0699	6.6752e−5	2.0887	8.1674e−5	2.1074
2	32	1.2163e−5	2.0066	1.6502e−5	2.0161	2.0134e−5	2.0203
	64	3.1149e−6	1.9653	4.1429e−6	1.9940	5.0316e−6	2.0005
	128	8.9300e−7	1.8024	1.0696e−6	1.9535	1.2734e−6	1.9823

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1D time-space fractional equations

Time-space fractional diffusion equation with RL derivative in time

$$\begin{cases} \partial_t U = {}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\left(L^{(\alpha)}U\right) + g(x,t), & (x,t) \in (a,b)\times(0,T], \\ U(x,0) = \phi_0(x), & x \in (a,b), \\ U(a,t) = U(b,t) = 0, & t \in (0,T], \end{cases}$$

$$L^{(\alpha)} = c(x,t)\,{}_{RL}\mathrm{D}_{a,x}^{\alpha} + d(x,t)\,{}_{RL}\mathrm{D}_{x,b}^{\alpha},\ 0 < \gamma \le 1,\ 1 < \alpha < 2, \text{ and } c,d > 0$$

$${}_{p}^{B}\delta_t^{(1-\gamma)}$$

Explicit Euler method (with GL formula)

$$\begin{cases} \delta_t u_i^{n+\frac{1}{2}} = \boxed{{}^{GL}\delta_t^{(1-\gamma)}}\left(L_{\Delta x,q}^{(\alpha,n)} u_i^n\right) + f_i^n, & i = 1,2,\cdots,N-1, \\ u_i^0 = \phi_0(x_i), & i = 0,1,2,\cdots,N, \\ u_0^n = u_N^n = 0, \end{cases}$$

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Implicit Euler method (with GL formula)

$$\delta_t^{(1-\gamma)}\left(L_{\Delta x,q}^{(\alpha,n-\frac{1}{2})}u_i^{n-\frac{1}{2}}\right)=\frac{1}{\Delta t^{1-\gamma}}\Big[b_0L_{\Delta x,q}^{(\alpha,n-\frac{1}{2})}u_i^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(b_{n-1-k}-b_{n-k})L_{\Delta x,q}^{(\alpha,k-\frac{1}{2})}u_i^{k-\frac{1}{2}}$$

$$-(b_n-B_n)L_{\Delta x,q}^{(\alpha,\frac{1}{2})}u_i^{\frac{1}{2}}-A_nL_{\Delta x,q}^{(\alpha,0)}u_i^0\Big]$$

Crank-Nicolson method

$$\begin{cases}\delta_t u_i^{n-\frac{1}{2}}=\delta_t^{(1-\gamma)}\left(L_{\Delta x,q}^{(\alpha,n-\frac{1}{2})}u_i^{n-\frac{1}{2}}\right)+f_i^n, & i=1,2,\cdots,N-1,\\ u_i^0=\phi_0(x_i), & i=0,1,2,\cdots,N,\\ u_0^n=u_N^n=0,\end{cases}$$

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Integration method I

$$\begin{cases} \delta_t u_i^{n-\frac{1}{2}} = \delta_t^{(1-\gamma)}\left(L_{\Delta x,q}^{(\alpha,n)} u_i^n\right) + f_i^n, \quad i = 1,2,\cdots,N-1, \\ u_i^0 = \phi_0(x_i), \quad i = 0,1,2,\cdots,N, \\ u_0^n = u_N^n = 0, \end{cases}$$

$$\delta_t^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,n)} u_i^n) = \frac{1}{\Delta t^{1-\gamma}}\left[\sum_{k=1}^{n} b_{n-k}^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,k-1)} u_i^{k-1}) - \sum_{l=1}^{n-1} b_{n-k-1}^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,k-1)} u_i^{k-1})\right],$$

$$b_k^{(1-\gamma)} = \frac{1}{\Gamma(1+\gamma)}[(k+1)^\gamma - k^\gamma].$$

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Integration method II

$$\begin{cases} \delta_t u_i^{n-\frac{1}{2}} = \delta_t^{(1-\gamma)}\left(L_{\Delta x,q}^{(\alpha,n)} u_i^n\right) + f_i^n, \quad i = 1,2,\cdots,N-1, \\ u_i^0 = \phi_0(x_i), \quad i = 0,1,2,\cdots,N, \\ u_0^n = u_N^n = 0, \end{cases}$$

$$\delta_t^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,n)} u_i^n) = \frac{1}{\Delta t^{1-\gamma}}\left[\sum_{k=1}^{n} b_{n-k}^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,k)} u_i^k) - \sum_{l=1}^{n-1} b_{n-k-1}^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,k)} u_i^k)\right],$$

$$b_k^{(1-\gamma)} = \frac{1}{\Gamma(1+\gamma)}[(k+1)^\gamma - k^\gamma].$$

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Remark

The stability and convergence analysis of these methods are more complicated than their counterparts of the integer order equations.
If $c(x,t) = d(x,t) = K_\gamma > 0,$ **then the implicit method, the Crank-Nicolson and the integration method II are unconditionally stable and convergent of order** $O(\Delta t + \Delta x^p)$, $O(\Delta t^{2-\gamma} + \Delta x^p)$ **and** $O(\Delta t + \Delta x^p)$, **respectively.**

Example

$$\begin{cases} \partial_t U = {}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\left({}_{RL}\mathrm{D}_{0,x}^{\alpha} U + {}_{RL}\mathrm{D}_{x,1}^{\alpha} U\right) + g(x,t), & (x,t) \in (0,1)\times(0,1], \\ U(x,0) = 0, \quad x \in (0,1), \\ U(0,t) = U(1,t) = 0, \quad t \in (0,1], \end{cases}$$

$$0 < \gamma < 1, 1 < \alpha < 2$$

$$U(x,t) = (t^{2.5} + t)x^4(1-x)^4$$

Numerical solution of fractional PDEs

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1D time-space fractional equations

Time-space fractional diffusion equation with RL derivative in time

L^2 **error at** $t = 1$ **for the Crank-Nicolson method with** $\Delta t = 0.001$

q	N	$(\gamma,\alpha) = (0.2, 1.2)$	order	$(\gamma,\alpha) = (0.5, 1.5)$	order	$(\gamma,\alpha) = (0.8, 1.8)$	order
3	8	2.9761e−4		3.1824e−4		2.6556e−4	
	16	7.5926e−5	1.9708	7.8473e−5	2.0198	6.3540e−5	2.0633
	32	1.9605e−5	1.9534	1.9822e−5	1.9851	1.5808e−5	2.0070
	64	5.1805e−6	1.9201	5.0220e−6	1.9808	3.9573e−6	1.9980
	128	1.5276e−6	1.7618	1.2836e−6	1.9680	9.9178e−7	1.9964
4	8	1.7941e−4		3.0921e−4		2.9140e−4	
	16	7.1000e−5	1.3374	1.1228e−4	1.4615	8.2738e−5	1.8164
	32	3.0397e−5	1.2239	3.2374e−5	1.7942	2.1694e−5	1.9313
	64	9.4881e−6	1.6798	8.6250e−6	1.9082	5.5311e−6	1.9716
	128	2.7937e−6	1.7640	2.2403e−6	1.9448	1.3959e−6	1.9863

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

1D time-space fractional equations

Time-space fractional diffusion equation with RL derivative in time

L^2 error at $t = 1$ for the integration method II with $N = 1000$

q	$1/\Delta t$	$(\gamma,\alpha) = (0.2, 1.2)$	order	$(\gamma,\alpha) = (0.5, 1.5)$	order	$(\gamma,\alpha) = (0.8, 1.8)$	order
3	8	6.7781e−4		7.8483e−4		8.3669e−4	
	16	3.5113e−4	0.9489	4.0995e−4	0.9369	4.3435e−4	0.9458
	32	1.8027e−4	0.9619	2.1054e−4	0.9614	2.2147e−4	0.9717
	64	9.1999e−5	0.9704	1.0703e−4	0.9760	1.1188e−4	0.9852
	128	4.6759e−5	0.9764	5.4081e−5	0.9849	5.6238e−5	0.9923
4	8	6.7781e−4		7.8482e−4		8.3669e−4	
	16	3.5112e−4	0.9489	4.0994e−4	0.9369	4.3434e−4	0.9459
	32	1.8026e−4	0.9619	2.1053e−4	0.9614	2.2147e−4	0.9717
	64	9.1987e−5	0.9705	1.0702e−4	0.9761	1.1187e−4	0.9852
	128	4.6746e−5	0.9766	5.4072e−5	0.9850	5.6234e−5	0.9924

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-fractional equations

Time-fractional diffusion equation with RL derivative in time

$$\begin{cases} \partial_t U = {}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\left(K_1\partial_x^2 U + K_2\partial_y^2 U\right) + f(x,y,t), & (x,y,t)\in(a,b)\times(c,d)\times(0,T], \\ U(x,y,0) = \phi_0(x,y), & (x,y)\in(a,b)\times(c,d), \\ U(a,y,t) = U_a(y,t),\ U(b,y,t) = U_b(y,t), & (y,t)\in(c,d)\times(0,T], \\ U(x,c,t) = U_c(x,t),\ U(x,d,t) = U_d(x,t), & (x,t)\in(a,b)\times(0,T], \end{cases}$$

$0 < \gamma < 1$

$K_1, K_2 > 0$

$\Omega = (a,b)\times(c,d)\times[0,T]$ $\Delta t = T/n_T,\ \Delta x = (b-a)/N_x$ and $\Delta y = (d-c)/N_y$

$t_k = k\Delta t,\ x_i = a + i\Delta x$ $y_j = c + j\Delta y$ $t_{k+\frac{1}{2}} = (t_k + t_{k+1})/2$

$U^n = U^n(\cdot) = U(x,y,t_n)$ $U_{i,j}^n = U(x_i, y_j, t_n)$

$$\delta_x U_{i+\frac{1}{2},j}^n = \frac{U_{i+1,j}^n - U_{i,j}^n}{\Delta x}, \qquad \delta_y U_{i,j+\frac{1}{2}}^n = \frac{U_{i,j+1}^n - U_{i,j}^n}{\Delta y}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-fractional equations

Time-fractional diffusion equation with RL derivative in time

$$\delta_x^2 U_{i,j}^n = \frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{\Delta x^2}, \quad \delta_y^2 U_{i,j}^n = \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{\Delta y^2}$$

$$\delta_t U_{i,j}^{n+\frac{1}{2}} = \frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t}, \qquad U_{i,j}^{n+\frac{1}{2}} = \frac{U_{i,j}^n + U_{i,j}^{n+1}}{2}$$

$$\partial_t U(x_i, y_j, t_n) = \left[{}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\left(K_1 \partial_x^2 U + K_2 \partial_y^2 U\right)\right]_{(x,y,t)=(x_i,y_j,t_n)} + f(x_i, y_j, t_n)$$

$$\partial_t U(x_i, y_j, t_n) = \frac{U(x_i, y_j, t_n) - U(x_i, y_j, t_{n-1})}{\Delta t} + O(\Delta t) = \delta_t U_{i,j}^{n-\frac{1}{2}} + O(\Delta t),$$

$$\left({}_{RL}\mathrm{D}_{0,t}^{1-\gamma} \partial_x^2 U\right)(x_i, y_j, t_n) = {}^{GL}\delta_t^{(1-\gamma)}(\partial_x^2 U^n(x_i, y_j)) + O(\Delta t),$$

$$\left({}_{RL}\mathrm{D}_{0,t}^{1-\gamma} \partial_y^2 U\right)(x_i, y_j, t_n) = {}^{GL}\delta_t^{(1-\gamma)}(\partial_y^2 U^n(x_i, y_j)) + O(\Delta t),$$

$$\partial_x^2 U(x_i, y_j, t_n) = \delta_x^2 U_{i,j}^n + O(\Delta x^2), \quad \partial_y^2 U(x_i, y_j, t_n) = \delta_y^2 U_{i,j}^n + O(\Delta y^2)$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-fractional equations

Time-fractional diffusion equation with RL derivative in time

$$\delta_t U_{i,j}^{n-\frac{1}{2}} = {}^{GL}\delta_t^{(1-\gamma)}(K_1\delta_x^2 U_{i,j}^n + K_2\delta_y^2 U_{i,j}^n) + f_{i,j}^n + O(\Delta t + \Delta x^2 + \Delta y^2)$$

Implicit method

$$\begin{cases} \delta_t u_{i,j}^{n-\frac{1}{2}} = {}^{GL}\delta_t^{(1-\gamma)}(K_1\delta_x^2 u_{i,j}^n + K_2\delta_y^2 u_{i,j}^n) + f_{i,j}^n, \\ u_{i,j}^0 = \phi_0(x_i, y_j), \quad i = 0,1,2,\cdots,N_x,\ j = 0,1,2,\cdots,N_y, \\ u_{0,j}^n = U_a(y_j, t_n), \quad u_{N_x,j}^n = U_b(y_j, t_n), \quad j = 0,1,2,\cdots,N_y, \\ u_{i,0}^n = U_c(x_i, t_n), \quad u_{i,N_y}^n = U_d(x_i, t_n), \quad i = 0,1,2,\cdots,N_x, \end{cases}$$

$$u_{i,j}^n = u_{i,j}^{n-1} + \sum_{k=0}^{n} \omega_{n-k}^{(1-\gamma)} \left[\mu_1(u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k) + \mu_2(u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k)\right] + \Delta t f_i^n$$

$$\mu_1 = K_1\Delta t^\gamma/\Delta x^2 \text{ and } \mu_2 = K_2\Delta t^\gamma/\Delta y^2$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-fractional equations

Time-fractional diffusion equation with RL derivative in time

$$\underline{\mathbf{u}}^n = \begin{bmatrix} u^n_{1,1} & u^n_{1,2} & \cdots & u^n_{1,N_y-2} & u^n_{1,N_y-1} \\ u^n_{2,1} & u^n_{2,2} & \cdots & u^n_{2,N_y-2} & u^n_{2,N_y-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u^n_{N_x-1,1} & u^n_{N_x-1,2} & \cdots & u^n_{N_x-1,N_y-2} & u^n_{N_x-1,N_y-1} \end{bmatrix}_{(N_x-1)\times(N_y-1)}$$

$$B_{N,2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{u}^n_{cd} = \begin{bmatrix} u^n_{1,0} & u^n_{1,N_y} \\ u^n_{2,0} & u^n_{2,N_y} \\ \vdots & \vdots \\ u^n_{N_x-1,0} & u^n_{N_x-1,N_y} \end{bmatrix} \quad S_N = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix}_{N\times N}$$

$$\mathbf{u}^n_{ab} = \begin{bmatrix} u^n_{0,1} & u^n_{0,2} & \cdots & u^n_{0,N_y-1} \\ u^n_{N_x,1} & u^n_{N_x,2} & \cdots & u^n_{N_x,N_y-1} \end{bmatrix}_{2\times(N_y-1)}$$

Numerical solution of fractional PDEs

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2D time-fractional equations

Time-fractional diffusion equation with RL derivative in time

$$\underline{\mathbf{u}}^n - \left(\mu_1 S_{N_x-1}\underline{\mathbf{u}}^n + \mu_2 \underline{\mathbf{u}}^n S^T_{N_y-1}\right) = RHS^n$$

$$RHS^n = \underline{\mathbf{u}}^n \quad \sum_{k=0}^{n-1} \quad \Big)_{-1}\Big) + \Big)_{1,2}\Big) + \Delta t \mathbf{F}^n$$

$$\mathrm{vec}(M) = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_J \end{pmatrix}, \quad M = (m_1, m_2, \cdots, m_J)$$

$$(\mathbf{F}^n)_{i,j} = f(x_i, y_j, t_n), \quad \mathbf{F}^n \in \mathbb{R}^{(N_x-1)\times(N_y-1)}$$

$$A\,\mathrm{vec}(\underline{\mathbf{u}}^n) = \mathrm{vec}(RHS^n)$$

$$A = E_{N_y-1} \otimes E_{N_x-1} - \omega_0^{(1-\gamma)}\left(\mu_1 E_{N_y-1} \otimes S_{N_x-1} + \mu_2 S_{N_y-1} \otimes E_{N_x-1}\right)$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-fractional equations

Time-fractional diffusion equation with RL derivative in time

$$(\mathbf{u},\mathbf{v})_N = \Delta x \Delta y \sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1} u_{i,j}v_{i,j}, \quad \|\mathbf{u}\|_N = \sqrt{(\mathbf{u},\mathbf{u})_N}$$

$$\mathbf{u},\mathbf{v} \in \mathbb{R}^{(N_x+1)\times(N_y+1)}$$

$$\|\mathbf{u}^n\|_N^2 \le \|\mathbf{u}^0\|_N^2 + C \max_{0\le k\le n_T} \|\mathbf{F}^k\|_N^2$$

$$\|\mathbf{e}^n\|_N^2 \le \|\mathbf{e}^0\|_N^2 + C \max_{0\le k\le n_T} \|\mathbf{R}^k\|_N^2 \le C(\Delta t + \Delta x^2 + \Delta y^2)$$

coefficient matrix A of size $(N_x-1)(N_y-1)\times(N_x-1)(N_y-1)$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-fractional equations

Time-fractional diffusion equation with RL derivative in time

$$\delta_t U_{i,j}^{n-\frac{1}{2}} = {}^{GL}\delta_t^{(1-\gamma)}(K_1\delta_x^2 U_{i,j}^n + K_2\delta_y^2 U_{i,j}^n) + f_{i,j}^n + O(\Delta t + \Delta x^2 + \Delta y^2)$$

$$-(\omega_0^{(1-\gamma)}\Delta t^\gamma)^2 L_{\Delta x} L_{\Delta y}\delta_t U_{i,j}^{n-\frac{1}{2}} = O(\Delta t^{2\gamma})$$

$$\begin{aligned}
\delta_t U_{i,j}^{n-\frac{1}{2}} &= {}^{GL}\delta_t^{(1-\gamma)}(K_1\delta_x^2 U_{i,j}^n + K_2\delta_y^2 U_{i,j}^n) + f_{i,j}^n + O(\Delta t + \Delta x^2 + \Delta y^2) \\
&= \Delta t^{\gamma-1}\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma)}(L_{\Delta x} + L_{\Delta y})U_{i,j}^k + f_{i,j}^n + O(\Delta t + \Delta x^2 + \Delta y^2) \\
&= -(\omega_0^{(1-\gamma)}\Delta t^\gamma)^2 L_{\Delta x} L_{\Delta y}\delta_t U_{i,j}^{n-\frac{1}{2}} + O(\Delta t^{2\gamma}) \\
&\quad + \Delta t^{\gamma-1}\sum_{k=0}^{n}\omega_{n-k}^{(1-\gamma)}(L_{\Delta x} + L_{\Delta y})U_{i,j}^k + f_{i,j}^n + O(\Delta t + \Delta x^2 + \Delta y^2)
\end{aligned}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-fractional equations

Time-fractional diffusion equation with RL derivative in time

PR-ADI finite difference method

$$\delta_t u_{i,j}^{n-\frac{1}{2}} + K_1 K_2(\omega_0^{(1-\gamma)} \quad \delta_x^2 u_{i,j}^n + K_2 \delta_y^2 u_{i,j}^n) + f_{i,j}^n,$$

$$u_{i,j}^0 = \phi_0$$

$$u_{0,j}^n = U$$

$$(1 - K_1 \omega_0^{(1-\gamma)} \Delta t^\gamma \delta_x^2) u_{i,j}^* = (RHS)_{i,j}^n, \quad i = 1, 2, \cdots, N_x - 1,$$

$$(1 - K_2 \omega_0^{(1-\gamma)} \Delta t^\gamma \delta_y^2) u_{i,j}^n = u_{i,j}^*, \quad j = 1, 2, \cdots, N_y - 1.$$

$$u_{i,0}^* = (1 - K_2 \omega_0^{(1-\gamma)} \Delta t^\gamma \delta_y^2) u_{i,0}^n, \quad u_{i,N_y}^* = (1 - K_2 \omega_0^{(1-\gamma)} \Delta t^\gamma \delta_y^2) u_{i,N_y}^n$$

$$\|\mathbf{u}\|_N^2 \le \|\mathbf{u}^0\|_N^2 + \Delta t^\gamma \left(K_1 \|\delta_x \mathbf{u}^0\|_N^2 + K_2 \|\delta_y \mathbf{u}^0\|_N^2\right) + \Delta t^{2\gamma} \|\delta_x \delta_y \mathbf{u}^0\|_N^2 + C \max_{0 \le k \le n_T} \|\mathbf{f}^k\|_N^2$$

$$\|\mathbf{e}^n\|_N^2 \le \|\mathbf{e}^0\|_N^2 + \Delta t^\gamma \left(K_1 \|\delta_x \mathbf{e}^0\|_N^2 + K_2 \|\delta_y \mathbf{e}^0\|_N^2\right) + \Delta t^{2\gamma} \|\delta_x \delta_y \mathbf{e}^0\|_N^2 + C \max_{0 \le k \le n_T} \|\mathbf{R}^k\|_N^2$$

$$\le C(\Delta t + \Delta t^{2\gamma} + \Delta x^2 + \Delta y^2).$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-fractional equations

Time-fractional diffusion equation with Caputo derivative

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\gamma} U = K_1 \partial_x^2 U + K_2 \partial_y^2 U + g(x,y,t), & (x,y,t) \in (a,b)\times(c,d)\times(0,T], \\ U(x,y,0) = \phi_0(x,y), \quad (x,y) \in (a,b)\times(c,d), & K_1, K_2 > 0 \quad 0 < \gamma < 1 \\ U(a,y,t) = U_a(y,t),\ U(b,y,t) = U_b(y,t), & (y,t) \in (c,d)\times(0,T], \\ U(x,c,t) = U_c(x,t),\ U(x,d,t) = U_d(x,t), & (x,t) \in (a,b)\times(0,T], \end{cases}$$

$${}_C\mathrm{D}_{0,t}^{\gamma} U = (L_x + L_y)U + g(x,y,t) \qquad L_x U = K_1 \partial_x^2 U \text{ and } L_y U = K_1 \partial_y^2 U$$

$$U^n - \frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}(L_x + L_y)U^n = -\sum_{k=0}^{n-1} \frac{\alpha_{n,k}}{\alpha_{n,n}} U^k + \frac{\Delta t^{\gamma}}{\alpha_{n,n}} \sum_{k=0}^{n-1} \theta_{n,k}(L_x + L_y)U^k + \frac{\Delta t^{\gamma} G^n}{\alpha_{n,n}} + \frac{\Delta t^{\gamma} R^n}{\alpha_{n,n}}$$

$$+ \quad \left(\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}\right)^2 L_x L_y (U^n - U^{n-1}) = O(\Delta t^{1+2\gamma}) \quad \Longrightarrow$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-fractional equations

Time-fractional diffusion equation with Caputo derivative

$$U^n$$

$$= -$$

$$\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_x\right)\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_y\right)U^n$$

$$=\left(\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}\right)^2 L_xL_yU^{n-1}-\sum_{k=0}^{n-1}\frac{\alpha_{n,k}}{\alpha_{n,n}}U^k+\frac{\Delta t^{\gamma}}{\alpha_{n,n}}\sum_{k=0}^{n-1}\theta_{n,k}(L_x+L_y)U^k$$

$$+2\gamma)$$

$$\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_{\Delta x}\right)\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_{\Delta y}\right)U^n_{i,j}$$

$$=\left(\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}\right)^2 L_{\Delta x}L_{\Delta y}U^{n-1}_{i,j}-\sum_{k=0}^{n-1}\frac{\alpha_{n,k}}{\alpha_{n,n}}U^k_{i,j}+\frac{\Delta t^{\gamma}}{\alpha_{n,n}}\sum_{k=0}^{n-1}\theta_{n,k}(L_{\Delta x}+L_{\Delta y})U^k_{i,j}$$

$$+\frac{\Delta t^{\gamma}G^n_{i,j}}{\alpha_{n,n}}+\frac{\Delta t^{\gamma}R^n_{i,j}}{\alpha_{n,n}}+O(\Delta t^{\gamma}(\Delta t^{1+\gamma}+\Delta x^2+\Delta y^2)).$$

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2D time-fractional equations

Time-fractional diffusion equation with Caputo derivative

$$\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_{\Delta x}\right)\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_{\Delta y}\right)u_{i,j}^{n}=(RHS)_{i,j}^{n}$$

D-ADI finite difference method

$$\alpha_{n,k}$$

$$\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_{\Delta x}\right)u_{i,j}^{*}=(RHS)_{i,j}^{n},\quad i=1,2,\cdots,N_x-1,$$

$$\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_{\Delta y}\right)u_{i,j}^{n}=u_{i,j}^{*},\quad j=1,2,\cdots,N_y-1.$$

$$u_{0,j}^{*}=\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_{\Delta y}\right)u_{0,j}^{n},\quad u_{N_x,j}^{*}=\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_{\Delta y}\right)u_{N_x,j}^{n}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-space fractional equations
Fractional diffusion equation with Caputo time derivative

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\gamma} U = d_+(x,y,t)\,{}_{RL}\mathrm{D}_{a,x}^{\alpha} U + d_-(x,y,t)\,{}_{RL}\mathrm{D}_{x,b}^{\alpha} U \\ \qquad + c_+(x,y,t)\,{}_{RL}\mathrm{D}_{c,y}^{\alpha} U + c_-(x,y,t)\,{}_{RL}\mathrm{D}_{y,d}^{\alpha} U + g(x,y,t), \\ \qquad (x,y,t) \in (a,b)\times(c,d)\times(0,T], \\ U(x,y,0) = \phi_0(x,y), \quad (x,y) \in (a,b)\times(c,d), \\ U(a,y,t) = U(b,y,t) = 0, \quad (y,t) \in (c,d)\times(0,T], \\ U(x,c,t) = U(x,d,t) = 0, \quad (x,t) \in (a,b)\times(0,T], \end{cases}$$

$$0 < \gamma \le 1,\ 1 < \alpha < 2 \text{ and } c_+(x,y,t), c_-(x,y,t), d_+(x,y,t), d_-(x,y,t) \ge 0$$

$$L_x^{(\alpha)} = d_+(x,y,t)\,{}_{RL}\mathrm{D}_{a,x}^{\alpha} + d_-(x,y,t)\,{}_{RL}\mathrm{D}_{x,b}^{\alpha}$$

$${}_C\mathrm{D}_{0,t}^{\gamma} U = (L_x^{(\alpha)} + L_y^{(\alpha)})U + g(x,y,t)$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-space fractional equations
Fractional diffusion equation with Caputo time derivative

$$\frac{1}{\Delta t^{\gamma}}\sum_{k=0}^{n}\alpha_{n,k}u_{i,j}^{k}+\frac{\theta_{n,n}^{2}\Delta t^{1+\gamma}}{\alpha_{n,n}}L_{\Delta x,q}^{(\alpha,n)}L_{\Delta y,q}^{(\alpha,n)}\delta_t u_{i,j}^{n-\frac{1}{2}}=\sum_{k=0}^{n}\theta_{n,k}(L_{\Delta x,q}^{(\alpha,k)}+L_{\Delta y,q}^{(\alpha,k)})u_{i,j}^{k}+G_{i,j}^{n}$$

$$\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_{\Delta x,q}^{(\alpha,n)}\right)\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L_{\Delta y,q}^{(\alpha,n)}\right)u_{i,j}^{n}=(RHS)_{i,j}^{n}$$

$$L_{\Delta x,q}^{(\alpha,k)}U_{i,j}^{n}=\begin{cases}d_{+}(x_i,y_j,t_k)_L\delta_x^{(\alpha)}U_{i+1,j}^{n}+d_{-}(x_i,y_j,t_k)_R\delta_x^{(\alpha)}U_{i-1,j}^{n}, & q=1,\\ d_{+}(x_i,y_j,t_k)_L\delta_x^{(\alpha,1)}U_{i,j}^{n}+d_{-}(x_i,y_j,t_k)_R\delta_x^{(\alpha,1)}U_{i,j}^{n}, & q=2,\\ d_{+}(x_i,y_j,t_k)_L\delta_x^{(\alpha,2)}U_{i,j}^{n}+d_{-}(x_i,y_j,t_k)_R\delta_x^{(\alpha,2)}U_{i,j}^{n}, & q=3,\\ {}_{RZ}\delta_x^{(\alpha)}U_{i,j}^{n}, \quad q=4,\ d_{+}=d_{-}=c_{+}=c_{-}=-\dfrac{1}{2\cos(\alpha\pi/2)}=c_{\alpha}, & \end{cases}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-space fractional equations

Fractional diffusion equation with Caputo time derivative

$$ {}_L\delta_x^{(\alpha)} U_{i,j}^n = \frac{1}{\Delta x^\alpha} \sum_{k=0}^{i} \omega_k^{(\alpha)} U_{i-k,j}, \quad {}_R\delta_x^{(\alpha)} U_{i,j} = \frac{1}{\Delta x^\alpha} \sum_{k=0}^{N_x-i} \omega_k^{(\alpha)} U_{i+k,j}, \quad \omega_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}, $$

$$ {}_L\delta_x^{(\alpha,1)} U_{i,j}^n = \frac{\alpha}{2} {}_L\delta_x^{(\alpha)} U_{i+1,j}^n + \frac{2-\alpha}{2} {}_L\delta_x^{(\alpha)} U_{i,j}^n, $$

$$ {}_R\delta_x^{(\alpha,1)} U_{i,j}^n = \frac{\alpha}{2} {}_R\delta_x^{(\alpha)} U_{i-1,j}^n + \frac{2-\alpha}{2} {}_R\delta_x^{(\alpha)} U_{i,j}^n, $$

$$ {}_L\delta_x^{(\alpha,2)} U_{i,j}^n = \frac{2+\alpha}{4} {}_L\delta_x^{(\alpha)} U_{i+1,j}^n + \frac{2-\alpha}{4} {}_L\delta_x^{(\alpha)} U_{i-1,j}^n, $$

$$ {}_R\delta_x^{(\alpha,2)} U_{i,j}^n = \frac{2+\alpha}{4} {}_R\delta_x^{(\alpha)} U_{i-1,j}^n + \frac{2-\alpha}{4} {}_R\delta_x^{(\alpha)} U_{i+1,j}^n, $$

$$ {}_{RZ}\delta_x^{(\alpha)} U_{i,j}^n = -\frac{1}{\Delta x^\alpha} \sum_{k=-N_x+i}^{i} g_k u_{i-k}^n, \quad g_k = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\alpha/2-k+1)\Gamma(\alpha/2+k+1)}. $$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-space fractional equations
Fractional diffusion equation with Caputo time derivative

D-ADI finite difference method

$$\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L^{(\alpha,n)}_{\Delta x,q}\right)u^{*}_{i,j}=(RHS)^{n}_{i,j},\quad i=1,2,\cdots,N_x-1,$$

$$O(\Delta x^{p}+\Delta y^{p})$$

$$\left(1-\frac{\theta_{n,n}\Delta t^{\gamma}}{\alpha_{n,n}}L^{(\alpha,n)}_{\Delta y,q}\right)u^{n}_{i,j}=u^{*}_{i,j},\quad j=1,2,\cdots,N_y-1.$$

$$O(\Delta t)$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-space fractional equations
Fractional diffusion equation with RL time derivative

$$\begin{cases} \partial_t U = {}_{RL}\mathrm{D}_{0,t}^{1-\gamma}\left(L_x^{(\alpha)} + L_y^{(\alpha)}\right)U + g(x,y,t), & (x,y,t) \in (a,b)\times(c,d)\times(0,T], \\ U(x,y,0) = \phi_0(x,y), & (x,y) \in (a,b)\times(c,d), \\ U(a,y) = U(b,y) = 0, & (y,t) \in (c,d)\times(0,T], \\ U(x,c) = U(x,d) = 0, & (x,t) \in (a,b)\times(0,T], \end{cases}$$

$0 < \gamma \leq 1$, and $1 < \alpha < 2$

$$L_x^{(\alpha)} = d_+(x,y)\,{}_{RL}\mathrm{D}_{a,x}^{\alpha} + d_-(x,y)\,{}_{RL}\mathrm{D}_{x,b}^{\alpha}$$

$$L_y^{(\alpha)} = c_+(x,y)\,{}_{RL}\mathrm{D}_{c,y}^{\alpha} + c_-(x,y)\,{}_{RL}\mathrm{D}_{y,d}^{\alpha}$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-space fractional equations
Fractional diffusion equation with RL time derivative

CN finite difference method

$$\begin{cases} \delta_t u_{i,j}^{n-\frac{1}{2}} = \delta_t^{(1-\gamma)}(L_{\Delta x,q}^{(\alpha,0)} + L_{\Delta y,q}^{(\alpha,0)})u_{i,j}^{n-\frac{1}{2}} + f(x_i, y_j, t_{n-\frac{1}{2}}), \quad n = 2,3,\cdots,n_T, \\ u_{i,j}^0 = \phi_0(x_i, y_j), \quad i = 0,1,2,\cdots,N_x, \; j = 0,1,2,\cdots,N_y, \\ u_{0,j}^n = u_{N_x,j}^n = 0, \quad j = 0,1,2,\cdots,N_y, \\ u_{i,0}^n = u_{i,N_y}^n = 0, \quad i = 0,1,2,\cdots,N_x, \end{cases}$$

$$O(\Delta t^{1+\gamma} + \Delta x^p + \Delta y^p)$$

$$\delta_t^{(1-\gamma)} u_{i,j}^{n-\frac{1}{2}} = \frac{1}{\Delta t^{1-\gamma}} \left[b_0 u_{i,j}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (b_{n-1-k} - b_{n-k}) u_{i,j}^{k-\frac{1}{2}} - (b_{n-1} - B_{n-1}) u_{i,j}^{\frac{1}{2}} - A_{n-1} u_{i,j}^0 \right]$$

$$A_n = B_n - \frac{\gamma(n+1/2)^{\gamma-1}}{\Gamma(1+\gamma)} \qquad b_n = \frac{1}{\Gamma(1+\gamma)}[(n+1)^\gamma - n^\gamma], \quad B_n = \frac{2}{\Gamma(1+\gamma)}[(n+1/2)^\gamma - n^\gamma]$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-space fractional equations
Fractional diffusion equation with RL time derivative

Corresponding ADI method

$$\begin{cases} \delta_t u_{i,j}^{n-\frac{1}{2}} + (a_{n,n}\Delta t^{\gamma})^2 L_{\Delta x,q}^{(\alpha,0)} L_{\Delta y,q}^{(\alpha,0)} \delta_t u_{i,j}^{n-\frac{1}{2}} = \delta_t^{(1-\gamma)} (L_{\Delta x,q}^{(\alpha,0)} + L_{\Delta y,q}^{(\alpha,0)}) u_{i,j}^{n-\frac{1}{2}} + f(x_i, y_j, t_{n-\frac{1}{2}}), \\ u_{i,j}^{0} = \phi_0(x_i, y_j), \quad i = 0,1,2,\cdots,N_x, \ j = 0,1,2,\cdots,N_y, \\ u_{0,j}^{n} = u_{N_x,j}^{n} = 0, \quad j = 0,1,2,\cdots,N_y, \\ u_{i,0}^{n} = u_{i,N_y}^{n} = 0, \quad i = 0,1,2,\cdots,N_x, \end{cases}$$

$$O(\Delta t^{1+\gamma} + \Delta t^{2\gamma} + \Delta x^{p} + \Delta y^{p})$$

$$a_{n,n} = \frac{b_0}{2} = \frac{1}{2\Gamma(1+\gamma)} \text{ for } n > 1 \text{ and } a_{n,n} = \frac{1}{2} B_0 = \frac{2^{1-\gamma}}{\Gamma(1+\gamma)} \text{ for } n = 1$$

$$d_{\pm}(x,y,t) = K_1 \text{ and } c_{\pm}(x,y,t) = K_2 \text{ and } q = 1$$

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

2D time-space fractional equations
Fractional diffusion equation with RL time derivative

$$\left(E_{N_x-1}+\frac{\mu_1 b_0}{2}S^{(\alpha)}_{N_x-1}\right)\underline{\mathbf{u}}^n\left(E_{N_y-1}+\frac{\mu_1 b_0}{2}S^{(\alpha)}_{N_y-1}\right)$$
$$=\left(E_{N_x-1}-\frac{\mu_1 b_0}{2}S^{(\alpha)}_{N_x-1}\right)\underline{\mathbf{u}}^{n-1}\left(E_{N_y-1}-\frac{\mu_1 b_0}{2}S^{(\alpha)}_{N_y-1}\right)+RHS^n,\quad n>1$$

- Solve $\left(E_{N_x-1}+\frac{\mu_1 b_0}{2}S^{(\alpha)}_{N_x-1}\right)\mathbf{u}^*=\left(E_{N_x-1}-\frac{\mu_1 b_0}{2}S^{(\alpha)}_{N_x-1}\right)\underline{\mathbf{u}}^{n-1}\left(E_{N_y-1}-\frac{\mu_1 b_0}{2}S^{(\alpha)}_{N_y-1}\right)+RHS^n$ to obtain $\mathbf{u}^*$;
- Solve $\left(E_{N_y-1}+\frac{\mu_1 B_0}{2}S^{(\alpha)}_{N_y-1}\right)(\underline{\mathbf{u}}^n)^T=(\mathbf{u}^*)^T$ to obtain $\underline{\mathbf{u}}^n$.

Numerical solution of fractional PDEs

FDM for fractional PDEs (FPDEs)

Example

2D time-space fractional equations

Fractional diffusion equation with RL time derivative

$$ {}_C\mathrm{D}^{\gamma}_{0,t}U = ({}_{RL}\mathrm{D}^{\alpha}_{0,x} + {}_{RL}D^{\alpha}_{x,1})U + ({}_{RL}\mathrm{D}^{\alpha}_{0,y} + {}_{RL}\mathrm{D}^{\alpha}_{y,1})U + g(x,y,t), \quad 0<\gamma<1, $$

$$ (x,y,t) \in (0,1)\times(0,1)\times(0,1], $$

$$ U(x,y,t) = (t^{2.5}+t+1)x^4(1-x)^4y^4(1-y)^4 $$

L^2 error at $t=1$ for the ADI method when $N=N_x=N_y$ with $\Delta t = 0.001$

q	N	$\alpha=1.2$	order	$\alpha=1.5$	order	$\alpha=1.8$	order
	8	1.1109e−6		1.2108e−6		1.0371e−6	
	16	2.8707e−7	1.9523	3.0289e−7	1.9991	2.5046e−7	2.0499
3	32	7.3656e−8	1.9625	7.6649e−8	1.9825	6.2291e−8	2.0075
	64	1.8704e−8	1.9775	1.9296e−8	1.9899	1.5465e−8	2.0100
	8	6.4331e−7		1.1712e−6		1.1338e−6	
	16	2.7603e−7	1.2207	4.3286e−7	1.4360	3.2506e−7	1.8024
4	32	1.1582e−7	1.2530	1.2524e−7	1.7892	8.5447e−8	1.9276
	64	3.5216e−8	1.7175	3.3269e−8	1.9124	2.1678e−8	1.9788

Thanks for your attention

Any question?