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The Logarithms - From Calculation to Functional Equations.

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0. Introduction

John Napier (1550-1617) and Jost Bürgi (1552- 1632) together share the merits to be the inventor of the logarithms. The first printed publications about the logarithms of both, Napier[1] and Bürgi[2] are preserved in the socalled *"Bibliotheca Mathematica"* at the Library of the University of Graz.. This *Bibliotheca Mathematica* is a special part of the rara and incunabla collection, which has its origin in the private library of Paulus Guldin (1576-1643). Guldin was professor for mathematics at the (former Jesuit) University of Graz, where he died *"Anno 1643 Ætatis 67. Societatis 42 relictis plurimis libris et instrumentis"* as one can read from the subtitle of an old oil painted portrait of Guldin.

Especially the possession of *Bürgi's "Progreß Tabulen"* is important in view of several facts. First, there are preserved only very fiew issues of the *Progreß Tabulen*, since they are printed two years after the beginning of the Thirty Years' War, in Praha, the initial place of this war. So, the main part of the edition seems to be destroied by these war events. On the title page of the *Progreß Tabulen* a user's guide for these tables (*"sambt gründlichem unterricht ⋯ "*) is announced. But this user's guide never was printed at Bürgi's time. The first printing was done by Dr. Gieswald in the year 1856, following the manuscript of the library of the city of Danzig/Gdansk ([Gieswald 1856]). So it seems that in Graz is the second of the two unique hand written guide to the Progreß Tabulen, slightly different from the version of Danzig.

Shortly after the contributions of Bürgi and Napier appeared the logarithmic papers of Johannes Kepler (1571- 1630), the *"Chilias Logarithmorum"*, Marburg

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[1] *John Napier: Mirifici Logarithmorum canonis descriptio, Eusque usus, in utraque Trigonometria, ut etiam in omni Logistica Mathematica, Authore ac Inventore, IOANNE NEPERO, Barone Merchistonii, Edinburgi 1614.*

[2] *Jost Bürgi: Arithmetische und Geometrische Progreß Tabulen/ sambt gründlichem unterricht/ wie solche nützlich in allerley Rechnungen zugebrauchen/ und verstanden werden sol. Gedruckt/ In der Alten Stadt Prag/ bey Paul Sessen/ der Löblichen Universität Buchdruckern/Im Jahr/ 1620.*

1624 and the *"Supplementum Chiliadis Logarithmorum"*, Marburg 1625 (see [Kepler 1624 and 1625]). Also Johannes Kepler spent a period of his life in Graz. It was the beginning of his career after his study. But these logarithmic papers have been written by Kepler in Linz in Austria, and have been finished in 1621 about. He knew the "Logarithms" of both, Bürgi and Napier.

As a matter of fact Kepler, a professional mathematician, was the first who propound the logarithm as a function - the function of the natural logarithm, as one says nowaday. Indeed he proposed the logarithmic functional equation, but not only the general wellknown one, namely, in our terminology

$$f(x \cdot y) = f(x) + f(y) \tag{1}$$

but the restricted one

$$f(x^2) = 2 \cdot f(x). \tag{2}$$

Each of the two functional equations characterize the natural logarithm as the only one (up to a constant) continuous differentiable solution of the considered functional equation.

Kepler investigated equation (1) and (2), and determined their solution, which he called "mensura". He used the formula $f(x^{2^n}) = 2^n \cdot f(x)$, derived from (2), for calculating his "Logarithmus".

It seems worthwile to me, to reveal the logarithmic papers of Johannes Kepler and to show his mathematical merits in the theory of functional equations. There are several reference books on the history of mathematics, where the work of Kepler on the logaritms is represented (see e.g. [Hammer], [Naux], or [Tropfke]), but in more or less short way, and not from the point of view of the theory of functional equations. So I hope to present some new and interesting facts on this very impressive period of the development of the mathematic. First I will give a short survey on the theory of the both ancessors of Johannes Kepler.

I. The Logarithms of Bürgi and Napier.

Bürgi and Napier both, use the same mathematical principle for their tables. The two entry rows of the tables are mathematical sequences. One row is an arithmetical sequence

$$x_n = n \cdot s; n = 0, 1, \cdots$$

the other is a geometrical one

$$y_n = z^n \cdot q; n = 0, 1, \cdots$$

where s, z and q are fixed constants, individually chosen. Bürgi uses in his series the constants

$$s = 10, z = 10^8 \text{ and } q = 1 + 10^{-4},$$

Napier takes

$$s = 1 + 0.5 \cdot 10^{-7}, z = 10^7 \text{ and } q = 1 - 10^{-7}.$$

Napier calls the number x_n the "Logarithmus" of y_n . Rules like multiplication, division, calculation of the geometrical mean (regula detri) and root extraction are demonstrated in a intuitive way. If one denotes the "Logarithmus" of Napier by $L_N(y)$, using the defining identities

$$x = L_N(y) \text{ if and only if } y = 10^7(1 - 10^{-7})^{x/s}$$

one gets the approximative formula

$$L_N(y) \approx 10^7 \cdot log\frac{10^7}{y}.$$

For further explanations of Napiers tables see for example [Tropfke 1980], p. 303f and [Hammer 1960], p. 469 ff.

Ia. The Unterricht of Bürgi.

Bürgi quotes in his *"Unterricht"* (a user's guide to the Progreß Tabulen, see [Gieswald 1856], p. 321) a method of Simon Jacob Moritius Zons and others. This method is nothing else but the method of Michael Stifel ($\sim 1487 - 1567$). In his book Arithmetica integra, Nürnberg 1544, Stifel shows how to calculate with exponents (even negative ones) of the number 2. Stifel wrote: *"Man könnte ein ganz neues Buch über die wunderbaren Eigenschaften dieser Zahlen schreiben, aber ich muß mich an dieser Stelle bescheiden und mit geschlossenen Augen daran vorübergehen"*[3] (see [Tropfke 1921], p. 171 ff.).

In Bürgis tables the arithmetical sequence consists of

$$0, 10, 20, 30, \cdots.$$

Bürgi calls these numbers the red numbers (*"rote Zahlen"*) and they also are printed in this colour.

The geometrical sequence is

$$100000000, 100010000, 100020001, 100030003, \cdots.$$

These are the black numbers (*"schwarzen Zahlen"*). Let us denote the "Logarithmus" of Bürgi by $L_B(y)$. Then in contemporary mathematical notation one gets

$$L_B(y) = 10^5 \cdot {}_a log\frac{y}{10^8}.$$

[3] *It would be possible to write a very new book on the miraculous properties of these numbers, but at this point I have to be modest and fail to see it.*

Here $a = (1+10^{-4})^{10000}$ and $_a log$ denotes the logarithm to the base a. This number $a = 2.71814595\cdots$ is almost equal to the Euler number $e = 2.7182818285\cdots$.

In the introduction to the users guide (*"Vorrede an den Treuherzigen Leser"*) one can read:

Betrachtendt derowegen die Aigenschafft und Correspondenz der Progressen als der Arithmetischen mit der Geometrischen, das was in der ist Multiplizieren ist in jener nur Addieren, und was ist in der dividieren, in Jehner Subtrahieren, und was in der ist Radicem quadratam Extrahieren, in Jener ist nur halbieren, Radicem Cubicam Extrahieren, nur in 3 dividieren, Radicem Zensi in 4 dividieren, Sursolidam in 5. Und also fort in Anderen quantitaten, so habe Ich nichts Nützlichres erachtet, dan dise Tabulen.[4] (Issue of Graz, compare also with [Gieswald 1856], p. 320).

In the sequel follows the users guide (*"Kurzer Bericht der Progreßs Tabulen wie dieselbige nützlich in Allerley Rechnung zugebrauchen"*). Here he quotes Simon Jacob Moritius Zons, and shows some calculations with the geometrical sequence to the base 2. Afterwards follow many examples how to use the Progreß Tabulen. We give one example, comparing the issues of Danzig and of Graz:

Issue of Danzig ([Gieswald 1856], p. 327):

Aus einer gegebenen Zahlen Radicem quadratam extrahiern. Man sol zum Exempel Radicem quadratam auß 4015374 extrahiern, wird also erstlich punctiert wie bei der extraction breuchlich ist und steht also $\dot{4}0\dot{1}5\dot{3}7\dot{4}$ *und weil alhier fünf punkten seindt, so wirdt sein Radix auch 5 Ziffern haben, die rothe Zahl dieser obgeführten ist 139020 dieße halbirt kombt 69510 dessen Schwarze Zahl ist* $2003\overset{\circ}{8}3982$ *oder soll verstanden werden* $20038\frac{3982}{1000}$.

Issue of Graz:

Aus einer gegebenen Zahlen Radicem quadratam zu extrahieren. Man sol zum Exempel Radicem quadratam aus 4015374 Extrahieren, wird also erstlich punctiert wie beÿ der extraction bräuchlich ist und steht also $\dot{4}0\dot{1}5\dot{3}7\dot{4}$ *und weil alhir vier Punkten seindt, so wirdt sein Radix auch vier Ziffern haben, die rothe Zahl dieser obgeführten ist 139020 dieße halbiret kombt 69510 dessen Schwarze Zahl ist* $2003\overset{\circ}{8}3982$ *oder soll verstanden werden* $20038\frac{3982}{1000}$.

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[4] *Considering the properties and the correspondence of both sequences, the arithmetical sequence and the geometrical one, what in this one is multiplication in the other is addition, in this one division in that one subtraction, and what in this one is square root extraction in the other one is to halve, cube root extraction only division by 3, forth root division by 4 and fifth root by 5. And so on in the other quantitities, thus I have considered nothing more useful than these tables.*

There are misprints in both versions. Actually the following calculation is to be performed:

$$\sqrt{401537400} = 20038.398140.$$

Remarkable is the exactness of these calculations, which cannot be improved by normal pocket calculators. Remarkable further is, that Bürgi uses a sign (the small circle $^{\circ}$) for the decimal point.

The theoretical motivation of the tables of the two authors is different. Bürgi uses simply algebraic laws, respecting the addition law of exponents. Napier makes use of the physical visualisation of motion for introducing these two sequences.

Of course both authors use the addition law for the exponents but, one cannot say that they solved the functional equation (1), or a similar one for the logarithm. This is firstly done by Johannes Kepler.

II. Johannes Kepler and the Logarithm.

Johannes Kepler came across with the logarithms when he was engaged to establish a big volume of astronomical tables, the socalled "Tabulae Rudolphinae". At about the year 1616 he got the knowledge of the new method of Napier, calculating with the logarithms. But not before 1619 Johannes Kepler obtained for his own an issue of Napier's *"Mirifici Logarithmorum Canonis Descriptio"*.

But the tables of Napier were not conform to the purpose of Kepler. On the other hand, Kepler did not know the theoretical foundation of Napier's logarithms, which was published posthumous in 1619 by Napier's son Robert. Johannes Kepler therefore decided to establish new theoretical fundamentals for Napier's logarithms and to construct tables, which are more suitable for his own purpose. The results are Kepler's *"Chilias logarithmorum"*, the theoretical part of his logarithmic work, consisting of three postulates, 30 propositions, several corollaries and one definition together with the numerical tables. It follows the *"Supplementum Chilias logarithmorum"*, a kind of user's guide.

In order to establish his own theory of Napier's logarithms, Johannes Kepler came to slight different numbers than Napier. Kepler refered this to the fact, that he had used smaller intervals for the interpolation than Napier, therefore his calculations were getting more exact. This should be true too. But in reality Johannes Kepler used another theoretical method to derive the logarithms. So he got for his logarithmic function L_K, as we will see below, the folowing one

$$L_K(y) = 10^7 \cdot log \frac{10^7}{y}.$$

As it was pointed out, the logarithmic papers of Johannes Kepler were inspired by John Napier. But Kepler must have known also the method of Bürgi, since they lived both for several years in Praha together, namely from 1605 until 1612.

It is known that they had several common scientific interests, and that they worked together in mathematics. But the only reference in connection with the logarithms by Kepler to Bürgi is the following one in the Tabulae Rudolphinae (see [Kepler 1627], p. 48): *"qui etiam apices logistici Iusto Byrgio multis annis ante editionem Neperianam viam praeiverunt ad hos ipsissimos Logarithmos. Etsi homo cunctator et secretorum suorum custos foetum in partu destituit, non ad usus publicos educavit"*.[5]

The method of Johannes Kepler in the "Chilias logarithmorum" (see [Kepler 1624]) is, roughly speeking, the following. He introduces a function on the set of all ratios, say real positive numbers (Postulatum I: *Omnes proportiones $\cdots$ quantitate metiri seu exprimere*), which is called measure (mensura). This measure satisfies the functional equation

$$M(x \cdot y) = M(x) + M(y) \tag{1}$$

or even the restricted one ($y = x$)

$$M(x^2) = 2 \cdot M(x). \tag{2}$$

It is not clear, which one of the two functional equations are used primarily. Equation (2) appears in the first place in I. Propositio. Using only (2), Kepler shows in the EXEMPLUM SECTIONIS how to calculate the Logarithmus which is defined lateron in the DEFINITIO. Equation (1) one can find expressis verbis in XIX. Propositio. In Postulatum II Kepler requires, one can say, that the measure should be sufficiently smooth. In Postulatum III he states, that the measure M, which is a solution of (2) and therefore is of the form

$$M(x) = c \cdot log(x),$$

should have as derivative in a fixed point z the value 1. This yields for the measure M the representation

$$M(x) = z \cdot log(x). \tag{6}$$

Kepler takes for the constant z in his papers different values. In the EXEMPLUM SECTIONIS he takes for z the value 10^{20}, in the DEFINITIO the value 1000, and finally in the logarithmic tables and in the Supplementum (see [Kepler 1625]) the value 10^7 . Thus Kepler's mensura is, up to the decimals the function of the natural logaritms. In his DEFINITIO (see [Kepler 1625], p. 297) he introduces his *"Logaritmus"* as

$$L_K(y) = M\Big(\frac{z}{y}\Big)$$

[5] *these logistic apices showed Jobst Bürgi, many years before Napier's edition, the way to even the same Logarithms. But this lingerer and secret monger did neglect his foetus and did not educate it for the public use.*

hence if $z = 10^7$

$$L_K(y) = 10^7 \cdot log\frac{10^7}{y}.$$

All the propositions, examples, corollaries and remarks between Postulatum I ([Kepler 1625], p. 280) and DEFINITIO build the proof that the measure M can be computed in a unique way. That is, Johannes Kepler gives a constructive proof of existence and uniqueness of the solutions of the functional equation (2). In order to do this, he proves many of identities and inequalities and uses the same tricks as they are used nowaday in the elementary theory of functional equations.

The style of the representation of the mathematics in the *"Chilias logarithmorum"* seems sometimes to give rise for doubts, how to interprete the particular statements. Sometimes these doubts come from the fact that Johannes Kepler really introduced new things, for those the vocabulary was not founded. He personally writes in the "Chilias": *"At cùm in re insolatâ laboremus penuria vocabulorum"*[6].

On the other hand, in generally the formulation of the mathematical statements and their proofs is astonishingly clear. So, if there is some vagueness in the representation of some statements, then this may also have its origin in the very complicate history of printing of the logarithmic works of Johannes Kepler. For more details of the history of Keplers logarithmic works see [Hammer], p. 461ff. and 469ff., [Naux], p. 128ff. [Belyj Ju.A. and D. Trifunovic] and [Gronau].

Concluding these explanations, finally one can say that Johannes Kepler was the first, who introduced the real function of the natural logarithm, he called it "mensura". He propound a functional equation for it, and solved this functional equation in the constructive way, showing how to compute the values of the solution of this functional equation.

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[6] *Unusual things effort more work due to the lack of denominations.*

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