Numerical solution of fractional PDEs

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First lecture (37 slides)

Numerical solution of fractional PDEs

Fractional calculus' origin

Gottfried Wilhelm Leibniz
1646-1716

Leibniz replied: "It leads to a paradox, from which one day useful consequences will be drawn."

Guillaume de l'Hôpital
1661-1704

Fractional calculus is not a new topic, in reality it has almost the same history as that of the classical one. It goes back to **30 September 1695** when **G. W. l'Hôpital**'s and **G. de Leibniz** had some conversations via mail.

Numerical solution of fractional PDEs

A timeline of fractional calculus during 1650-1950

A poster about the old history of fractional calculus, J. A. Tenreiro Machado, Virginia Kiryakova, Francesco Mainardi, *Fractional Calculus and Applied Analysis* 13(4), 447-454, 2010.

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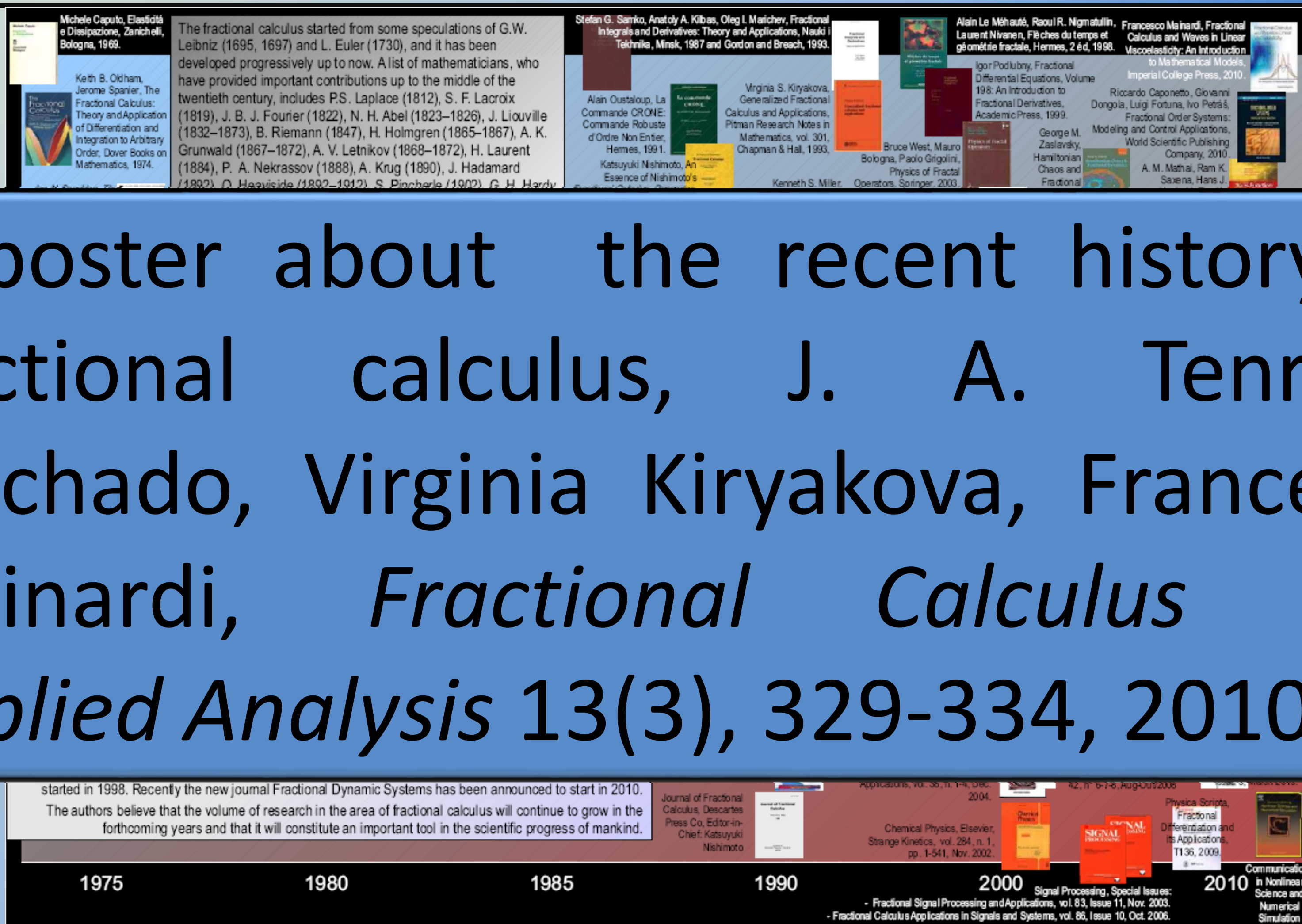
A timeline of fractional calculus during 1975-2010



A poster about the recent history of fractional calculus, J. A. Tenreiro Machado, Virginia Kiryakova, Francesco Mainardi, *Fractional Calculus and Applied Analysis* 13(3), 329-334, 2010.

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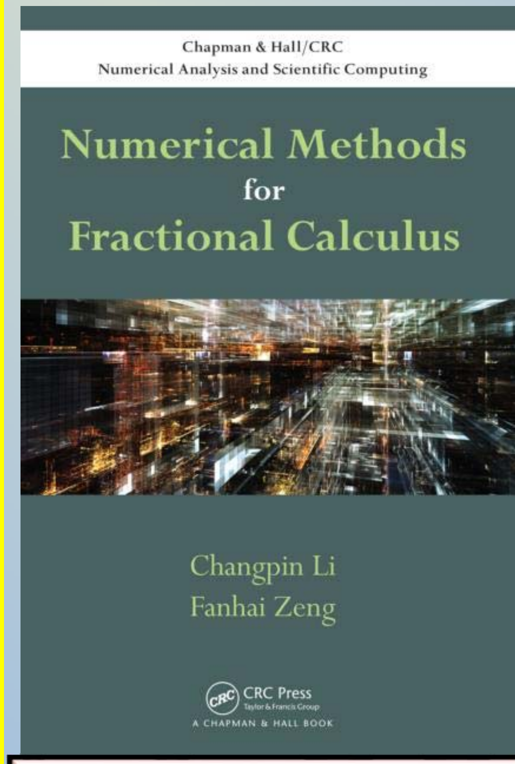
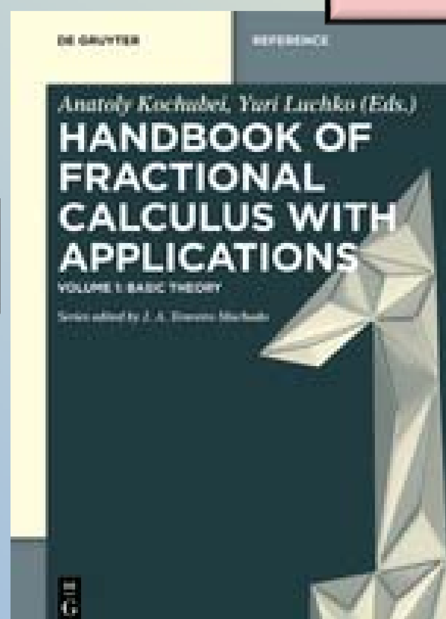
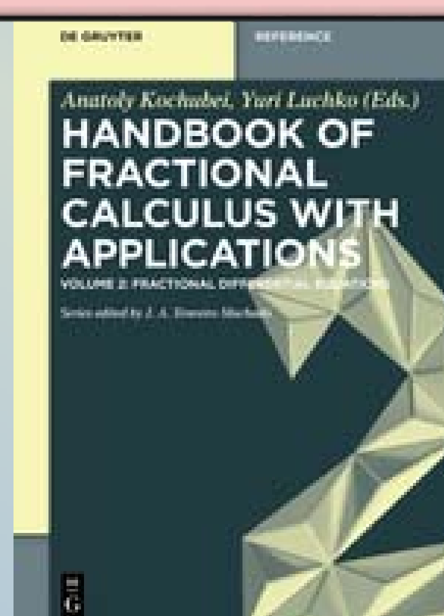
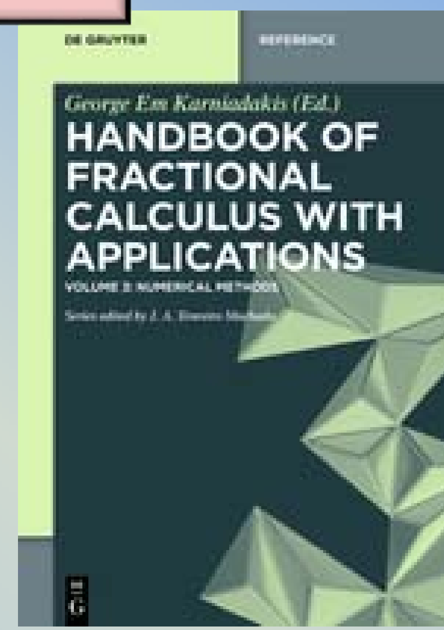


Main reference

Other references







Numerical solution of fractional PDEs

- Introduction to fractional calculus
- Numerical methods for fractional integral and derivatives
- Numerical methods for fractional ODEs
- FDM for fractional PDEs
- FEM for fractional PDEs

Introduction to fractional calculus

What is *fractional calculus*?

Definition

Fractional calculus, as its name suggests, refers to fractional integration and fractional differentiation. In other words, the theory of integrals and derivatives of arbitrary order, may unify and generalize the notion of integer-order differentiation and integration. It does not mean the **calculus of fractions**, neither does it mean **a fraction of calculus**.

Fractional calculus is often regarded as a branch of mathematical analysis which deals with integro-differential equations where the integrals are of the convolution type and exhibit (weakly singular) kernels of the power-law type.

Numerical solution of fractional PDEs

Introduction to fractional calculus

- **Fractional calculus**
  - Fractional integration
    - Riemann-Liouville integral
  - Fractional differentiation
    - Grünwald-Letnikov derivative
    - Riemann-Liouville derivative
    - Caputo derivative
    - Riesz derivative

Numerical solution of fractional PDEs

Introduction to fractional calculus

Riemann-Liouville fractional integral

n-fold definite integral

$$\int_a^x dx \int_a^x dx \cdots \int_a^x f(x)dx = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt$$

left- and right-hand sided Riemann–Liouville fractional integrals

$$\mathrm{D}_{a,t}^{-\alpha} f(t) = {}_{RL}\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s$$

$\alpha > 0$

$f \in L^1(a,b)$

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, ds$$

$$\mathrm{D}_{t,b}^{-\alpha} f(t) = {}_{RL}\mathrm{D}_{t,b}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\,\mathrm{d}s$$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Riemann-Liouville fractional integral

Some useful properties

left and right Riemann–Liouville fractional integral

$$D_{a,t}^{-\alpha} D_{a,t}^{-\beta} f(t) = D_{a,t}^{-\beta} D_{a,t}^{-\alpha} f(t) = D_{a,t}^{-\alpha-\beta} f(t)$$

$\alpha, \beta > 0$

$$D_{t,b}^{-\alpha} D_{t,b}^{-\beta} f(t) = D_{t,b}^{-\beta} D_{t,b}^{-\alpha} f(t) = D_{t,b}^{-\alpha-\beta} f(t)$$

$$\lim_{t\to a} D_{a,t}^{-\alpha} f(t) = \lim_{t\to b} D_{t,b}^{-\alpha} f(t) = 0, \qquad \forall \alpha > 0$$

If $f(t)$ is continuous on $[a,b]$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Grünwald-Letnikov fractional derivative

the *original formula* for derivatives of order n

$$D_x^n(f(t)) = \lim_{h\to 0} \frac{\sum_{j=0}^{n}(-1)^j \binom{n}{j} f(x-jh)}{h^n}$$

left and right Grünwald–Letnikov derivatives

$${}_{GL}\mathrm{D}_{a,t}^{\alpha} f(t) = \lim_{\substack{h\to 0 \\ Nh=t-a}} h^{-\alpha} \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} f(t-jh)$$

$\alpha > 0$

$${}_{GL}\mathrm{D}_{t,b}^{\alpha} f(t) = \lim_{\substack{h\to 0 \\ Nh=b-t}} h^{-\alpha} \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} f(t+jh)$$

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Introduction to fractional calculus

Riemann-Liouville fractional derivative

left and right Riemann–Liouville derivatives

$$
{}_{RL}\mathrm{D}_{a,t}^{\alpha} f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\mathrm{D}_{a,t}^{-(m-\alpha)} f(t)\right]
= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_a^t (t-s)^{m-\alpha-1} f(s)\,\mathrm{d}s
$$

$\alpha > 0$ $\qquad$ $m-1 \le \alpha < m$

$$
{}_{RL}\mathrm{D}_{t,b}^{\alpha} f(t) = (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\mathrm{D}_{t,b}^{-(m-\alpha)} f(t)\right]
= \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_t^b (s-t)^{m-\alpha-1} f(s)\,\mathrm{d}s
$$

The left-inverse operators to the corresponding fractional RL integrals

Numerical solution of fractional PDEs

Introduction to fractional calculus

Riemann-Liouville fractional derivative

Some useful properties

$$\mathrm{D}_{a,t}^{-\alpha}\left({}_{RL}\mathrm{D}_{a,t}^{\alpha}f(t)\right) = f(t) - \sum_{j=1}^{m}\left[{}_{RL}\mathrm{D}_{a,t}^{\alpha-j}f(t)\right]_{t=a}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}$$

$m-1 \le \alpha < m$, m *is a positive integer.*

$$\mathrm{D}_{t,b}^{-\alpha}\left({}_{RL}\mathrm{D}_{t,b}^{\alpha}f(t)\right) = f(t) - \sum_{j=1}^{m}\left[{}_{RL}\mathrm{D}_{t,b}^{\alpha-j}f(t)\right]_{t=b}\frac{(b-t)^{\alpha-j}}{\Gamma(\alpha-j+1)}$$

$$\mathrm{D}_{a,t}^{-\alpha}\,{}_{RL}\mathrm{D}_{a,t}^{\alpha}f(t) = f(t), \qquad \mathrm{D}_{t,b}^{-\alpha}\,{}_{RL}\mathrm{D}_{t,b}^{\alpha}f(t) = f(t)$$

$f^{(j)}(a) =$

$$\mathrm{D}_{a,t}^{-\beta}\left({}_{RL}\mathrm{D}_{a,t}^{\alpha}f(t)\right) = {}_{RL}\mathrm{D}_{a,t}^{\alpha-\beta}f(t) - \sum_{j=1}^{m}\left[{}_{RL}\mathrm{D}_{a,t}^{\alpha-j}f(t)\right]_{t=a}\frac{(t-a)^{\beta-j}}{\Gamma(1+\beta-j)},$$

$$\mathrm{D}_{t,b}^{-\beta}\left({}_{RL}\mathrm{D}_{t,b}^{\alpha}f(t)\right) = {}_{RL}\mathrm{D}_{t,b}^{\alpha-\beta}f(t) - \sum_{j=1}^{m}\frac{(b-t)^{\beta-j}}{\Gamma(1+\beta-j)}.$$

$${}_{RL}\mathrm{D}_{a,t}^{\alpha}\mathrm{D}_{a,t}^{-\alpha}f(t) = f(t)$$

$${}_{RL}\mathrm{D}_{t,b}^{\alpha}\mathrm{D}_{t,b}^{-\alpha}f(t) = f(t)$$

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Introduction to fractional calculus

Riemann-Liouville fractional derivative

Some useful properties

$$
{}_{RL}\mathrm{D}_{a,t}^{\alpha}\left({}_{RL}\mathrm{D}_{a,t}^{\beta}f(t)\right) = {}_{RL}\mathrm{D}_{a,t}^{\alpha+\beta}f(t) - \sum_{j=1}^{n}\left[{}_{RL}\mathrm{D}_{a,t}^{\beta-j}f(t)\right]_{t=a}\frac{(t-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)}
$$

$$
m-1 \le \alpha < m,\ n-1 \le \beta < n
$$

$$
{}_{RL}\mathrm{D}_{t,b}^{\alpha}\left({}_{RL}\mathrm{D}_{t,b}^{\beta}f(t)\right) = {}_{RL}\mathrm{D}_{a,t}^{\alpha+\beta}f(t) - \sum_{j=1}^{n}\left[{}_{RL}\mathrm{D}_{t,b}^{\beta-j}f(t)\right]_{t=b}\frac{(b-t)^{-\alpha-j}}{\Gamma(1-\alpha-j)}
$$

$$
\left[{}_{RL}\mathrm{D}_{a,t}^{\beta-j}f(t)\right]_{t=a} = \left[{}_{RL}\mathrm{D}_{b,t}^{\beta-j}f(t)\right]_{t=b}\left[{}_{RL}\mathrm{D}_{a,t}^{\alpha-k}f(t)\right]_{t=a} = \left[{}_{RL}\mathrm{D}_{b,t}^{\alpha-k}f(t)\right]_{t=b} = 0
$$

$$
\Downarrow
$$

$$
{}_{RL}\mathrm{D}_{a,t}^{\alpha}\left({}_{RL}\mathrm{D}_{a,t}^{\beta}f(t)\right) = {}_{RL}\mathrm{D}_{a,t}^{\beta}\left({}_{RL}\mathrm{D}_{a,t}^{\alpha}f(t)\right) = {}_{RL}\mathrm{D}_{a,t}^{\alpha+\beta}f(t)
$$

$$
{}_{RL}\mathrm{D}_{t,b}^{\alpha}\left({}_{RL}\mathrm{D}_{t,b}^{\beta}f(t)\right) = {}_{RL}\mathrm{D}_{t,b}^{\beta}\left({}_{RL}\mathrm{D}_{t,b}^{\alpha}f(t)\right) = {}_{RL}\mathrm{D}_{t,b}^{\alpha+\beta}f(t)
$$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Caputo fractional derivative

left and right Caputo derivatives

$$
{}_C\mathrm{D}^{\alpha}_{a,t} f(t) = \mathrm{D}^{-(m-\alpha)}_{a,t}\left[f^{(m)}(t)\right] = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s)\,\mathrm{d}s
$$

$\alpha > 0$ $\quad$ $m-1 < \alpha \leq m$

$$
{}_C\mathrm{D}^{\alpha}_{t,b} f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_t^b (s-t)^{m-\alpha-1} f^{(m)}(s)\,\mathrm{d}s
$$

$$
\lim_{t\to a} {}_C\mathrm{D}^{\alpha}_{a,t} f(t) = \lim_{t\to b} {}_C\mathrm{D}^{\alpha}_{t,b} f(t) = 0, \quad \alpha > 0
$$

$f(t)$ is sufficiently smooth and $f^{(m)}$ is bounded

Numerical solution of fractional PDEs

Introduction to fractional calculus

Caputo fractional derivative

Some useful properties

$$
{}_C\mathrm{D}_{a,t}^{\beta}\left(\mathrm{D}_{a,t}^{-\alpha} f(t)\right)=\begin{cases}\mathrm{D}_{a,t}^{-(\alpha-\beta)} f(t), \quad \beta\leq\alpha \text{ or } \alpha<\beta,\ \alpha\in\mathbb{N},\\ {}_C\mathrm{D}_{a,t}^{\beta-\alpha} f(t)+\displaystyle\sum_{k=0}^{n-m}\frac{f^{(k)}(a)}{\Gamma(k+1+\alpha-\beta)}(t-a)^{k+\alpha-\beta},\\ \quad \alpha<\beta,\ m-1<\alpha<m,\ n-1<\beta<n,\ m,n\in\mathbb{N}.\end{cases}
$$

$$
{}_C\mathrm{D}_{t,b}^{\beta}\left(\mathrm{D}_{t,b}^{-\alpha} f(t)\right)=\begin{cases}\mathrm{D}_{t,b}^{-(\alpha-\beta)} f(t), \quad \beta\leq\alpha \text{ or } \alpha<\beta,\ \alpha\in\mathbb{N},\\ {}_C\mathrm{D}_{t,b}^{\beta-\alpha} f(t)+\displaystyle\sum_{k=0}^{n-m}\frac{f^{(k)}(b)}{\Gamma(k+1+\alpha-\beta)}(b-t)^{k+\alpha-\beta},\\ \quad \alpha<\beta,\ m-1<\alpha<m,\ n-1<\beta<n,\ m,n\in\mathbb{N}.\end{cases}
$$

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Introduction to fractional calculus

Caputo fractional derivative

Some useful properties

$$ {}_{C}\mathrm{D}_{a,t}^{\beta}\,{}_{C}\mathrm{D}_{a,t}^{m} f(t) = {}_{C}\mathrm{D}_{a,t}^{m+\beta} f(t), \quad \beta > 0, m \in \mathbb{N} $$

$$ {}_{RL}\mathrm{D}_{a,t}^{m}\left({}_{RL}\mathrm{D}_{a,t}^{\beta} f(t)\right) = {}_{RL}\mathrm{D}_{a,t}^{m+\beta} f(t) $$

$$ \begin{aligned} {}_{C}\mathrm{D}_{a,t}^{m}\,{}_{C}\mathrm{D}_{a,t}^{\beta} f(t) &= {}_{RL}\mathrm{D}_{a,t}^{m}\,{}_{RL}\mathrm{D}_{a,t}^{-(n-\beta)} f^{(n)}(t) = {}_{RL}\mathrm{D}_{a,t}^{m-n-\beta} f^{(n)}(t) \\ &= {}_{RL}\mathrm{D}_{a,t}^{m+\beta} f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(1+j-m-\beta)}(t-a)^{j-m-\beta} \end{aligned} $$

$$ \begin{aligned} {}_{RL}\mathrm{D}_{a,t}^{\beta}\left({}_{RL}\mathrm{D}_{a,t}^{m} f(t)\right) &= {}_{RL}\mathrm{D}_{a,t}^{\beta+m} f(t) - \sum_{j=1}^{m} \frac{f^{(m-j)}(a)}{\Gamma(1-\beta-j)}(t-a)^{-\beta-j} \\ &= {}_{RL}\mathrm{D}_{a,t}^{\beta+m} f(t) - \sum_{j=0}^{m-1} \frac{f^{(j)}(a)}{\Gamma(1+j-\beta-m)}(t-a)^{j-\beta-m} \end{aligned} $$

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Introduction to fractional calculus

Riesz fractional derivative

Riesz derivative

$$\frac{\partial^\alpha f(t)}{\partial |t|^\alpha}$$

$$_{RZ}\mathrm{D}_t^\alpha f(t) = c_\alpha \left({}_{RL}\mathrm{D}_{a,t}^\alpha f(t) + {}_{RL}\mathrm{D}_{t,b}^\alpha f(t) \right)$$

$$c_\alpha = -\frac{1}{2\cos(\alpha\pi/2)}, \alpha \neq 2k+1, k = 0, 1, \cdots$$

Remark

In the above definitions, the initial value a is often set to zero. When we say the fractional integral (or Riemann–Liouville integral), the Grünwald–Letnikov derivative, the Riemann–Liouville derivative, and the Caputo derivative, we often mean the left fractional integral/derivative.

Numerical solution of fractional PDEs

Introduction to fractional calculus

Some relations between fractional derivatives

$$f \in C^m[a,b]$$

$$_{RL}\mathrm{D}_{a,t}^{\alpha} f(t) = {}_{GL}\mathrm{D}_{a,t}^{\alpha} f(t), \quad {}_{RL}\mathrm{D}_{t,b}^{\alpha} f(t) = {}_{GL}\mathrm{D}_{t,b}^{\alpha} f(t)$$

$f \in C^{m-1}[a,t]$ and $f^{(m)}$ is integrable on $[a,t]$

$$_{RL}\mathrm{D}_{a,t}^{\alpha} f(t) = {}_{C}\mathrm{D}_{a,t}^{\alpha} f(t) + \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)}$$

$$m-1 < \alpha < m$$

$$f^{(k)}(a) = 0 \; (k = 0, 1, 2, \cdots, m-1, m-1 < \alpha < m)$$

$$_{RL}\mathrm{D}_{a,t}^{\alpha} f(t) = {}_{C}\mathrm{D}_{a,t}^{\alpha} f(t)$$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Some relations between fractional and ordinary derivatives

$f(t)$ is suitably smooth, $m-1<\alpha<m$, m is a positive integer

$$\lim_{\alpha\to m^-} {}_{RL}\mathrm{D}^{\alpha}_{a,t} f(t) = f^{(m)}(t), \qquad \lim_{\alpha\to (m-1)^+} {}_{RL}\mathrm{D}^{\alpha}_{a,t} f(t) = f^{(m-1)}(t);$$

$$\lim_{\alpha\to m^-} {}_{C}\mathrm{D}^{\alpha}_{a,t} f(t) = f^{(m)}(t), \qquad \lim_{\alpha\to (m-1)^+} {}_{C}\mathrm{D}^{\alpha}_{a,t} f(t) = f^{(m-1)}(t) - f^{(m-1)}(0)$$

Remark

Obviously, the Riemann–Liouville derivative is reduced to the classical derivative when the fractional order α approaches an integer for the fixed t, but it is not the case for the Caputo derivative if the homogeneous initial conditions are not satisfied.

Numerical solution of fractional PDEs

Introduction to fractional calculus

Some other properties of fractional derivatives

Leibniz rule

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Big(g(t)f(t)\Big) = \sum_{k=0}^{n}\binom{n}{k} g^{(k)}(t) f^{(n-k)}(t)$$

Leibniz rule for fractional differentiation

$$f(t) = 1$$

$${}_{RL}\mathrm{D}_{a,t}^{\alpha}(g(t)f(t)) = \sum_{k=0}^{\infty}\binom{\alpha}{k} g^{(k)}(t) f^{(\alpha-k)}(t)$$

$$f^{(\alpha-k)}(t) = {}_{RL}\mathrm{D}_{a,t}^{\alpha-k} f(t)$$

$${}_{RL}\mathrm{D}_{a,t}^{\alpha} g(t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} g(t) + \sum_{k=1}^{\infty}\binom{\alpha}{k}\frac{(t-a)^{k-\alpha}}{\Gamma(1-\alpha+k)} g^{(k)}(t)$$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Some other properties of fractional derivatives

$$ {}_{RL}\mathrm{D}^{\alpha}_{a,t} g(t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} g(t) + \sum_{k=1}^{\infty} \binom{\alpha}{k} \frac{(t-a)^{k-\alpha}}{\Gamma(1-\alpha+k)} g^{(k)}(t) $$

$$ g(t) = F(h(t)) $$

$$ \frac{\mathrm{d}^k}{\mathrm{d}t^k} F(h(t)) = k! \sum_{m=1}^{k} F^{(m)}(h(t)) \sum \prod_{r=1}^{k} \frac{1}{a_r!} \left(\frac{h^{(r)}(t)}{r!} \right)^{a_r} $$

$$ {}_{RL}\mathrm{D}^{\alpha}_{a,t} F(h(t)) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} F(h(t)) $$

fractional derivative of a composite function

$$ + \sum_{k=1}^{\infty} \binom{\alpha}{k} \frac{k!(t-a)^{k-\alpha}}{\Gamma(1-\alpha+k)} \sum_{m=1}^{k} F^{(m)}(h(t)) \sum \prod_{r=1}^{k} \frac{1}{a_r!} \left(\frac{h^{(r)}(t)}{r!} \right)^{a_r} $$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Some other properties of fractional derivatives

$$f(t) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{\Gamma(1+k)}(t-a)^k + \mathrm{D}_{a,t}^{-n} f^{(n)}(t)$$

Applying the Riemann–Liouville derivative operator

$${}_{RL}\mathrm{D}_{a,t}^{\alpha} f(t) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha} + \mathrm{D}_{a,t}^{\alpha-n} f^{(n)}(t), \quad \alpha < n$$

$$\lim_{t \to a+0} {}_{RL}\mathrm{D}_{a,t}^{\alpha} f(t) = \begin{cases} 0, & \alpha < 0, \\ f(a), & \alpha = 0, \\ \infty, & \alpha > 0. \end{cases}$$

$f(t) = 1$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Laplace transform of the RL fractional integral

Laplace transform of a given function $f(t)$

$$F(s) = L\{f(t); s\} = \int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t$$

$|f(t)| \le Me^{\mu t}$ holds for all $t > T$

$$f(t) * g(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s = \int_0^t f(s)g(t-s)\,\mathrm{d}s$$

$$L\{f(t) * g(t); s\} = F(s)G(s)$$

$$g(t) = t^{\alpha-1}$$

$$\mathrm{D}_{0,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t)$$

Laplace transform of the fractional integral

$$L\{\mathrm{D}_{0,t}^{-\alpha} f(t); s\} = \frac{1}{\Gamma(\alpha)} L\{t^{\alpha-1} * f(t); s\} = s^{-\alpha} L\{f(t); s\} = s^{-\alpha} F(s)$$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Laplace transform of the RL fractional derivative

$$L\{f^{(n)}(t);s\} = s^n L\{f(t);s\} - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n L\{f(t);s\} - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0)$$

$$g(t) = \mathrm{D}_{0,t}^{-(m-\alpha)} f(t) \longrightarrow {}_{RL}\mathrm{D}_{0,t}^{\alpha} f(t) = g^{(m)}(t)$$

$$\longrightarrow L\{{}_{RL}\mathrm{D}_{0,t}^{\alpha} f(t);s\} = L\{g^{(m)}(t);s\} = s^m L\{g(t);s\} - \sum_{k=0}^{m-1} s^k g^{(m-k-1)}(0)$$

$$L\{g(t);s\} = L\{\mathrm{D}_{0,t}^{-(m-\alpha)} f(t);s\} = s^{-(m-\alpha)} L\{f(t);s\}$$

Laplace transform of the Riemann–Liouville derivative

$$L\{{}_{RL}\mathrm{D}_{0,t}^{\alpha} f(t);s\} = s^{\alpha} L\{f(t);s\} - \sum_{k=0}^{m-1} s^k \left[{}_{RL}\mathrm{D}_{0,t}^{\alpha-k-1} f(t)\right]_{t=0}, \quad m-1 \le \alpha < m$$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Laplace transform of the Caputo fractional derivative

$$ {}_C\mathrm{D}_{0,t}^{\alpha} f(t) = \mathrm{D}_{0,t}^{-(m-\alpha)} g(t), \quad g(t) = f^{(m)}(t) $$

$$\begin{aligned} L\{{}_C\mathrm{D}_{0,t}^{\alpha} f(t); s\} &= L\{\mathrm{D}_{0,t}^{-(m-\alpha)} g(t); s\} = s^{-(m-\alpha)} L\{g(t); s\} \\ &= s^{-(m-\alpha)} \left[s^m L\{f(t); s\} - \sum_{k=0}^{m-1} s^{m-k-1} f^{(k)}(0) \right] \\ &= s^{\alpha} L\{f(t); s\} - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0). \end{aligned}$$

Laplace transform of the Caputo derivative

$$ L\{{}_C\mathrm{D}_{0,t}^{\alpha} f(t); s\} = s^{\alpha} L\{f(t); s\} - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0), \quad m-1 < \alpha \le m $$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Fourier transform of the RL fractional integral

$$F\{f(t);\omega\} = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,\mathrm{d}t$$

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t-s)g(s)\,\mathrm{d}s = \int_{-\infty}^{\infty} f(s)g(t-s)\,\mathrm{d}s$$

$$F\{f(t) * g(t);\omega\} = F\{f(t);\omega\}F\{g(t);\omega\}$$

$$h_+(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t = 0. \end{cases}$$

$0 < \alpha < 1$

$$F\{h_+(t);\omega\} = (i\omega)^{-\alpha} \qquad \mathrm{D}_{-\infty,t}^{-\alpha} f(t) = h_+(t) * f(t)$$

$$F\{h_+(-t);\omega\} = (-i\omega)^{-\alpha} \qquad \mathrm{D}_{t,\infty}^{-\alpha} f(t) = h_+(-t) * f(t)$$

left

$$\begin{aligned} F\{\mathrm{D}_{-\infty,t}^{-\alpha} f(t);\omega\} &= F\{h_+(t) * f(t);\omega\} = F\{h_+(t);\omega\}F\{f(t);\omega\} \\ &= (i\omega)^{-\alpha} F\{f(t);\omega\}. \end{aligned}$$

$$F\{\mathrm{D}_{t,\infty}^{-\alpha} f(t);\omega\} = F\{h_+(-t);\omega\}F\{f(t);\omega\} = (-i\omega)^{-\alpha} F\{f(t);\omega\}$$

right

Numerical solution of fractional PDEs

Introduction to fractional calculus

Fourier transform of the fractional derivative

If $f^{(k)}(t)$ $(k = 0, 1, 2, \cdots, n-1)$ vanish as $t \to \pm\infty$

$$F\{f^{(n)}(t);\omega\} = (i\omega)^n F\{f(t);\omega\}$$

$f(t)$ is sufficiently smooth and $f^{(k)}(-\infty)$ $(k = 0, 1, \cdots, m-1)$ are bounded

$$\left.\begin{array}{r} {}_{GL}\mathrm{D}^{\alpha}_{t,\infty} f(t) \\ {}_{RL}\mathrm{D}^{\alpha}_{t,\infty} f(t) \\ {}_{C}\mathrm{D}^{\alpha}_{t,\infty} f(t) \end{array}\right\} = (-1)^m \mathrm{D}^{-(m-\alpha)}_{t,\infty} f^{(m)}(t), \quad m-1 < \alpha < m$$

$$\begin{aligned} & F\{{}_{RL}\mathrm{D}^{\alpha}_{-\infty,t} f(t);\omega\} = F\{{}_{RL}\mathrm{D}^{-(m-\alpha)}_{-\infty,t} f^{(m)}(t);\omega\} \\ =& (i\omega)^{-(m-\alpha)} F\{f^{(m)}(t);\omega\} = (i\omega)^{-(m-\alpha)} (i\omega)^m F\{f(t);\omega\} \\ =& (i\omega)^{\alpha} F\{f(t);\omega\}. \end{aligned}$$

$$F\{{}_{RL}\mathrm{D}^{\alpha}_{t,\infty} f(t);\omega\} = (-i\omega)^{\alpha} F\{f(t);\omega\}$$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Directional integral and derivative in $\mathbb{R}^2$

α-th order fractional integral in the direction of θ

$$\mathrm{D}_\theta^{-\alpha} u(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \xi^{\alpha-1} u(x-\xi\cos\theta, y-\xi\sin\theta)\,\mathrm{d}\xi$$

$$\alpha > 0,\ \theta \in [0, 2\pi)$$

$$\mathrm{D}_0^{-\alpha} u(x,y) = \mathrm{D}_{-\infty,x}^{-\alpha} u(x,y), \qquad \mathrm{D}_{\pi/2}^{-\alpha} u(x,y) = \mathrm{D}_{-\infty,y}^{-\alpha} u(x,y),$$

$$\mathrm{D}_\pi^{-\alpha} u(x,y) = \mathrm{D}_{x,\infty}^{-\alpha} u(x,y), \qquad \mathrm{D}_{3\pi/2}^{-\alpha} u(x,y) = \mathrm{D}_{y,\infty}^{-\alpha} u(x,y).$$

α-th order fractional derivative in the direction θ

$$\mathrm{D}_\theta^{\alpha} u(x,y) = \mathrm{D}_\theta^{n} \mathrm{D}_\theta^{-(n-\alpha)} u(x,y)$$

$$n-1 \le \alpha < n,\ \theta \in [0, 2\pi)$$

$$\mathrm{D}_\theta^{n} u(x,y) = \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)^n u(x,y)$$

Introduction to fractional calculus

Directional integral and derivative in $\mathbb{R}^2$

$$\alpha, \beta > 0,\ \theta \in [0, 2\pi),\ u \in L^2(\mathbb{R}^2)$$

$$\mathrm{D}_\theta^{-\alpha}\mathrm{D}_\theta^{-\beta}u(x,y) = \mathrm{D}_\theta^{-\alpha-\beta}u(x,y)$$

$$\alpha > 0,\ \theta \in [0, 2\pi),\ u \in L^2(\mathbb{R}^2)$$

$$\mathrm{D}_\theta^{\alpha}\mathrm{D}_\theta^{-\alpha}u(x,y) = u(x,y)$$

$$F\{u(x,y);\omega\} = \int_{\mathbb{R}^2} e^{-i(\omega_1 x+\omega_2 y)}u(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$F\{\mathrm{D}_\theta^{-\alpha}u(x,y);\omega\} = (i\omega_1\cos\theta + i\omega_2\sin\theta)^{-\alpha}F\{u(x,y);\omega\}$$

$$u \in C_0^\infty(\Omega), \Omega \in \mathbb{R}^2 \ and\ \alpha > 0$$

$$F\{\mathrm{D}_\theta^{\alpha}u(x,y);\omega\} = (i\omega_1\cos\theta + i\omega_2\sin\theta)^{\alpha}F\{u(x,y);\omega\}$$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Partial fractional derivative

Partial RL derivative of order $\alpha_1 + \alpha_2$

$$
{}_{RL}\mathrm{D}^{\alpha_1+\alpha_2}_{x^{\alpha_1}y^{\alpha_2}}u(x,y) = \frac{1}{\Gamma(m-\alpha_1)\Gamma(n-\alpha_2)}\frac{\partial^{m+n}}{\partial x^m \partial y^n} \times \int_0^x \int_0^y (x-s)^{m-\alpha_1-1}(y-\tau)^{n-\alpha_2-1}u(s,\tau)\,\mathrm{d}\tau\,\mathrm{d}s
$$

$m-1<\alpha_1<m$, $n-1<\alpha_2<n$, m, n *are positive integers*

Partial Caputo derivative of order $\alpha_1 + \cdots + \alpha_l$

$$
{}_{C}\mathrm{D}^{\alpha_1+\alpha_2+\cdots+\alpha_\ell}_{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_\ell^{\alpha_\ell}}u(x,y) = \frac{1}{\prod_{k=1}^{\ell}\Gamma(m_k-\alpha_k)}\int_0^{x_1}\cdots\int_0^{x_\ell}(x_\ell-\xi_\ell)^{m_\ell-\alpha_\ell-1}\cdots(x_1-\xi_1)^{m_1-\alpha_1-1} \times \frac{\partial^{m_1+m_2+\cdots+m_\ell}}{\partial\xi_1^{m_1}\partial\xi_2^{m_2}\cdots\partial\xi_\ell^{m_\ell}}u(\xi_1,\cdots,\xi_\ell)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_\ell,
$$

$m_k-1<\alpha_k<m_k$ $(k=1,2,\cdots,\ell)$, m_k *are positive integers*

Numerical solution of fractional PDEs

Introduction to fractional calculus

Fractional calculus versus classical calculus

$$\lim_{\alpha\to(n-1)^+} {}_C\mathrm{D}_{0,t}^{\alpha}x(t) = x^{(n-1)}(t) - x^{(n-1)}(0), \quad \lim_{\alpha\to n^-} {}_C\mathrm{D}_{0,t}^{\alpha}x(t) = x^{(n)}(t)$$

$\alpha \in (n-1, n)$

The Caputo derivative is not the mathematical extension of a typical derivative.

$$x(t) = \begin{cases} 1-t, t \in [0,1], \\ t-1, t \in (1, 1+t_0). \end{cases}$$

$x'(t)$ exists on the domain $[0,1) \cup (1, 1+t_0]$

${}_{RL}\mathrm{D}_{0,t}^{\alpha}x(t)$ $(\alpha \in (0,1))$ exists on the interval $(0, 1+t_0]$

The RL derivative is not the mathematical generalization of the typical derivative.

Remark

Overall, fractional calculus, closely related to classical calculus, is not direct generalization of classical calculus in the sense of rigorous mathematics.

Numerical solution of fractional PDEs

Introduction to fractional calculus

Geometrical and physical meaning of fractional integral/derivative

Recalling … etry, it means the area of the … $f(x) \geq 0$. From the viewpoint …) indicates the velocity at time x. The geometrical and physical meaning of the derivative is well known … dicate … On the other h … nt at … city at time t, $s''(t)$ the acceleration at time t. Now, we give a possible interpretation of the fractional calculus.

$$D_{a,t}^{-\alpha} f(t) = {}_{RL}D_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, \mathrm{d}s$$

$$D_{a,t}^{-\alpha} f(t) = \int_a^t f(\tau)\mathrm{d}Y_\alpha(\tau)$$

$$Y_\alpha(\tau) = \begin{cases} -\frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)}, & \tau \in [a,t], \\ 0, & \tau < a. \end{cases}$$

This is the standard *Stieltjes integral*. $Y_\alpha(\tau)$ is a monotonously increasing function in $(-\infty, t]$. The positive number α is an index characterizing the singularity: the smaller α, the stronger singularity the integral. If $Y_\alpha(\tau) = \tau$, the above integral is reduced to a typical one. So $D_{a,t}^{-\alpha} f(t)$ indicates the generalized area in the sense of length $Y_\alpha(\tau)$ (geometrical meaning) or the generalized displacement in the sense of $Y_\alpha(\tau)$ if $f(t)$ means the velocity at time t (physical meaning).

Numerical solution of fractional PDEs

Introduction to fractional calculus

Geometrical and physical meaning of fractional integral/derivative

$$
{}_{RL}\mathrm{D}_{a,t}^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t f(\tau)\mathrm{d}Y_{1-\alpha}(\tau)
$$

$$
Y_{1-\alpha}(\tau) = \begin{cases} -\frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)}, & \tau \in [a,t], \\ 0, & \tau < a. \end{cases}
$$

Obviously, $Y_{1-\alpha}(\tau)$ is a monotonously increasing function in $(-\infty, t]$. So ${}_{RL}\mathrm{D}_{a,t}^{\alpha} f(t)$ indicates the generalized slope in the sense of length $Y_{1-\alpha}(\tau)$ if $f(t)$ means the slope (geometrical meaning) or the generalized velocity in the sense of length $Y_{1-\alpha}(\tau)$ if $f(t)$ means the velocity (physical meaning). If $Y_{1-\alpha}(\tau) = \tau$, it is reduced to the classical case.

$$
{}_{C}\mathrm{D}_{a,t}^{\alpha} f(t) = \int_a^t f'(\tau)\mathrm{d}Y_{1-\alpha}(\tau)
$$

$$
Y_{1-\alpha}(\tau) = \begin{cases} -\frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)}, & \tau \in [a,t], \\ 0, & \tau < a. \end{cases}
$$

So ${}_{C}\mathrm{D}_{a,t}^{\alpha} f(t)$ indicates the generalized displacement of the curve $f(t)$ in the sense of length $Y_{1-\alpha}(\tau)$ (physical meaning) if $f(t)$ means the displacement, or represents the generalized curve in the sense of length $Y_{1-\alpha}(\tau)$ if $f(t)$ is a curve (geometrical meaning). If $Y_{1-\alpha}(\tau) = \tau$, the above integral is reduced to the typical one.

Numerical solution of fractional PDEs

Introduction to fractional calculus

Fractional initial value problems

Caputo fractional differential equation

$$ {}_C\mathrm{D}_{0,t}^{\alpha} x(t) = f(x,t),\ n-1 < \alpha < n \in Z^+ $$

$$ {}_C D_{0,t}^{\alpha} x(t) = \mathrm{D}_{0,t}^{-(n-\alpha)} D^n x(t) = \mathrm{D}_{0,t}^{-(n-\alpha)} x^{(n)}(t) $$

Ordinary differential equation

$$ x^{(n)}(t) = {}_{RL}\mathrm{D}_{0,t}^{n-\alpha} f(x,t) = \frac{1}{\Gamma(\alpha-n+1)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-\tau)^{\alpha-n} f(x(\tau),\tau)\mathrm{d}\tau =: F(x,t) $$

Necessary (for being well-posed) initial conditions

$$ x^{(k)}(0) = x_0^{(k)},\ k = 0, 1, \ldots, n-1 $$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Fractional initial value problems

RL fractional differential equation

$$_{RL}\mathrm{D}_{0,t}^{\alpha}x(t) = f(x,t), \; n-1 < \alpha < n \in Z^{+}$$

$$_{RL}\mathrm{D}_{0,t}^{\alpha}x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\, {}_{RL}\mathrm{D}_{0,t}^{-(n-\alpha)}x(t)$$

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left({}_{RL}\mathrm{D}_{0,t}^{-(n-\alpha)}x(t)\right) = f(x,t)$$

Necessary (for being well-posed) initial conditions

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left({}_{RL}\mathrm{D}_{0,t}^{-(n-\alpha)}x(t)\right)_{t=0} = {}_{RL}\mathrm{D}_{0,t}^{k+\alpha-n}x(t)|_{t=0} = x_0^{(k)}, \; k = 0, 1, \ldots, n-1$$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Two-point fractional boundary value problem

$$ {}_{RL}\mathrm{D}_{a,x}^{\alpha} y(x) = f(x,y),\ x \in (a,b),\ 1 < \alpha < 2 $$

Necessary (for being well-posed) boundary conditions

i) ${}_{RL}\mathrm{D}_{a,x}^{\alpha-2} y(x)|_{x=a} = c_1,\ {}_{RL}\mathrm{D}_{x,b}^{\alpha-2} y(x)|_{x=b} = c_2,$

ii) ${}_{RL}\mathrm{D}_{a,x}^{\alpha-1} y(x)|_{x=a} = c_3,\ {}_{RL}\mathrm{D}_{x,b}^{\alpha-1} y(x)|_{x=b} = c_4,$

iii) ${}_{RL}\mathrm{D}_{a,x}^{\alpha-2} y(x)|_{x=a} = c_1,\ {}_{RL}\mathrm{D}_{x,b}^{\alpha-1} y(x)|_{x=b} = c_4,$

iv) ${}_{RL}\mathrm{D}_{a,x}^{\alpha-1} y(x)|_{x=a} = c_3,\ {}_{RL}\mathrm{D}_{x,b}^{\alpha-2} y(x)|_{x=b} = c_2.$

Numerical solution of fractional PDEs

Introduction to fractional calculus

Fractional initial-boundary value problem

$${}_{RL}\mathrm{D}^{\alpha}_{0,t}u(x,t) = {}_{RL}\mathrm{D}^{\beta}_{a,x}u(x,t),\ t>0,\ x\in(a,b),\ \alpha\in(0,1), \beta\in(1,2)$$

Boundary conditions

i) *Dirichlet type*

$${}_{RL}\mathrm{D}^{\beta-2}_{a,x}u(x,t)|_{x=a} = \xi_1(t),\ {}_{RL}\mathrm{D}^{\beta-2}_{x,b}u(x,t)|_{x=b} = \xi_2(t).$$

ii) *Neumann type*

$${}_{RL}\mathrm{D}^{\beta-1}_{a,x}u(x,t)|_{x=a} = \xi_1(t),\ {}_{RL}\mathrm{D}^{\beta-1}_{x,b}u(x,t)|_{x=b} = \xi_2(t).$$

iii) *Rubin type*

$$\left({}_{RL}\mathrm{D}^{\beta-1}_{a,x}u(x,t) - \sigma_1\,{}_{RL}\mathrm{D}^{\beta-2}_{a,x}u(x,t)\right)_{x=a} = \xi_1(t),\ \sigma_1>0,$$

$$\left({}_{RL}\mathrm{D}^{\beta-1}_{x,b}u(x,t) + \sigma_2\,{}_{RL}\mathrm{D}^{\beta-2}_{x,b}u(x,t)\right)_{x=b} = \xi_2(t),\ \sigma_2>0.$$

$\alpha\in(1,2)$ **Initial conditions** ${}_{RL}\mathrm{D}^{\alpha-2}_{0,t}u(x,t)|_{t=0} = \phi(x),\ {}_{RL}\mathrm{D}^{\alpha-1}_{0,t}u(x,t)|_{t=0} = \psi(x)$

Thanks for your attention

Any question?