

NONCOMMUTATIVE RESOLUTIONS OF MONOMIAL CURVES

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ABSTRACT.

1. INTRODUCTION

2. PRELIMINARIES

We write $\mathbb{N} = \{1, 2, 3, \dots\}$ for the set of natural numbers. For $a, b \in \mathbb{Z}$, we denote by $\llbracket a, b \rrbracket$ the set of all integers from a to b , including both endpoints.

Throughout, we fix an arbitrary ground field k . We begin by recalling some basic facts about toric algebras, focusing in particular on the one-dimensional case. See [CLS11] for further details.

Let S be an affine semigroup, that is, a finitely generated submonoid of $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. The **toric algebra** associated with S is the semigroup algebra

$$R := k[S] = \text{span}_k\{x^a \mid a \in S\}.$$

The algebra R is a finitely generated k -subalgebra of the Laurent polynomial ring $k[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$. In particular, it is a Noetherian domain and inherits the natural $\mathbb{Z}^d$ -grading. Its Krull dimension is equal to the rank of $\text{gp}(S)$, where $\text{gp}(S)$ denotes the subgroup of $\mathbb{Z}^d$ generated by S [CLS11, Prop. 1.1.14]. Consequently, R is one-dimensional if and only if $\text{gp}(S) \cong \mathbb{Z}$, in which case we may assume without loss of generality that $\text{gp}(S) = \mathbb{Z}$. Every submonoid of $\mathbb{Z}$ is either a group, a submonoid of $\mathbb{N}_0$, or a submonoid of $-\mathbb{N}_0$ [BG25, § 2]. Hence, when R is one-dimensional, S is, up to isomorphism of monoids, either $\mathbb{Z}$ or a submonoid of $\mathbb{N}_0$. Accordingly, either $R \cong k[x^{\pm 1}]$ or $R \cong k[x^{a_1}, \dots, x^{a_n}]$ for some $a_1, \dots, a_n \in \mathbb{N}_0$.

When k is algebraically closed, the algebra $R = k[S]$ is the coordinate ring of the associated affine toric variety $X = \text{Spec}(R)$. In dimension one, the corresponding affine toric variety is either the one-dimensional torus $\text{Spec}(k[x^{\pm 1}]) = \mathbb{A}_k^1 \setminus \{0\}$, or an **affine monomial curve** $\text{Spec}(k[x^{a_1}, \dots, x^{a_n}])$, parametrized by

$$(2.1) \quad \varphi: \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^n, \quad t \mapsto (t^{a_1}, \dots, t^{a_n}).$$

2.1. Numerical monoids. Let S be a submonoid of $\mathbb{N}_0$. Writing $S = \text{gcd}(S) \cdot S'$, where S' is a submonoid of $\mathbb{N}_0$ with $\text{gcd}(S') = 1$, yields an isomorphism of monoids between S and S' . Thus, it suffices to consider submonoids of $\mathbb{N}_0$ with greatest common divisor 1, called **numerical monoids**. Crucially, the gap set $\mathbb{N}_0 \setminus S$ of a numerical monoid S is finite. For further background on numerical monoids, see [RG09].

For a numerical monoid $S = \langle a_1, \dots, a_n \rangle \subseteq \mathbb{N}_0$, the parametrization φ from (2.1) of the associated affine monomial curve $X = \text{Spec}(R)$ restricts to an isomorphism of varieties $\mathbb{A}_k^1 \setminus \{0\} \xrightarrow{\sim} X \setminus \{0\}$. Hence X is nonsingular away from the origin. At the origin, it is singular unless $S = \mathbb{N}_0$, in which case $R = k[x]$ and $X = \mathbb{A}_k^1$.

2.2. Normality, seminormality, and weak normality. A domain Λ is called **normal** if it is integrally closed in its field of fractions $\text{Frac}(\Lambda)$, and **seminormal** if, for every $\lambda \in \text{Frac}(\Lambda)$, $\lambda^2 \in \Lambda$ and $\lambda^3 \in \Lambda$ imply that $\lambda \in \Lambda$. The normalization $\bar{\Lambda}$ and the seminormalization ${}^+\Lambda$ of Λ are, respectively, the intersections of all intermediate normal and seminormal rings Γ satisfying $\Lambda \subseteq \Gamma \subseteq \text{Frac}(\Lambda)$. If Λ has prime characteristic p , then Λ is called **weakly normal** if, for every $\lambda \in \text{Frac}(\Lambda)$, the condition $\lambda^p \in \Lambda$ implies $\lambda \in \Lambda$. Clearly, normality implies seminormality. In prime characteristic p , normality implies weak normality, and weak normality in turn implies seminormality. Indeed, suppose that Λ is weakly normal and let $\lambda \in \text{Frac}(\Lambda)$ satisfy $\lambda^2, \lambda^3 \in \Lambda$. Write $p = 2k + 3\ell$ for some $k, \ell \in \mathbb{N}_0$. Then

$$\lambda^p = (\lambda^2)^k (\lambda^3)^\ell \in \Lambda,$$

and hence $\lambda \in \Lambda$.

For a toric algebra $R = k[S]$, the field of fractions in the definitions above may be replaced by $k[\text{gp}(S)]$. Indeed, $\text{gp}(S) \cong \mathbb{Z}^{d'}$ for some $d' \in \mathbb{N}_0$, so $k[\text{gp}(S)]$ is isomorphic to a Laurent polynomial ring and is therefore normal. Consequently, every element of $\text{Frac}(R)$ that is integral over R already belongs to $k[\text{gp}(S)]$.

mention normality, ... for affine monoids?

In the case of a numerical monoid S , the corresponding toric algebra R satisfies ${}^+R = k[x]$. Indeed, let $n_1 > \dots > n_k$ be the elements of $\mathbb{N}_0 \setminus S$. Since n_1 is the largest gap of S , we have $(x^{n_1})^2, (x^{n_1})^3 \in R \subseteq {}^+R$, and hence $x^{n_1} \in {}^+R$ by seminormality. Repeating this argument successively for the remaining gaps shows that $x^{n_1}, \dots, x^{n_k} \in {}^+R$, and therefore $k[x] \subseteq {}^+R$. Since $k[x]$ is normal, hence seminormal, the reverse inclusion follows. As $k[x]$ is normal, it also follows that $\bar{R} = k[x]$, and consequently $\text{Frac}(R) = k(x)$. Thus, unless $S = \mathbb{N}_0$, the ring R is neither normal nor seminormal.

2.3. Graded Krull–Schmidt theorem. The category of finitely generated $\mathbb{Q}^d$ -graded R -modules satisfies the **Krull–Schmidt property**. That is, in the graded category, every such module decomposes as a finite direct sum of indecomposable modules, uniquely up to graded isomorphism and permutation of the summands [Shi17, § 2].

We are interested in the summands of such graded decompositions only up to degree shifts, and therefore introduce the following notation. Let $M, N, M_1, \dots, M_k$ be $\mathbb{Q}^d$ -graded R -modules. We write $M \cong_{\text{gr}} N$ if M and N are isomorphic as $\mathbb{Q}^d$ -graded R -modules. Moreover, we write

$$M \cong_{\text{sh}} M_1 \oplus \dots \oplus M_k$$

if there exist $d_i \in \mathbb{Q}^d$ such that $M \cong_{\text{gr}} M_1(d_1) \oplus \dots \oplus M_k(d_k)$, where $M_i(d_i)$ denotes the degree shift of M_i by d_i . The notation is compatible with refinements of decompositions. If $M \cong_{\text{sh}} M_1 \oplus \dots \oplus M_k$ and $M_1 \cong_{\text{sh}} M'_1 \oplus \dots \oplus M'_\ell$, then $M \cong_{\text{sh}} M'_1 \oplus \dots \oplus M'_\ell \oplus M_2 \oplus \dots \oplus M_k$.

3. FROBENIUS MAPS ON TORIC ALGEBRAS

Let $R = k[S]$ be the toric algebra associated with an affine monoid $S \subseteq \mathbb{Z}^d$. Fix a positive integer q , and write $q^{-1}S = \{a/q \mid a \in S\} \subseteq \mathbb{Q}^d$ for the rescaling of S by the factor $1/q$. Consider the semigroup algebra

$$R^{1/q} := k[q^{-1}S] = \text{span}_k \{x^{a/q} \mid a \in S\}$$

regarded as a k -subalgebra of $k[\mathbb{Q}^d] = \text{span}_k\{x^\alpha \mid \alpha \in \mathbb{Q}^d\}$, which carries a natural $\mathbb{Q}^d$ -grading. While the algebra $R^{1/q}$ is defined for arbitrary $q \in \mathbb{N}$ in any characteristic, the notation is motivated by the case $q = p^e$ with k perfect of prime characteristic p , where $R^{1/q}$ is the ring of q -th roots of elements in R . Further, for $r = \sum_{a \in \mathbb{Q}^d} k_a x^a \in k[\mathbb{Q}^d]$, we set ${}^q r := \sum_{a \in \mathbb{Q}^d} k_a x^{qa}$, and denote by ${}^q F$ the resulting k -algebra endomorphism

$${}^q F: R \rightarrow R, \quad r \mapsto {}^q r,$$

referred to as the q -th pseudo-Frobenius endomorphism of R .

Remark 3.1. Note that even in prime characteristic p with $q = p^e$, the q -th pseudo-Frobenius endomorphism ${}^q F$ need not agree with the usual e -times iterated Frobenius endomorphism $F^e: R \rightarrow R$, mapping r to r^q .

Proposition 3.2. *Assume that k is a field of prime characteristic p and that $q = p^e$. Then the e -times iterated Frobenius endomorphism F^e and the q -th pseudo-Frobenius endomorphism ${}^q F$ agree precisely when k is finite of order p^f with $f \mid e$.*

Proof. First, note that for $r = \sum_{a \in S} k_a x^a \in R$, we have $F^e(r) = \sum_{a \in S} k_a^q x^{qa}$. Hence, ${}^q F$ and F^e agree precisely when F^e fixes $k \subseteq R$ pointwise.

Suppose first that k is finite of order p^f . By Lagrange's theorem, $k^{p^f} = k$ for every $k \in k$. If $f \mid e$, then repeated application of this identity gives $k^{p^e} = k$ for all $k \in k$. Now assume that $f \nmid e$. Since the multiplicative group of a finite field is cyclic [DF04, § 9, Prop. 18], there exists an element $k \in k \setminus \{0\}$ of order $p^f - 1$. We claim that $k^{p^e} \neq k$. Suppose, for contradiction, that $k^{p^e} = k$. Then $k^{p^e - 1} = 1$, and hence $p^f - 1 \mid p^e - 1$. It follows that

$$p^f - 1 = \gcd(p^e - 1, p^f - 1) = p^{\gcd(e, f)} - 1,$$

where the last equality is a standard identity; see, for example, [Mol08, Example 1.14]. Therefore, $f = \gcd(e, f)$, and hence $f \mid e$, contradicting our assumption. Finally, if k is infinite, the polynomial $x^q - x \in k[x]$ cannot vanish on all of k , so there exists some $k \in k$ such that $k^q \neq k$. $\square$

Definition 3.3. Let M be a $\mathbb{Q}^d$ -graded R -module. The q -th pseudo-Frobenius twist of M , denoted ${}^q M$, is the R -module obtained by restricting scalars along the q -th pseudo-Frobenius endomorphism ${}^q F$. That is, ${}^q M$ is the R -module with the same underlying abelian group as M , but with R -action given by $r \star_q m := {}^q r \cdot m$ for all $r \in R$ and $m \in M$.

Remark 3.4. The q -th pseudo-Frobenius twist of M carries a $\mathbb{Q}^d$ -graded R -module structure when endowed with the homogeneous components $({}^q M)_\alpha := M_{q\alpha}$ for $\alpha \in \mathbb{Q}^d$. Indeed,

$$R_\alpha \star_q ({}^q M)_\beta \subseteq R_{q\alpha} M_{q\beta} \subseteq M_{q(\alpha+\beta)} = ({}^q M)_{\alpha+\beta}$$

for all $\alpha, \beta \in \mathbb{Q}^d$.

Lemma 3.5. *There is a canonical $\mathbb{Q}^d$ -graded R -module isomorphism*

$$R^{1/q} \xrightarrow{\sim} {}^q R,$$

defined by $r = \sum_{a \in S} k_a x^{a/q} \mapsto {}^q r = \sum_{a \in S} k_a x^a$.

Proof. By abuse of notation, we also denote by ${}^qF$ the map $R^{1/q} \rightarrow {}^qR$ defined above. This map is clearly additive, bijective and preserves the $\mathbb{Q}^d$ -grading. To verify that it is compatible with the respective R -module structures, let $r = \sum_{a \in S} k_a x^a \in R$ and $m = \sum_{a \in S} \ell_a x^{a/q} \in R^{1/q}$. Then

$${}^qF(rm) = {}^qF\left(\sum_{a,b \in S} k_a \ell_b x^{a+\frac{b}{q}}\right) = \sum_{a,b \in S} k_a \ell_b x^{qa+b},$$

which is precisely $r \star_q {}^qF(m)$ in ${}^qR$. $\square$

Lemma 3.6. *Let $M, M_1, \dots, M_k$ be $\mathbb{Q}^d$ -graded R -modules. If it holds that $M \cong_{\text{sh}} M_1 \oplus \dots \oplus M_k$, then ${}^qM \cong_{\text{sh}} {}^qM_1 \oplus \dots \oplus {}^qM_k$.*

Proof. For simplicity of notation, we prove the result only for $k = 2$. By assumption, there are $d_1, d_2 \in \mathbb{Q}^d$ such that $M \cong_{\text{gr}} M_1(d_1) \oplus M_2(d_2)$. As the construction of the pseudo-Frobenius twist respects direct sums, we have

$${}^qM \cong_{\text{gr}} {}^q(M_1(d_1)) \oplus {}^q(M_2(d_2)) = ({}^qM_1)(d_1/q) \oplus ({}^qM_2)(d_2/q),$$

where the last equality follows from the computation

$$\begin{aligned} ({}^q(M_1(d_1)) \oplus {}^q(M_2(d_2)))_\alpha &= (M_1(d_1))_{q\alpha} \oplus (M_2(d_2))_{q\alpha} \\ &= (M_1)_{q\alpha+d_1} \oplus (M_2)_{q\alpha+d_2} \\ &= ({}^qM_1)_{\alpha+\frac{d_1}{q}} \oplus ({}^qM_2)_{\alpha+\frac{d_2}{q}} \\ &= (({}^qM_1)(d_1/q))_\alpha \oplus (({}^qM_2)(d_2/q))_\alpha. \end{aligned}$$

$\square$

Definition 3.7. We say that R is q -**split** if the pseudo-Frobenius map ${}^qF: R \rightarrow {}^qR$ splits as a map of R -modules, that is, if there exists an R -module homomorphism $\sigma: {}^qR \rightarrow R$ such that $\sigma \circ {}^qF = \text{id}_R$. Moreover, we call R q -**weakly normal** if, for every $r \in k[\text{gp}(S)]$, ${}^qr \in R$ implies $r \in R$.

Remark 3.8. By [NO04, Prop. 2.3.1], the map ${}^qF$ splits as an R -module homomorphism if and only if it splits in the category of $\mathbb{Q}^d$ -graded R -modules. In other words, whenever ${}^qF$ splits, the splitting map σ may be chosen to be $\mathbb{Q}^d$ -graded. Hence, R is q -split if and only if it is a graded direct summand of ${}^qR$, that is, if there exists a $\mathbb{Q}^d$ -graded R -module N such that ${}^qR \cong_{\text{gr}} R \oplus N$.

In prime characteristic p , the Frobenius endomorphism yields ${}^pr \in R$ if and only if $r^p \in R$, from which we obtain the following.

Corollary 3.9. *Assume that k is a field of prime characteristic p . Then R is p -weakly normal if and only if R is weakly normal. In particular, R is seminormal in this case.*

It is known that, in prime characteristic, the Frobenius map splits as a map of R -modules if and only if R is weakly normal. The argument of [Yas09, Prop. 5.3] extends verbatim to the characteristic-independent setting upon replacing the Frobenius map by the pseudo-Frobenius map. For the reader's convenience, we include the argument here.

Proposition 3.10. [HR76, Prop. 5.31], [Yas09, Prop. 5.3] *R is q -split if and only if it is q -weakly normal.*

Proof. Suppose R is q -split, with $\sigma: {}^qR \rightarrow R$ an R -module homomorphism satisfying $\sigma \circ {}^qF = \text{id}_R$. Let $r \in \mathbf{k}[\text{gp}(\mathbf{S})]$ be such that ${}^qr \in R$, and choose $s \in R \setminus \{0\}$ with $sr \in R$. Then

$$sr = \sigma({}^qF(sr)) = \sigma({}^qs^qr) = \sigma(s \star_q ({}^qr)) = s\sigma({}^qr),$$

and cancelling s yields $r = \sigma({}^qr) \in R$.

Conversely, suppose that R is q -weakly normal. Let $\sigma: {}^qR \rightarrow R$ be the map that sends $f \in {}^qR$ to the unique $g \in R$ with $f = {}^qg$, if such a g exists, and to 0 otherwise. Clearly $\sigma \circ {}^qF = \text{id}_R$, so it remains to show that σ is a homomorphism of R -modules. As σ is additive, it suffices to check that $\sigma(x^a \star_q x^b) = x^a \sigma(x^b)$ for $a, b \in \mathbf{S}$. Suppose first that $b = qc$ for some $c \in \mathbf{S}$. Then

$$\sigma(x^a \star_q x^b) = \sigma(x^{q(a+c)}) = x^{a+c} = x^a \sigma(x^b).$$

Now suppose that b is not of the form $b = qc$ for some $c \in \mathbf{S}$, meaning that $\sigma(x^b) = 0$. We claim that $qa + b$ is also not of the form $qa + b = qc$ for some $c \in \mathbf{S}$. Indeed, suppose that there exists a $c \in \mathbf{S}$ such that $qa + b = qc$. Then ${}^q(x^{c-a}) = x^b \in R$. By assumption, this implies that $x^{c-a} \in R$, and hence $c - a \in \mathbf{S}$. Consequently, $b = q(c - a)$, contradicting our assumption on b . It follows that $\sigma(x^a \star_q x^b) = \sigma(x^{qa+b}) = 0$. $\square$

Corollary 3.11. [BLR06, Cor. 6.3] *R is normal if and only if R is q -weakly normal for all $q \in \mathbb{N}$ if and only if R is q -split for all $q \in \mathbb{N}$.*

Proof. By [Hoc72, Prop. 1], the toric algebra $R = \mathbf{k}[\mathbf{S}]$ is normal if and only if $\mathbf{S}$ is normal, that is, if for every $a \in \text{gp}(\mathbf{S})$ and $q \in \mathbb{N}$, the condition $qa \in \mathbf{S}$ implies $a \in \mathbf{S}$. This is precisely the monoid-theoretic formulation of R being q -weakly normal for all $q \in \mathbb{N}$. By the previous proposition, this is equivalent to R being q -weakly normal for every $q \in \mathbb{N}$. $\square$

Lemma 3.12. [BS13, Lemma 5.0.3] *Let $q = p_1^{e_1} \cdots p_k^{e_k}$, where the p_i are distinct primes and $e_i \in \mathbb{N}$. Then R is q -split if and only if R is $(p_1^{\ell_1} \cdots p_k^{\ell_k})$ -split for all $(\ell_1, \dots, \ell_k) \in \mathbb{N}_0^k$.*

Proof. Assume that R is q -split, i.e., there exists a R -module homomorphism $\sigma: {}^qR \rightarrow R$ such that $\sigma \circ {}^qF = \text{id}_R$. We first claim that, for each $i \in \llbracket 1, k \rrbracket$, R is $p_i^{\ell_i}$ -split for every $\ell_i \in \mathbb{N}_0$. To this end, fix an $i \in \llbracket 1, k \rrbracket$ and consider the R -module homomorphism $f_i: {}^{p_i}R \rightarrow {}^qR$, defined by $r \mapsto {}^{q/p_i}r$. Setting $\sigma_i := \sigma \circ f_i$, we obtain that

$$\sigma_i({}^{p_i}F(r)) = \sigma(f_i({}^{p_i}r)) = \sigma({}^qr) = r$$

for all $r \in R$, so R is p_i -split. Now, an induction on ℓ_i shows that, as maps of sets, $\sigma_i^{\ell_i} \circ {}^{p_i^{\ell_i}}F = \text{id}_R$ for all $\ell_i \in \mathbb{N}_0$, where $\sigma_i^{\ell_i}$ denotes the ℓ_i -fold composition of σ_i . Moreover, $\sigma_i^{\ell_i}$, regarded as a map ${}^{p_i^{\ell_i}}R \rightarrow R$, is readily verified to be R -linear, providing the claim.

Finally, fix $(\ell_1, \dots, \ell_k) \in \mathbb{N}_0^k$, and set $q' := p_1^{\ell_1} \cdots p_k^{\ell_k}$. By the first part, R is $p_i^{\ell_i}$ -split for each $i \in \llbracket 1, k \rrbracket$. Hence, there exist R -module homomorphisms $\sigma'_i: {}^{p_i^{\ell_i}}R \rightarrow R$ such that $\sigma'_i \circ {}^{p_i^{\ell_i}}F = \text{id}_R$. Setting $\sigma' := \sigma'_1 \circ \cdots \circ \sigma'_k$, a direct verification shows that σ' gives the required splitting of ${}^{q'}F$. $\square$

4. R -MODULE DECOMPOSITION OF $R^{1/q}$

We now specialize to the case where $\mathbf{S} = \langle a_1, \dots, a_n \rangle \subseteq \mathbb{N}_0$ is a numerical monoid and $R = \mathbf{k}[\mathbf{S}]$ is the corresponding one-dimensional toric algebra. To exclude the trivial case, we assume that $\mathbf{S}$ is proper, that is, $\mathbf{S} \subsetneq \mathbb{N}_0$. Following Shibuta [Shi17], we sketch, for a fixed $q \in \mathbb{N}$, how $R^{1/q} = \mathbf{k}[x^{a_1/q}, \dots, x^{a_n/q}]$ decomposes as a $\mathbb{Q}_{\geq 0}$ -graded R -module. Observe that

$$\Gamma_q := \left\{ (x^{a_1/q})^{e_1} \cdots (x^{a_n/q})^{e_n} \mid e_i \in \llbracket 0, q-1 \rrbracket \right\}$$

is a homogeneous generating set for $R^{1/q}$. We partition Γ_q by degree modulo $\mathbb{Z}$: writing each degree as a/q with $a \in \mathbb{N}_0$, this amounts to grouping elements by the residue of a modulo q . This yields

$$\Gamma_q = \Gamma_{q,0} \sqcup \Gamma_{q,1} \sqcup \cdots \sqcup \Gamma_{q,q-1},$$

where $\Gamma_{q,r} := \{x^{a/q} \in \Gamma_q \mid a \equiv r \pmod{q}\}$. Let $I_{q,r}$ be the R -submodule of $\mathbf{k}[\mathbb{Q}_{\geq 0}] = \mathbf{k}[x^q \mid q \in \mathbb{Q}_{\geq 0}]$ generated by $\Gamma_{q,r}$. As a $\mathbf{k}$ -vector space,

$$(4.1) \quad I_{q,r} = \text{span}_{\mathbf{k}}\{x^{a/q} \mid a \in \mathbf{S}, a \equiv r \pmod{q}\},$$

and, by construction, summing over r gives the $\mathbb{Q}_{\geq 0}$ -graded R -module decomposition

$$R^{1/q} = \bigoplus_{r \in \llbracket 0, q-1 \rrbracket} I_{q,r},$$

where each summand is indecomposable (see Lemma 4.4).

Definition 4.1. Let $A \subseteq \mathbb{N}$. The **gap module defined by A** is the $\mathbb{N}_0$ -graded R -submodule of $\overline{R} = \mathbf{k}[x]$ given by

$$I(A) := \langle x^\alpha \mid \alpha \in A \cup \{0\} \rangle_R.$$

Remark 4.2. Clearly, in Definition 4.1, the set A may always be chosen as a subset of the gap set $\mathbb{N}_0 \setminus \mathbf{S}$. In particular, there are only finitely many gap modules. Moreover, we may assume that A consists of elements that are minimal with respect to the partial order $\preceq_{\mathbf{S}}$ induced by $\mathbf{S}$, where $\alpha \preceq_{\mathbf{S}} \beta$ means $\beta - \alpha \in \mathbf{S}$. We write

$$\text{Min}(\mathbf{S}) := \{A \subseteq \mathbb{N}_0 \setminus \mathbf{S} \mid A \text{ is minimal with respect to } \preceq_{\mathbf{S}}\}.$$

Corollary 4.3. Every $\mathbb{Q}_{\geq 0}$ -graded indecomposable summand of $R^{1/q}$ is, up to a shift in degree, isomorphic to a gap module.

Proof. Let $\frac{m_{q,r}}{q}$ denote the minimal degree occurring among the elements of $\Gamma_{q,r}$, and set

$$A_{q,r} := \left\{ \alpha \in \mathbb{N} \mid \frac{m_{q,r}}{q} + \alpha \in \Gamma_{q,r} \right\},$$

from which it follows that $I_{q,r} = x^{\frac{m_{q,r}}{q}} \cdot I(A_{q,r})$. $\square$

In particular, as q ranges over $\mathbb{N}$, only finitely many isomorphism classes of $\mathbb{Q}_{\geq 0}$ -graded indecomposable summands of $R^{1/q}$ occur, up to shifts in degree. More precisely, for every $q \in \mathbb{N}$, there exist $m_A \in \mathbb{N}_0$ for $A \in \text{Min}(\mathbf{S})$ such that

$$R^{1/q} \cong_{\text{sh}} \bigoplus_{A \in \text{Min}(\mathbf{S})} I(A)^{\oplus m_A}.$$

Lemma 4.4. *For each $A \in \text{Min}(\mathbf{S})$, the gap module $I(A)$ has rank 1 and is therefore indecomposable. Moreover, the assignment*

$$A \mapsto [I(A)]$$

defines a bijection between $\text{Min}(\mathbf{S})$ and the set of R -module isomorphism classes of $\mathbb{N}_0$ -graded R -submodules of $\overline{R} = \mathbf{k}[x]$.

Proof. For simplicity, set $Q := \text{Frac}(R) = \mathbf{k}(x)$ and fix $A \in \text{Min}(\mathbf{S})$. Observe that $Q \otimes_R I(A)$ is naturally isomorphic, as a Q -vector space, to $I(A)_{(0)} \subseteq Q$. Since $I(A)$ contains 1, it follows that $I(A)_{(0)} = Q$ and $I(A)$ has rank 1.

For the second part, each $I(A)$ with $A \in \text{Min}(\mathbf{S})$ is an $\mathbb{N}_0$ -graded R -submodule of $\overline{R}$, so the map is well-defined. To prove injectivity, suppose that $\varphi: I(A) \rightarrow I(B)$ is an R -module isomorphism for some $A, B \in \text{Min}(\mathbf{S})$. Localizing at (0) yields an isomorphism of Q -vector spaces $\varphi_{(0)}: I(A)_{(0)} \rightarrow I(B)_{(0)}$, where both source and target are equal to Q . Hence φ , and thus also $\varphi_{(0)}$, is given by multiplication by some $g_0 \in Q$, so $\varphi(f) = g_0 f$ for all $f \in I(A)$. A simple degree argument shows that $g_0 \in \mathbf{k} \setminus \{0\}$. Therefore,

$$I(B) = \varphi(I(A)) = g_0 \cdot I(A) = I(A).$$

Since A and B are both minimal with respect to $\preceq_{\mathbf{S}}$, it follows that $A = B$. We now prove that the assignment is surjective. Let $I \subseteq \overline{R}$ be an arbitrary $\mathbb{N}_0$ -graded R -submodule. Because R is Noetherian and $\overline{R}$ is finitely generated as an R -module, I is finitely generated, i.e., $I = \langle x^\beta \mid \beta \in B \rangle_R$ for some finite subset $B \subseteq \mathbb{N}_0$. Define $m := \min(B)$ and $A' := \{\beta - m \mid \beta \in B\} \setminus \{0\}$. Consequently, $I = x^m \cdot I(A')$, which implies that $I \cong I(A')$. Surjectivity then follows from the equality $I(A') = I(A)$, where A is the set of elements of $A' \cap \mathbb{N}_0 \setminus \mathbf{S}$ that are minimal with respect to $\preceq_{\mathbf{S}}$. $\square$

Remark 4.5. We now present another perspective on the gap modules $I(A)$. For $A \in \text{Min}(\mathbf{S})$, the R -module $I(A)$ admits the $\mathbf{k}$ -vector space description

$$I(A) = \text{span}_{\mathbf{k}}\{x^\alpha \mid \alpha \in \mathbf{S}_A\},$$

where $\mathbf{S}_A := (A \cup \{0\}) + \mathbf{S}$. The set $\mathbf{S}_A$ is a normalized relative ideal of $\mathbf{S}$, that is, a subset of $\mathbb{N}_0$ containing 0 that is closed under addition by elements of $\mathbf{S}$. One readily checks that $A \mapsto \mathbf{S}_A$ defines a one-to-one correspondence between the elements of $\text{Min}(\mathbf{S})$ and the normalized relative ideals of $\mathbf{S}$.

Corollary 4.6. *For each $A \in \text{Min}(\mathbf{S})$, the gap module $I(A)$ is maximal Cohen–Macaulay. In particular, $R^{1/q}$ is maximal Cohen–Macaulay.*

Proof. By the Cohen–Seidenberg theorems, $\dim(R) = \dim(\overline{R}) = \dim(\mathbf{k}[x]) = 1$. Fix $A \in \text{Min}(\mathbf{S})$ and a maximal ideal $\mathfrak{m}$ of R . Since R is a domain of Krull dimension 1, every maximal ideal has height 1; in particular $\dim(R_{\mathfrak{m}}) = \text{ht}(\mathfrak{m}) = 1$. The localization $I(A)_{\mathfrak{m}}$ is a non-zero $R_{\mathfrak{m}}$ -submodule of $\mathbf{k}(x)$, hence torsion-free, so $1 \leq \text{depth}(I(A)_{\mathfrak{m}}) \leq \dim(R_{\mathfrak{m}}) = 1$. Hence, $I(A)$ is maximal Cohen–Macaulay.

By the depth lemma [BH93, Prop. 1.2.9], the depth of a finite direct sum is the minimum of the depths of its summands. Hence, a finite direct sum of maximal Cohen–Macaulay modules is again maximal Cohen–Macaulay. By Corollary 4.3, the module $R^{1/q}$ decomposes as a finite direct sum of gap modules $I(A)$, and the claim follows. $\square$

Lemma 4.7. *For $q \in \mathbb{N}$, it holds that ${}^q \overline{R} \cong_{\text{sh}} \overline{R}^{\oplus q}$.*

Proof. First, note that

$$\overline{R}^{1/q} = \mathbf{k}[x^{1/q}] = \bigoplus_{r \in \llbracket 0, q-1 \rrbracket} x^{r/q} \cdot \mathbf{k}[x] \cong_{\text{sh}} \overline{R}^{\oplus q}$$

as R -modules. By Lemma 3.5, $\overline{R}^{1/q}$ and ${}^q\overline{R}$ are isomorphic as $\mathbb{Q}_{\geq 0}$ -graded modules over $\overline{R}$, and hence, by restriction of scalars along $R \subseteq \overline{R}$, also as $\mathbb{Q}_{\geq 0}$ -graded modules over R . $\square$

Example 4.8. Let $S = \langle 3, 4 \rangle \subseteq \mathbb{N}_0$ and $R = \mathbf{k}[S] = \mathbf{k}[x^3, x^4]$.

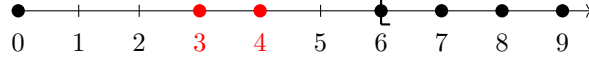


FIGURE 1. The numerical monoid $S = \langle 3, 4 \rangle$

The set

$$\{x^{(3/q)e_1 + (4/q)e_2} : e_i \in \llbracket 0, q-1 \rrbracket\}$$

generates $R^{1/q} = \mathbf{k}[x^{3/q}, x^{4/q}]$ as an R -module. For $q = 2$, this homogeneous generating set is $\{1, x^{3/2}, x^2, x^{7/2}\}$, and grouping the numerators of the exponents by residue modulo 2 yields the $\mathbb{Q}_{\geq 0}$ -graded R -module decomposition

$$R^{1/2} = \langle 1, x^2 \rangle_R \oplus x^{3/2} \cdot \langle 1, x^2 \rangle_R \cong_{\text{sh}} I(\{2\})^{\oplus 2}$$

For $q = 4$, a direct computation shows $R^{1/4} \cong_{\text{sh}} \overline{R}^{\oplus 4}$. Using the $\mathbb{Q}_{\geq 0}$ -graded isomorphism between $R^{1/q}$ and ${}^qR$, we compute

$$R^{1/8} \cong_{\text{gr}} {}^8R = {}^2({}^4R) \cong_{\text{gr}} {}^2(R^{1/4}).$$

Hence, by Lemma 3.6, it holds that $R^{1/8} \cong_{\text{sh}} ({}^2\overline{R})^{\oplus 4}$. The preceding lemma then yields $R^{1/8} \cong_{\text{sh}} \overline{R}^{\oplus 8}$. By induction, we obtain that $R^{1/2^e} \cong_{\text{sh}} \overline{R}^{\oplus 2^e}$ for all $e \geq 2$. Consequently, there exist summands appearing in the $\mathbb{Q}_{\geq 0}$ -graded direct sum decomposition of $R^{1/2^e}$ for some $e \in \mathbb{N}$ that do *not* appear in the decomposition of $R^{1/2^\ell}$ for large ℓ . This stands in contrast to the case where S is a normal affine monoid and $R = \mathbf{k}[S]$ is a normal toric algebra [Bru05]. $\blacksquare$

Proposition 4.9. *It holds that $q \in S$ if and only if every $\mathbb{Q}^d$ -graded indecomposable summand of $R^{1/q}$ is, up to a degree shift, isomorphic to the normalization $\overline{R} = \mathbf{k}[x]$ of R , that is, if $R^{1/q} \cong_{\text{sh}} \overline{R}^{\oplus q}$. In particular, this holds for all sufficiently large q .*

Proof. Assume that $q \in S$, and fix $r \in \llbracket 0, q-1 \rrbracket$. For simplicity, set

$$\text{supp}(I_{q,r}) := \{a/q \mid a \in S, a \equiv r \pmod{q}\}$$

and observe that $\text{supp}(I_{q,r}) \subseteq \{\frac{m_{q,r}}{q} + \alpha \mid \alpha \in \mathbb{N}_0\}$, where $\frac{m_{q,r}}{q}$ is the minimum of $\text{supp}(I_{q,r})$. Conversely, for each $\alpha \in \mathbb{N}_0$, the choice of q ensures $\alpha q \in S$, and hence $m_{q,r} + \alpha q \in S$. Since $m_{q,r} + \alpha q \equiv r \pmod{q}$, we obtain $\frac{m_{q,r}}{q} + \alpha \in \text{supp}(I_{q,r})$, and altogether

$$\text{supp}(I_{q,r}) = \left\{ \frac{m_{q,r}}{q} + \alpha \mid \alpha \in \mathbb{N}_0 \right\}.$$

It now follows from (4.1) that

$$I_{q,r} = \text{span}_k \left\{ x^{\frac{m_{q,r}}{q} + \alpha} \mid \alpha \in \mathbb{N}_0 \right\} = x^{\frac{m_{q,r}}{q}} \cdot \text{span}_k \{ x^\alpha \mid \alpha \in \mathbb{N}_0 \} \cong k[x].$$

Conversely, suppose that every indecomposable $\mathbb{Q}^d$ -graded summand of $R^{1/q}$ is, up to a degree shift, isomorphic to $\bar{R}$. In particular, $I_{q,0} \cong_{\text{sh}} \bar{R}$. Since both are gap modules, Lemma 4.4 yields $I_{q,0} = \bar{R} = k[x]$. The description of $I_{q,0}$ then implies that $q \in S$. $\square$

think about what can be said about decompositions of $R^{1/q}$ for $q \notin S$

Example 4.10. Revisiting Example 4.8, it remains to determine the $\mathbb{Q}_{\geq 0}$ -graded decomposition of $R^{1/5}$ in order to obtain a complete description of the decompositions of $R^{1/q}$ for $q \in \mathbb{N}$. A direct computation yields

$$R^{1/5} \cong_{\text{sh}} I(\{2\}) \oplus \bar{R}^{\oplus 4}.$$

Since $\bar{R} = k[x]$ corresponds to $I(\{1, 2\})$, the indecomposable $\mathbb{Q}_{\geq 0}$ -graded summands occurring in $R^{1/q}$ for some $q \in \mathbb{N}$ are, up to degree shift, precisely $I(\{2\})$ and $I(\{1, 2\})$. On the other hand,

$$\text{Min}(S) = \{\emptyset, \{1\}, \{2\}, \{5\}, \{1, 2\}\},$$

so not every potential summand $I(A)$ with $A \in \text{Min}(S)$ actually occurs. $\blacksquare$

Corollary 4.11. *Let $S \subsetneq \mathbb{N}_0$. Then, for every $q \in \mathbb{N}_{\geq 2}$, the toric algebra R is not q -split, or equivalently, R does not occur as a $\mathbb{Q}_{\geq 0}$ -graded direct summand of $R^{1/q}$.*

note: this can also be easily proven directly!

Proof. Fix $q \in \mathbb{N}_{\geq 2}$, and let p be a prime divisor of q . Suppose, for contradiction, that R is q -split. By Lemma 3.12, R is then p^e -split for every $e \in \mathbb{N}$. This contradicts Proposition 4.9, which shows that, for all sufficiently large e , R^{1/p^e} decomposes as a direct sum of shifted copies of $\bar{R}$, and therefore cannot contain R as a $\mathbb{Q}_{\geq 0}$ -direct summand. $\square$

Proposition 4.12. *Let $p \in \mathbb{N}$. If the indecomposable $\mathbb{Q}_{\geq 0}$ -graded summands of $R^{1/q}$ and $R^{\frac{1}{p^e q}}$ coincide up to degree shift, then the same holds for $R^{1/q}$ and $R^{\frac{1}{p^e q}}$ for every $e \in \mathbb{N}$.*

Proof. Decompose $R^{1/q} \cong_{\text{sh}} \bigoplus_{i \in [1, k]} I(A_i)^{\oplus m_i}$, where $A_1, \dots, A_k \in \text{Min}(S)$ are pairwise distinct and $m_i \in \mathbb{N}$. By Lemma 3.6, it follows that

$$R^{\frac{1}{p^e q}} \cong_{\text{gr}} {}^p(R^{1/q}) \cong_{\text{sh}} \bigoplus_{i \in [1, k]} ({}^p I(A_i))^{\oplus m_i}.$$

Thus, by assumption, the indecomposable $\mathbb{Q}_{\geq 0}$ -graded direct summands occurring in ${}^p I(A_i)$, for $i \in [1, k]$, are, up to degree shift, precisely $I(A_1), \dots, I(A_k)$. Iterating this argument shows that $R^{\frac{1}{p^e q}}$ has the same indecomposable $\mathbb{Q}_{\geq 0}$ -graded direct summands, up to degree shift, for every $e \in \mathbb{N}$. $\square$

5. GLOBAL DIMENSION OF ENDOMORPHISM ALGEBRAS

Throughout this section, let S again be a numerical monoid and $R = k[S]$ the corresponding toric algebra.

For a commutative Noetherian ring Λ and a finitely generated faithful Λ -module M , the endomorphism algebra $\text{End}_\Lambda(M)$ is called a **noncommutative resolution of singularities** of Λ if it has finite global dimension.

Theorem 5.1. *For $q \in S$, the global dimension of $\text{End}_R(R^{1/q})$ is equal to the Krull dimension of R . In particular, this is true for all sufficiently large q .*

Proof. For $q \in S$, Proposition 4.9 gives $R^{1/q} \cong \overline{R}^{\oplus q}$. By [Leu12, Cor. 2.3], multiplicities of direct summands do not affect the endomorphism ring up to Morita equivalence; hence $\text{End}_R(R^{1/q})$ is Morita equivalent to $\text{End}_R(\overline{R})$. As in the proof of Lemma 4.4, since $\overline{R}$ has rank 1, any R -module endomorphism of $\overline{R}$ is induced by multiplication with an element of $\overline{R}$, and thus $\text{End}_R(\overline{R}) \cong \overline{R}$. Consequently, we obtain

$$\text{gl. dim}(\text{End}_R(R^{1/q})) = \text{gl. dim}(\overline{R}) = \text{gl. dim}(k[x]) = 1 = \dim(R),$$

where the first equality follows from the Morita invariance of global dimension. $\square$

is there a notion of crepant for non-normal, non-Gorenstein rings?

for $\mathbb{A} := \bigoplus_{A \in \text{Min}(S)} I(A)$, think about $\text{gl. dim}(\text{End}_R(\mathbb{A}))$ and/or $\text{gl. dim}(\text{End}_R(R^{1/q}))$ for small q

Following the argument of Faber, Muller, and Smith [FMS19, Theorem 6.13] for the corresponding result on normal toric varieties, we obtain the following:

Theorem 5.2. *For a perfect field k of prime characteristic, the ring $D_k(R)$ of differential operators on R has finite global dimension.*

is there machinery to show equality for the global dimension of $D_k(R)$ and $\dim(R)$?

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