Aequationes Math. 68 (2004) 230–247
0001-9054/04/030230-18
DOI 10.1007/s00010-004-2765-3

Aequationes Mathematicae

Normal solutions of difference equations, Euler's functions and spirals

Detlef Gronau

Dedicated to Professor János Aczél on the occasion of his eightieth birthday

Summary. We consider the linear first order difference equation

$$f(x+1) = \alpha(x) \cdot f(x)$$

for the unknown function $f : I \subseteq \mathbb{R} \to \mathbb{C}$. We present, under some conditions on the given function α, a product representation of a specific solution of the considered difference equation that can be understood as *the normal* or *principal solution* in the concept of N. E. Nörlund, F. John, W. Krull and others. This normal solution is characterized by its asymptotic behavior near infinity. In that way we get a characterization of e.g. the gamma function, the q-gamma function and of several classical spirals.

Mathematics Subject Classification (2000). 39-03, 39B12.

Keywords. Difference equations, normal solution, principal solution, gamma function, beta function, q-gamma function, spirals.

0. Introduction

This note is inspired by the book [4] on spirals by Philip J. Davis and my intensive studies of the gamma function and its history (see [6]). Davis considers in [4] the beautiful *spiral of Theodorus* (named after Theodorus of Cyrene, $\sim$ 470–399 B.C.). Davis gives for this spiral a product representation as a special solution of the corresponding difference equation

$$f(x+1) = \alpha(x) \cdot f(x), \text{ with the initial condition } f(0) = 1, \tag{1}$$

for the unknown function $f : [0, \infty) \to \mathbb{C}$, where $\alpha(x) = 1 + i/\sqrt{x+1}$, $i = \sqrt{-1}$. Different characterizations of Davis's solution are given in [7].

Here we consider a class of first order difference equations of the form (1) which describes several known spirals but also other functions like the gamma function, the beta function and the q-gamma function. We give a product representation of a specific solution of the considered difference equation which can be understood as *the normal solution* or *the principal solution.*

The concept of normal solutions is, that one determines in a 'natural way' one specific solution of the considered difference equation (1), e.g. by a general formula or by using a limit process and/or geometrical conditions. *If this normal solution is nonzero for all values of its domain of definition, then all other solutions of equation* (1) *differ from this special solution by a function of period* 1.

Nörlund [19] uses the denomination 'Hauptlösung'= *principal solution*, John [10] says *special solution.* I prefer the notation of Krull [12] who uses the term *normal solution.* In this connection one should also mention the monographs Milne-Thomson [16] and Kucma [13] and the quoted references therein.

Normal or principal solutions had their origin in the so called 'interpolation problem', raised around 1700: The values of a solution f of a difference equation of the type (1) are uniquely defined for all integer. What would be a suitable interpolating function passing through the discrete sequence $f(1)$, $f(2)$, $f(3)$, ...? (See [6] and the quoted references there.)

So, for example, if we consider the difference equation of the terms of a geometric series

$$f(x+1) = q \cdot f(x),\ f(0) = a, \tag{2}$$

where $q \in \mathbb{C}$, $q \neq 0$, then we define its *normal solution* by

$$f(x) = a \cdot q^x. \tag{3}$$

Another example is

$$f(x+1) = c \cdot q^x \cdot f(x), \quad f(1) = a. \tag{4}$$

Its normal solution is

$$f(x) = a \cdot c^{x-1} \cdot q^{x(x-1)/2}. \tag{5}$$

(In the case that c and/or q are not positive reals one has to take the principal branch of the power function. In case that c (or q) is a negative real then one has to take instead of the power function c^x the real part of it, i.e. $|c|^x \cdot \cos(\pi x)$, in order to get a real normal solution of (4) or (2), respectively.)

Both examples have in common that, if one considers the equation for the discrete domain $\mathbb{N}_0$ or $\mathbb{N}$, respectively, then its unique solution is given in closed form by $f(n) = a \cdot q^n$ or $f(n) = a \cdot c^{n-1} \cdot q^{n(n-1)/2}$, respectively. In these cases it is easy to enlarge the domain of definition to the positive reals in a 'natural way'. Both functions (3) and (5) stood at the beginning of the interpolating problem. Especially the normal solution (3) of equation (2) was used by L. Euler to introduce his product representation of the gamma function (see next section).

The definition of the normal solution of the more general equation (see Theorem 1 and the following examples) in this note will be based on the definition of the normal solutions of (2) and (4) given here. The crucial point in this note is that all the classical spirals and functions treated in this note are normal solutions of their related difference equation.

In section 1 we consider a difference equation which is a generalization of the difference equation of the gamma function under some conditions on the coefficient of this equation, see Theorem 1 and Proposition 1. In section 2 we show that this concept of normal solutions applies as well to the gamma and beta functions as the Bohr–Mollerup Theorem. In section 3 and 4 we handle a difference equation which applies to spirals, see Proposition 2, and give some examples of spirals. In section 5 we treat a more general case which also applies, e.g., to the q-gamma function for $q > 1$, see Theorem 2. Of course all examples given here can be treated by this last theorem. But in order to enhance the readability of this note we preferred a step by step approach to this equation.

1. The difference equation

We first treat equation (1) for a wider class of α, which apply e.g. also for the gamma functional equation. In order to accommodate the assumptions to the gamma equation and also to other classical functions we will make a slight change in the initial condition and the domain of definition of the functions. Thus we consider

$$f(x+1) = \alpha(x) \cdot f(x), \quad f(1) = a, \tag{6}$$

for $x \in (0, \infty)$, where we make the general assumption that $\alpha(x) \neq 0$ for all $x \in (0, \infty)$.

First we remind a well known fact on the difference equation (6) which guarantees the existence of many solutions of this equation without any assumptions on the given function α. Indeed, the following folklore lemma gives the *general solution* of the difference equation (6).

Lemma 1. *Let $\varphi : (0, 1] \to \mathbb{C}$ be an arbitrary function with $\varphi(1) = a$. Then there exists one and only one solution f of* (6) *with $f(x) = \varphi(x)$ for $x \in (0, 1]$. Every solution of* (6) *can be described in this way.*

The gamma function, for example, is a special solution of equation (6) with $\alpha(x) = x$ and $a = 1$. We follow the way of Euler when he introduced the gamma function as an infinite product (see e.g. [6] or consult directly Euler [5]). In this sense we require that for fixed large $n \in \mathbb{N}$ the *normal solution* of (6) is approximated by a function, say $\widetilde{f}$, which is a solution of a difference equation similar to (6), but with constant coefficient:

$$\widetilde{f}(n+x+1) = \alpha(n) \cdot \widetilde{f}(n+x), \ x \geq 0, \quad \widetilde{f}(n) = f(n).$$

According to (3) this equation has the normal solution $\widetilde{f}(n+x) = \alpha(n)^x \cdot f(n)$, $x \geq 0$. Therefore we claim that $f(n+x) \approx \alpha(n)^x \cdot f(n)$ for large n. This means that we suppose

$$\lim_{n\to\infty} \frac{\alpha(n)^x \cdot f(n)}{f(n+x)} = 1. \tag{7}$$

From (6) we deduce for $n \in \mathbb{N}$

$$f(n) = f(1) \cdot \prod_{k=1}^{n-1} \alpha(k), \tag{8}$$

and for $x \in (0, \infty)$

$$f(n+x) = f(x) \cdot \prod_{k=0}^{n-1} \alpha(k+x). \tag{9}$$

Hence

$$\frac{\alpha(n)^x \cdot f(n)}{f(n+x)} = \alpha(n)^x \cdot \frac{f(1)}{f(x)} \cdot \frac{1}{\alpha(x)} \cdot \prod_{k=1}^{n-1} \frac{\alpha(k)}{\alpha(k+x)}, \tag{10}$$

therefore

$$f(x) = \frac{a}{\alpha(x)} \cdot \lim_{n\to\infty} \alpha(n+1)^x \cdot \prod_{k=1}^{n} \frac{\alpha(k)}{\alpha(k+x)}, \tag{11}$$

supposed that this limit exists. In this case a straightforward computation shows that we get by (11) a solution of (6), provided that

$$\lim_{n\to\infty} \frac{\alpha(n+1)}{\alpha(n+1+x)} = 1 \tag{12}$$

holds. This solution is nonzero for all x (since we supposed $\alpha(x) \neq 0$) and uniquely defined by the condition (7). So we can state

Theorem 1. *Suppose that the limit* (11) *exists for* $x > 0$ *and that* (12) *is satisfied, then the unique solution of* (6) *for* $x \in (0, \infty)$ *which satisfies* (7) *is of the form* (11). *The general solution of* (6) *is the product of the solution* (11) *with a function* p *of period* 1*, i.e.,* $p(x+1) = p(x)$.

We call the solution (11) introduced in Theorem 1 *the normal solution* of (6).

Remark 1. To investigate the convergence of (11) we have to do some rearrangements. Let

$$\begin{aligned} P(x) &:= \lim_{n\to\infty} \frac{(\alpha(n+1))^x}{\alpha(x)} \cdot \prod_{k=1}^{n} \frac{\alpha(k)}{\alpha(k+x)} \\ &= \lim_{n\to\infty} \frac{\alpha(1)^x}{\alpha(x)} \cdot \prod_{k=1}^{n} \left(\frac{\alpha(k+1)}{\alpha(k)}\right)^x \cdot \frac{\alpha(k)}{\alpha(k+x)} \\ &= \frac{\alpha(1)^x}{\alpha(x)} \cdot \prod_{k=1}^{\infty} (1 + \widehat{\alpha}(k,x)), \end{aligned} \tag{13}$$

where

$$\widehat{\alpha}(k,x) = \left(\frac{\alpha(k+1)}{\alpha(k)}\right)^x \cdot \frac{\alpha(k)}{\alpha(k+x)} - 1. \tag{14}$$

A necessary condition for the convergence of the infinite product (13) is that $\sum_{k=1}^{\infty} \widehat{\alpha}(k,x)$ converges (see e.g. [21] or [1], Appendix A).

In this case we have of course that $\lim_{n\to\infty} \widehat{\alpha}(k,x) = 0$. A sufficient condition that the product (13) converges is that $\sum_{k=1}^{\infty} \widehat{\alpha}(k,x)$ converges absolutely. In this case the limit of the product is 0 if and only if one of its factors is 0.

In some cases it may be more convenient to show that $\sum_{k=1}^{\infty} \widehat{\widehat{\alpha}}(k,x)$ converges absolutely, where

$$\widehat{\widehat{\alpha}}(k,x) = \left(\frac{\alpha(k)}{\alpha(k+1)}\right)^x \cdot \frac{\alpha(k+x)}{\alpha(k)} - 1, \tag{15}$$

which also implies the convergence of (11).

We now give a criterion for a wide class of equations. The conditions on α in the following proposition are, for example, fulfilled if α is *a polynomial, a rational function or the logarithmic function.*

Proposition 1. *Suppose that $\alpha(x) \neq 0$ for all $x \in (0,\infty)$, and it is twice differentiable for large x. Suppose further that $\frac{\alpha'(x)}{\alpha(x)} = O(1/x)$ and*

$$\tfrac{\alpha''(x+\varepsilon)}{\alpha(x)} = O(1/x^2) \tag{16}$$

for all $\varepsilon \in [0,\infty)$ holds.[1] *Then $\widehat{\widehat{\alpha}}(k,x) = O(1/x^2)$, i.e., the product representation* (13) *converges and the suppositions of Theorem 1 are fulfilled. With*

$$f(x) = a \cdot P(x) \tag{17}$$

we get the uniquely defined normal solution of (6).

Proof. By Taylor's formula we have $\alpha(k+x) = \alpha(k) + x\cdot\alpha'(k) + \frac{x^2}{2}\cdot\alpha''(k+\xi_x)$. Hence

$$\begin{aligned}
\widehat{\widehat{\alpha}}(k,x) &= \left(\frac{\alpha(k+1)}{\alpha(k)}\right)^{-x} \cdot \frac{\alpha(k+x)}{\alpha(k)} - 1 \\
&= \left(1 + \frac{\alpha'(k)}{\alpha(k)} + O(1/k^2)\right)^{-x} \cdot \left(1 + x\cdot\frac{\alpha'(k)}{\alpha(k)} + O(1/k^2)\right) - 1 \\
&= O(1/k^2).
\end{aligned}$$

Also condition (12) is fulfilled. □

[1] Here and in the sequel the Landau symbol $O(g(x))$ will always refer to $x \to \infty$.

Remark 2. The condition (16) in Proposition 1 needs only to be supposed for $\varepsilon \in [0, 1)$, which guarantees the convergence of (13) for $x < 1$. The convergence of (13) for $x \geq 1$ can be shown via the difference equation (6).

Remark 3. The normal solution given in Proposition 1 (especially in section 2 and in a similar way also the solution given in Proposition 2 below) is characterized as the unique solution of its difference equation (6) which satisfies the asymptotic equivalence

$$f(n + x) \approx (\alpha(n))^x \cdot f(n), \quad \text{for } n \to \infty.$$

2. The gamma and the beta function

In this section we deal with the classical gamma and beta function. We show that the concept of normal solutions and their characterization through the asymptotic behavior as mentioned in Remark 3 plays the same role as the classical Bohr–Mollerup Theorem.

2.1. The gamma function Γ

Here we have $\alpha(x) = x$ in (6), thus

$$f(x + 1) = x \cdot f(x) \quad f(1) = 1. \tag{18}$$

As consequence of Proposition 1 we get by formula (11)

$$f(x) = \lim_{n\to\infty} \frac{(n + 1)^x \cdot n!}{x(x + 1)...(x + n)}, \tag{19}$$

which yields the well known product representation for the gamma function

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x \cdot n!}{x(x + 1)...(x + n)}$$

since $\lim_{n\to\infty} \frac{(n+1)^x}{n^x} = 1$.

As it is well known there is also an integral representation of the gamma function, the so-called second Euler integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{20}$$

To show that the product representation (19) and the integral representation (20) coincide we can use the concept of normal solutions.

The integral $\int_0^\infty t^{x-1}e^{-t}dt$ satisfies also the difference equation (18) which can be shown by integration by parts. Hence $\int_0^\infty t^{n-1}e^{-t}dt = (n-1)!$. If condition (7) is satisfied for $\int_0^\infty t^{x-1}e^{-t}dt$ then, by the uniqueness statement in Proposition 1, it coincides with the normal solution (19). The proof of the validity of (7) for this integral is elementary, due to Pringsheim [20], who has shown the identity of (19) and (20) in a similar way (see also [15], § 141, p. 496). We give here, for the sake of completeness, a slightly simplified version of Pringsheim's proof.

Lemma. *The function* $f(x) = \int_0^\infty t^{x-1}e^{-t}dt$ *satisfies the condition*

$$\lim_{n\to\infty} \frac{f(n+x)}{n^x \cdot f(n)} = \lim_{n\to\infty} \frac{f(n+x)}{n^x \cdot (n-1)!} = 1, \tag{21}$$

Proof.

$$\begin{aligned}\frac{f(n+x)}{n^x \cdot (n-1)!} &= \frac{1}{n^x \cdot (n-1)!}\int_0^\infty t^{n+x-1}e^{-t}dt \\ &= 1 + \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\left(\left(\frac{t}{n}\right)^x - 1\right)e^{-t}dt.\end{aligned}$$

We suppose firstly for x the condition $0 \le x \le 1$. Then we split the last integral into two parts.

i.) For $0 \le t \le n$ we have $1 \ge \left(\frac{t}{n}\right)^x \ge \frac{t}{n}$, hence

$$\begin{aligned}0 &\ge \frac{1}{(n-1)!}\int_0^n t^{n-1}\left(\left(\frac{t}{n}\right)^x - 1\right)e^{-t}dt \ge \frac{1}{(n-1)!}\int_0^n t^{n-1}\left(\frac{t}{n} - 1\right)e^{-t}dt \\ &= -\frac{n^n}{e^n \cdot n!}.\end{aligned}$$

ii.) For $n \le t$ we have $1 \le \left(\frac{t}{n}\right)^x \le \frac{t}{n}$, hence

$$\begin{aligned}0 &\le \frac{1}{(n-1)!}\int_n^\infty t^{n-1}\left(\left(\frac{t}{n}\right)^x - 1\right)e^{-t}dt \le \frac{1}{(n-1)!}\int_n^\infty t^{n-1}\left(\frac{t}{n} - 1\right)e^{-t}dt \\ &= \frac{n^n}{e^n \cdot n!}.\end{aligned}$$

Using the power series expansion of e^n one gets easily for arbitrary large $k \in \mathbb{N}$, and n with $n \ge k^2$:

$$\frac{e^n n!}{n^n} > \sum_{i=1}^{k} \frac{n^i}{(n+1)...(n+i)} > k\frac{1}{(1+k/n)^k} > k\frac{1}{(1+1/k)^k} > \frac{k}{e},$$

hence $\lim_{n\to\infty} \frac{e^n \cdot n!}{n^n} = \infty$.[2] Thus we have proven (21) for $0 \le x \le 1$. For $x > 1$ condition (21) follows by repeated application of the difference equation (18). □

[2] For the estimate of the last limit one could also use Stirling's formula, see [15]. With this asymptotic equivalence $n! \approx n^n e^{-n}\sqrt{2\pi n}$ one can show even more: For a positive real a is the limit $\lim_{n\to\infty} \frac{a^n \cdot n!}{n^n}$ equal 0, if $a < e$, and ∞, of $a \ge e$.

2.2. Euler's first integral, the beta function

The beta function is given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ \ x,y > 0. \tag{22}$$

It is easy to see (integration by parts) that B satisfies

$$B(x+1,y) = \frac{x}{x+y}\cdot B(x,y) \text{ with } B(1,y) = \frac{1}{y}. \tag{23}$$

Thus $f(x) = B(x,y)$ is a solution of (6) with $\alpha(x) = \frac{x}{x+y}$ and $f(1) = a = 1/y$.

We can apply Proposition 1 to the equation

$$f(x+1) = \frac{x}{x+y}\cdot f(x), \quad f(1) = \frac{1}{y}. \tag{24}$$

Its normal solution is given by (11). It is uniquely determined by the condition (7), i.e., since $\lim_{n\to\infty}\alpha(n) = 1$, by

$$\lim_{n\to\infty}\frac{f(n)}{f(n+x)} = 1. \tag{25}$$

The function $f(x) = B(x,y)$, being a monotonic decreasing function satisfies

$$1 = \frac{f(n)}{f(n)} < \frac{f(n)}{f(n+x)} < \frac{f(n)}{f(n+1)} = \frac{n+y}{n} \quad \text{for } 0 < x < 1.$$

Hence (25) is satisfied for $0 \le x \le 1$. By repeated application of the difference equation (24) it follows that (25) is satisfied also for $x > 1$. Therefore, $B(x,y)$ is the normal solution of (23) given by

$$B(x,y) = \frac{a}{\alpha(x)}\cdot\lim_{n\to\infty}\alpha(n+1)^x\cdot\prod_{k=1}^{n}\frac{\alpha(k)}{\alpha(k+x)}.$$

This yields

$$\begin{aligned} B(x,y) = \lim_{n\to\infty} & \frac{n!(n+1)^x}{x(x+1)\cdots(x+n)}\cdot\frac{n!(n+1)^y}{y(y+1)\cdots(y+n)} \\ & \cdot\frac{(x+y)(x+y+1)\cdots(x+y+n)}{n!(n+1)^{x+y}}\cdot\frac{(n+1)^x}{(n+1+y)^x}. \end{aligned}$$

Each of these four terms of the limit converges separately. Its limits are $\Gamma(x)$, $\Gamma(y)$, $1/\Gamma(x+y)$, and 1, respectively. Hence we got an easy proof of the well known representation

$$B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}. \tag{26}$$

In the literature, usually, this identity is proved with the aid of the Bohr–Mollerup Theorem, showing that the function $g(x) = \frac{\Gamma(x+y)}{\Gamma(y)}B(x,y)$ is the unique log-convex solution of the gamma functional equation with $g(1) = 1$, whence, $g(x) = \Gamma(x)$ (see e.g. [22]).

From what is said above one can easily deduce:

Corollary. *Any* monotonic *solution of* (24) *is the beta function.*

3. Spirals

Spirals are very popular and therefore there exist numerous books and articles on spirals. Beside of Davis's book [4] we mention only two of them. Heitzer [8] is full of pictures of spirals in mathematics and nature. The book of Lawrence [14] gives a concise overview on special curves including spirals.

Spirals are often described in the complex plane by a sequence of points recursively defined. The problem lies in finding a continuous curve passing through these points and defined by the corresponding difference equation. This curve, also called the 'interpolating' curve, will be in our examples the normal solution of a first order difference equation for a function $f : I \to \mathbb{C}$ of the form

$$f(x+1) = \alpha(x) \cdot f(x), \quad f(0) = a, \tag{27}$$

where $I \subseteq \mathbb{R}$ is an interval and $a \in \mathbb{C}$, $a \neq 0$. Usually we will take for the domain of f the nonnegative reals, i.e., $I = [0, \infty)$. Here we chose the initial condition in the form $f(0) = a$ to make it easier to compare the results of this paper with others in the literature. Geometric properties of these normal solutions for the case of the spiral of Theodorus are exhibited in [7], see also Remark 4.

Similarly to (8) and (9) we deduce for any solution f of (27) and for $n \in \mathbb{N}$ and $x \in [0, \infty)$

$$f(n) \;=\; f(0) \cdot \prod_{k=0}^{n-1} \alpha(k)$$
$$f(n+x) \;=\; f(x) \cdot \prod_{k=0}^{n-1} \alpha(k+x).$$

Hence in analogy to (11)

$$f(x) = a \cdot \lim_{n\to\infty} (\alpha(n+1))^x \cdot \prod_{k=0}^{n} \frac{\alpha(k)}{\alpha(k+x)}, \tag{28}$$

supposed that this infinite product converges.

Proposition 2. *Suppose that* $\alpha : [0, \infty) \to \mathbb{C} \setminus \{0\}$ *is of the form* $\alpha(x) = c + x^{-\varepsilon}(d + \delta(x))$, *where* $c, d \in \mathbb{C}$, $c \neq 0$, $\varepsilon > 0$ *and* $\delta(x) = O(1/x)$. *Then we have* $\lim_{n\to\infty}(\alpha(n+1))^x = c^x$, *the product* (28) *converges and condition* (12) *is fulfilled.*

Hence we get as normal solution of (27)

$$f(x) = a \cdot c^x \cdot \prod_{k=0}^{\infty} \frac{\alpha(k)}{\alpha(k+x)}. \tag{29}$$

This normal solution is uniquely determined by the condition $f(n+x) \approx c^x \cdot f(n)$.

Proof. Of course we could also apply Proposition 1 for differentiable α. To make it more illustrative we deliver here a different version of the proof. By assumption we have $\alpha(x) = c + x^{-\varepsilon}(d + \delta(x))$, $\delta(x) = O(1/x)$.

$$\begin{aligned}\frac{\alpha(k+x)}{\alpha(k)} - 1 &= \frac{c + (k+x)^{-\varepsilon}(d + \delta(k+x)) - c - k^{-\varepsilon}(d + \delta(k))}{c + k^{-\varepsilon}(d + \delta(k))} \\ &= \frac{k^{-\varepsilon}\left[d((1 + x/k)^{-\varepsilon} - 1) + (1 + x/k)^{-\varepsilon}\delta(k+x) - \delta(k)\right]}{c + k^{-\varepsilon}(d + \delta(k))}\end{aligned}$$

Since $(1+x/k)^{-\varepsilon} - 1$, $\delta(k+x)$, and $\delta(k)$ are both of the form $O(1/k)$ we have that $\frac{\alpha(k+x)}{\alpha(k)} - 1 = O\left(\frac{1}{k^{\varepsilon+1}}\right)$. Thus the product in (28) converges and also (12) holds. Of course we also have $\lim_{n\to\infty} \alpha(n+x) = c$ for all x. Thus (29) follows from (28). □

4. Examples of spirals

In what follows we consider some well known spirals. One common property of these spirals is, that the conditions of Proposition 2 are fulfilled for their related difference equations. It turns out that these spirals are normal solutions of their difference equation.

4.1. The spiral of Theodorus

(See [4], [7] and [9].) The discrete points z_n of this spiral are constructed as shown in Figure 1.

If we consider the spiral in the complex plane we have

$$z_{n+1} = z_n + i \cdot \frac{z_n}{|z_n|}, \quad z_0 = 1,$$

where $i = \sqrt{-1}$. By induction we deduce from $|z_0| = 1$ and the recursion formula that $|z_n| = \sqrt{n+1}$, hence we can rewrite the recursion formula in the form

$$z_{n+1} = z_n \left(1 + i/\sqrt{n+1}\right), \quad z_0 = 1.$$

Thus, we get as defining difference equation for the interpolating spiral of Theodorus

$$f(x+1) = (1 + i/\sqrt{x+1}) \cdot f(x), \quad f(0) = 1. \tag{30}$$

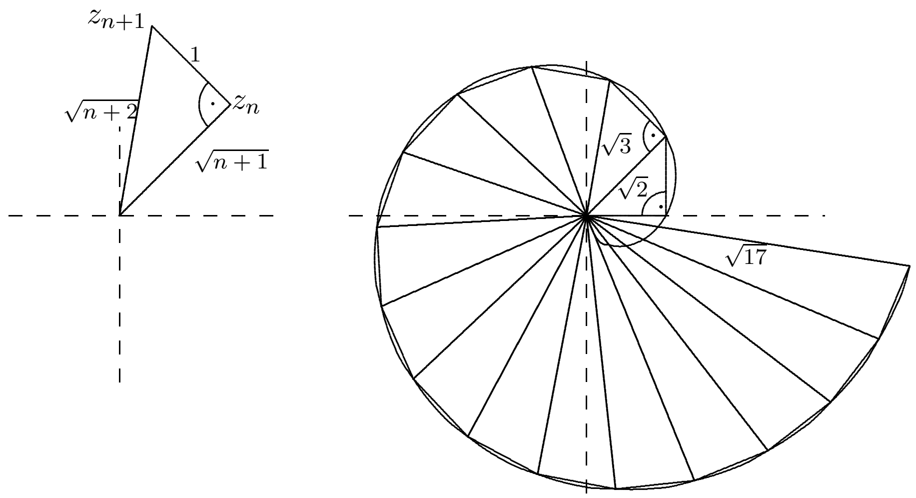


FIG. 1. The spiral of Theodorus.

With $\alpha(x) = 1 + i/\sqrt{x+1} = 1 + ix^{-1/2}(1 + 1/x)^{-1/2}$ we can apply Proposition 2 and we get as normal solution of (30)

$$f(x) = \prod_{k=1}^{\infty} \frac{1 + i/\sqrt{k}}{1 + i/\sqrt{x+k}}. \tag{31}$$

The infinite product in (31) converges for $x \in (-1, \infty)$ and fulfills equation (30) there. This is the function found by Davis in [4].

4.2. The spiral of Archimedes

This spiral is characterized by the property that in polar coordinates the radius and the argument add up by constant real positive values:

$$r_{n+1} = r_n + a,\ \varphi_{n+1} = \varphi_n + b \text{ for } z_n = r_n \cdot e^{i\varphi_n},\ n \in \mathbb{N}_0,\ z_0 = a.$$

Again we derive the difference equation

$$f(x+1) = \left(1 + \frac{1}{x+1}\right) e^{ib} \cdot f(x),\ f(0) = a. \tag{32}$$

For $\alpha(x) = (1 + 1/(x+1))e^{ib}$, according to Proposition 2 and formula (29), we get as normal solution of (32)

$$f(x) = a \cdot e^{ibx} \cdot \lim_{n\to\infty} (x+1) \cdot \frac{1}{1 + x/(n+1)} = a \cdot e^{ibx} \cdot (x+1).$$

This solution is valid for all $x \in \mathbb{R}$.

4.3. The logarithmic spiral

The logarithmic spiral goes back to René Descartes. Its polar equation is $r = e^{a\varphi}$, $a \in \mathbb{R}$. The related difference equation is

$$f(x+1) = e^{a+i} \cdot f(x), \; f(0) = 1 \tag{33}$$

Here we can see directly that the product (29) yields as normal solution of (33)

$$f(x) = e^{ax} \cdot e^{ix}.$$

This solution is defined of course for all $x \in \mathbb{R}$.

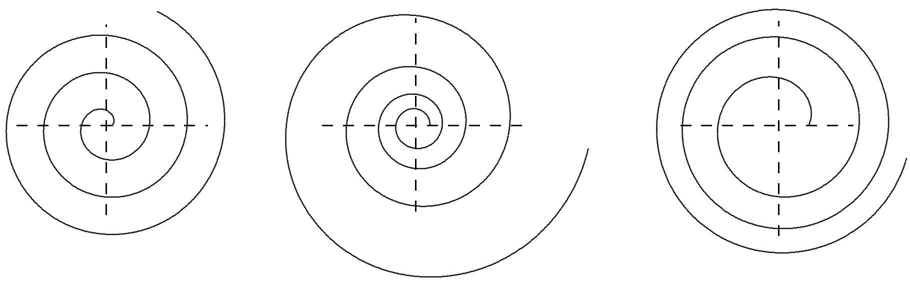


FIG. 2. Archimedes' spiral, the logarithmic spiral, and the sunflower spiral.

4.4. Sun flower spiral

In [18] a spiral of seeds in the sun flower is described, where in that paper the exciting properties of the discrete points of the spirals are exhibited. For these points we get the recursion formula

$$z_{n+1} = \sqrt{1 + \frac{1}{n+1}} \cdot e^{ib} \cdot z_n, \; n \in \mathbb{N}_0, \; z_0 = 1,$$

with a fixed real b. In analogy to the previous cases we get a difference equation for this spiral:

$$f(x+1) = \sqrt{1 + \frac{1}{x+1}} \cdot e^{ib} \cdot f(x), \; f(0) = 1. \tag{34}$$

Its normal solution is, according to Proposition 2,

$$f(x) = \sqrt{x+1}\; e^{ibx}.$$

4.5. The marigold spiral

(See Davis [4].) The discrete points z_n of this spiral are defined in the complex plane by $z_0 = 1$ and

$$z_{n+1} = c \cdot z_n + \overline{c} \cdot z_n / |z_n|\,, \ n > 0, \ \text{where } c = e^{\frac{\pi i}{4}}. \tag{35}$$

By induction one gets $|z_n| = \sqrt{n+1}$.

The discrete points yield a nice picture which reminds of a flower, the marigold flower (see Figure 3).

By some computation we derive from (35) the recursion formula

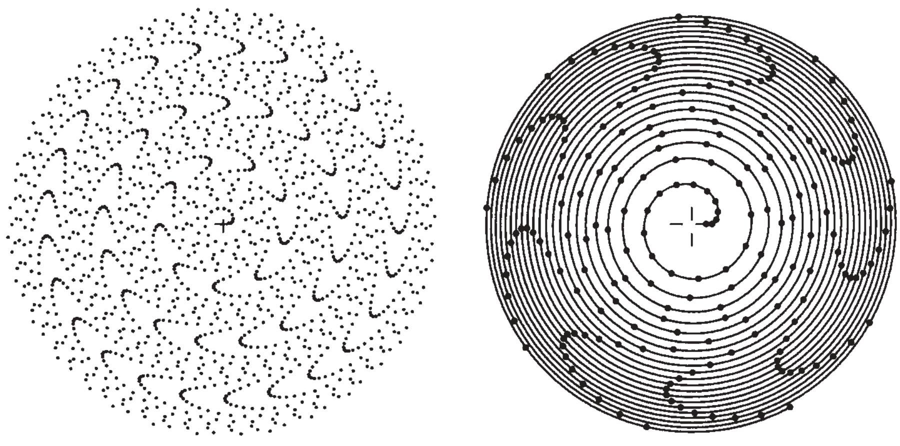
$$z_{n+1} = e^{\pi i/4} \cdot \left(1 - \frac{i}{\sqrt{n+1}}\right) \cdot z_n, \ n > 0, \tag{36}$$

For the interpolating function we therefore postulate the difference equation

$$f(x+1) = e^{\pi i/4} \cdot (1 - i/\sqrt{x+1}) \cdot f(x), \ f(0) = 1. \tag{37}$$

Its normal solution, according to Proposition 2, is given by

$$f(x) = \prod_{k=1}^{\infty} e^{x\pi i/4} \cdot \frac{1 - i/\sqrt{k}}{1 - i/\sqrt{x+k}}, \ x \in (0, \infty). \tag{38}$$



1500 points (discrete) *200 points (connected)*

Fig. 3. Marigold spiral, discrete points and the interpolating curve.

Remark 4. The limit in formula (38) as well as that of (31), however, is not very useful for computing this spiral, since the product therein converges very slowly.

Therefore we proceed as in [7], using for the solution f the ansatz in polar coordinates

$$f(x) = \psi(x) \cdot e^{i \cdot \varphi(x)},$$

with nonnegative real functions ψ and φ. This gives us two difference equations for the absolute value ψ and the argument φ, separately. Their monotonic (and in the case of φ continuous) solutions are uniquely defined. Some computations analogous to those in [7] yield the representation for the normal solution (38)

$$f(x) = \sqrt{x+1} \cdot e^{i\left(x\pi/4 + \sum_{k=0}^{\infty}\left(\arctan\sqrt{k+1} - \arctan\sqrt{x+k+1}\right)\right)}. \tag{39}$$

Formula (39) can relative easily be evaluated by computer. The data of Figure 3, as well as of Figure 2 are calculated by the mathematical program DERIVE on the author's desktop computer.

5. Further generalizations

We start with an example.

Example 1. *The q-gamma function* Γ_q (see e.g. Askey [2] or Andrew–Askey–Roy [1]). Here we have $\alpha(x) = \frac{1-q^x}{1-q}$ for fixed (real or complex) $q \neq 1$.

$$f(x+1) = \frac{1-q^x}{1-q} \cdot f(x),\ x > 0,\ f(1) = 1. \tag{40}$$

At first we treat the case $0 < |q| < 1$. The normal solution of this equation is, according to formula (11) and Theorem 1,

$$f(x) = \frac{1-q}{1-q^x} \cdot \lim_{n\to\infty} \left(\frac{1-q^{n+1}}{1-q}\right)^x \prod_{k=1}^{n} \frac{1-q^k}{1-q^{k+x}},$$

where the convergence follows directly since $0 < |q| < 1$. This normal solution is identical with

$$\Gamma_q(x) = (1-q)^{1-x} \frac{\prod_{k=0}^{\infty}(1-q^{k+1})}{\prod_{k=0}^{\infty}(1-q^x q^k)}, \tag{41}$$

the well known product representation for Γ_q.

To treat the q-gamma function for $|q| > 1$ we have to do some more generalizations.

We start again with a solution $f : (0,\infty) \to \mathbb{C}$ of the general equation (6), which exists due to Lemma 1, and the resulting product (10):

$$\frac{\alpha(n)^x \cdot f(n)}{f(n+x)} = \alpha(n)^x \cdot \frac{f(1)}{f(x)} \cdot \frac{1}{\alpha(x)} \cdot \prod_{k=1}^{n-1} \frac{\alpha(k)}{\alpha(k+x)}. \tag{10}$$

Define

$$P_n(x) = \frac{\alpha(n)^x}{\alpha(x)} \cdot \prod_{k=1}^{n-1} \frac{\alpha(k)}{\alpha(k+x)} \quad \text{and} \quad Q_n(x) = \left(\frac{\alpha(n)^x \cdot f(n)}{f(n+x)}\right)^{-1}.$$

Then we have

$$f(x) = a \cdot P_n(x) \cdot Q_n(x). \tag{42}$$

We get

$$P_n(x+1) = \alpha(x) \cdot \frac{\alpha(n)}{\alpha(x+n)} \cdot P_n(x)$$

and

$$\begin{aligned} f(x+1) &= a \cdot \alpha(x) \cdot \frac{\alpha(n)}{\alpha(x+n)} \cdot P_n(x) \cdot Q_n(x+1) \\ \alpha(x) \cdot f(x) &= \alpha(x) \cdot a \cdot P_n(x) \cdot Q_n(x), \end{aligned}$$

whence

$$Q_n(x+1) = \frac{\alpha(x+n)}{\alpha(n)} \cdot Q_n(x), \quad Q_n(1) = 1. \tag{43}$$

If $\lim_{n\to\infty} P_n(x) =: P(x)$ exists and $P(x) \neq 0$ for all x, then $\lim_{n\to\infty} Q_n(x) =: Q(x)$ and $\lim_{n\to\infty} \frac{\alpha(x+n)}{\alpha(n)} =: \beta(x)$ exist too and both are different from 0. In this case we have

$$P(x+1) = \frac{\alpha(x)}{\beta(x)} \cdot P(x).$$

Thus the following theorem holds.

Theorem 2. *Suppose that*

$$P(x) = \lim_{n\to\infty} P_n(x) \tag{44}$$

exists and is unequal 0*, then every solution of* (6) *has the form*

$$f(x) = a \cdot P(x) \cdot Q(x), \tag{45}$$

where Q *is a solution of the difference equation*

$$Q(x+1) = \beta(x) \cdot Q(x), \quad Q(1) = 1 \tag{46}$$

with

$$\beta(x) = \lim_{n\to\infty} \frac{\alpha(x+n)}{\alpha(n)}.$$

Conversely, each f *of the form* (45) *where* Q *satisfies* (46) *is a solution of* (6)*.*

We call the solution (45) a *normal solution* of (6), if Q has the form (3), or (5), or Q is a normal solution of (46), received by a foregoing application of Theorem 1 or Theorem 2.

Thus, we have shifted back the problem to a difference equation (46) for Q of the same type as (6) but with a possible simpler coefficient function.

Theorem 2 subsumes Theorem 1, hence is also applicable to the difference equation of the gamma and the q-gamma function ($0 < |q| < 1$) and, with some slight changes of the domain and of the initial condition, to the equations of spirals, as mentioned in the introduction.

As a further application of Theorem 2 we now can handle the case of the q-gamma function for $|q| > 1$.

Example 2. *The q-gamma function* Γ_q *for* $|q| > 1$. The behavior of solutions of (40) for $|q| > 1$ is different from that of the case $|q| < 1$ (see e.g. [3], [11], [17]). Here an application of Theorem 1 fails. We have to use Theorem 2. We have

$$\begin{aligned} P_n(x) &= \left(\frac{q^n - 1}{q - 1}\right)^x \frac{q - 1}{q^x - 1} \prod_{k=1}^{n-1} \frac{q^k - 1}{q^{k+x} - 1} \\ &= \frac{(q-1)^{1-x}(1-q^{-n})^x q^{nx}}{q^x - 1} \prod_{k=1}^{n-1} \frac{1 - q^{-k}}{q^x(1 - q^{-k-x})} \\ &= \frac{(q-1)^{1-x}(1-q^{-n})^x q^x}{q^x - 1} \prod_{k=1}^{n-1} \frac{1 - q^{-k}}{1 - q^{-k-x}} \\ &= \frac{(q-1)^{1-x}(1-q^{-n})^x}{1 - q^{-x}} \prod_{k=1}^{n-1} \frac{1 - q^{-k}}{1 - q^{-k-x}}. \end{aligned}$$

Thus $P_n(x)$ converges as an infinite product and is unequal to 0.

$$P(x) = (q-1)^{1-x} \prod_{k=0}^{\infty} \frac{1 - q^{-(k+1)}}{1 - q^{-(k+x)}}$$

and

$$\beta(x) = \lim_{n\to\infty} \frac{q^{n+x} - 1}{q^n - 1} = q^x.$$

So we get

$$f(x) = P(x) \cdot Q(x).$$

Q has to satisfy the difference equation

$$Q(x+1) = q^x \cdot Q(x),\ Q(1) = 1,$$

which is of the type (4). The normal solution of this equation is, according to (5)

$$Q(x) = q^{x(x-1)/2}.$$

In this way we get by Theorem 2 the well known product representation for the q-gamma function, $|q| > 1$, as the normal solution of (40):

$$f(x) = P(x) \cdot Q(x) = \Gamma_q = q^{x(x-1)/2}(q-1)^{1-x} \cdot \prod_{k=0}^{\infty} \frac{1 - q^{-(k+1)}}{1 - q^{-(k+x)}},$$

see [17], p. 278.

Remark 5. (Compare also with Remark 2.) The q-gamma function for $0<|q|<1$ is characterized as the only solution of its difference equation (6) which satisfies the asymptotic equivalence

$$f(n+x) \approx \left(\frac{1}{q-1}\right)^x \cdot f(n), \quad \text{for } n \to \infty.$$

For the q-gamma function for $|q| > 1$ we have the characterization to be the unique solution of its difference equation (40) for which

$$f(n+x) \approx \left(\frac{q^n-1}{q-1}\right)^x \cdot q^{x(x-1)/2} \cdot f(n)$$

holds.

Generally, the normal solution of (6), if it exists according to Theorem 2, is characterized by the asymptotic equivalence

$$f(n+x) \approx \alpha(n)^x \cdot Q(x) \cdot f(n),$$

where Q is the normal solution of (46).

Remark 6. It would be of interest to investigate when the normal solutions stated in this paper are also normal solutions in the sense of Nörlund, John or Krull. It is the case for the classical functions treated here. However, one should also remark that this concept of normal solutions here also applies for complex valued functions.

Acknowledgement. I would like to thank the referees for their constructive remarks, although I could not consider them all.

References

[1] G. E. Andrew, R. Askey and R. Roy, *Special functions*, Cambridge University Press, Cambridge, 1999.
[2] R. Askey, *The q-Gamma and q-Beta Functions*, Applicable Anal. *8* (1978), 125–141.
[3] B. Choczewski and A. Wach-Michalek, *A difference equation for q-gamma functions*, in: M. Bartušek and O. Došlý (eds.), Colloquium on Differential and Difference Equations, CDDE 2002 (Brno), 71–75, Folia Fac. Sci. Natur. Univ. Masaryk. Brun. Math., 13, Masaryk Univ., Brno, 2003.
[4] P. J. Davis, *Spirals from Theodorus to chaos*, A. K. Peters, Wellesley, Mass., 1993.
[5] L. Euler, *De termino generali serium hypergeometricarum*, Eneström No. 652, 1776. Opera omnia, seria I, 16a, 139–162.
[6] D. Gronau, *Why is the gamma function so as it is?* Teaching Mathematics and Computer Sciences *1* (2003), 43–54.
[7] D. Gronau, *The spiral of Theodorus*, Amer. Math. Monthly *111* (2004), no. 3, 230–237.
[8] J. Heitzer, *Spiralen, ein Kapitel phänomenaler Mathematik*, Lesehefte Mathematik, Ernst Klett, Schulbuchverlag, Leipzig, 1998.
[9] K. J. Heuvers, D. S. Moak and B. Boursaw, *The functional equation of the square root spiral*, in: T. M. Rassias (ed.), *Functional Equations and Inequalities*, Kluwer, Dordrecht, 2000, 111–117.
[10] F. John, *Special solutions of certain difference equations*, Acta Math. *71* (1939), 175–189.

[11] H.-H. Kairies, *Einbettungen von* $\log \Gamma$ *durch Scharen spezieller Krull-Normallösungen*, Aequationes Math. *29* (1985), 28–35.
[12] W. Krull, *Bemerkungen zur Differenzengleichung* $g(x+1) - g(x) = \varphi(x)$*, I. + II.*, Math. Nachr. *1* (1948), 365–376 + *2* (1949), 251–262.
[13] M. Kuczma, *Functional Equations in a Single Variable*, Monografie Matematyczne 46, PWN – Polish Scientific Publishers, Warsaw, 1968.
[14] J. D. Lawrence, *A catalog of special plane curves*, Dover Publ. Inc., New York, 1973.
[15] H. v. Mangold and K. Knopp, *Höhere Mathematik*, Band 3, 15. Aufl., S. Hirzel Verlag, Stuttgart, 1990.
[16] L. M. Milne-Thomson, *The calculus of finite differences*, Chelsea Publ, New York, 1933/1981.
[17] D. S. Moak, *The q-gamma function for* $q > 1$, Aequationes Math. *20* (1980), 278–285.
[18] M. Naylor, *Golden,* $\sqrt{2}$*, and* π *Flowers: A Spiral Story*, Math. Magazine, *75*, No. 3 (2002), 163–172.
[19] N. E. Nörlund, *Vorlesung über Differenzenrechung*, Springer, Berlin, 1924.
[20] A. Pringsheim, *Zur Theorie der Gamma-Funktionen*, Math. Ann. *31* (1888), 455–481.
[21] R. A. Rankin, *An Introduction to Mathematical Analysis*, Pergamon Press, Oxford etc., 1963.
[22] W. Rudin, *Principles of mathematical analysis*, Third Edition, Mc Graw-Hill, Inc., New York, 1985.

D. Gronau
Institut für Mathematik
Universität Graz
Heinrichstr. 36
A–8010 Graz
Austria
e-mail: gronau@uni-graz.at

Manuscript received: November 26, 2003 and, in final form, July 21, 2004.