A)

stationary heat equation (see Methode der finiten Elemente für Ingen)

$(-\lambda u')' + \bar{q}\, u = f + \bar{q}\, u_A$

$\bar{q}$ heat flux at the boundary $= 0$ (cause no convection and [illegible]

u_A environment temp.

$(-\lambda u')' = f$

In the strong setting u has to be in $C^2(0,1) \cap C[0,1]$, ← boundary cond

f has to be $\in C(0,1)$ and $\lambda \in C^1(0,1)$

In the weak setting we can multiply the DE with a „test func"

$v \in H^1(0,1)$ $\quad H^1(a,b) = \{ u \in L_2(a,b) \mid \exists v \in L_2(a,b) \int_{(a,b)} u w' dx = -\int_{(a,b)} w v \, dx \;\; \forall w \in C_1(a,b) \}$

$v(0) = v(1) = 0$ $\quad \subseteq L^2(a,b)$ $\quad w(a) = w(b) = 0$

and integrate

$$\int_0^1 (-\lambda u')' v \, dx = \int_0^1 f v \, dx$$

partial integrate

$$-[\lambda u']_0^1 + \int_0^1 \lambda u' v' \, dx = \int_0^1 f v \, dx$$

$$\int_0^1 \lambda u' v' \, dx = \int_0^1 f v \, dx$$

$u \in H^1(0,1)$ $u(0) = 0, u(1) = 1$

since $\int_0^1 f v$ should exist $f \in L^2(0,1)$

since $u', v' \in L^2$ in order for $\int_0 \lambda u' v'$ to exist $\lambda \in L^\infty(a,b)$

$f = c$ $\quad \lambda =$ constant

$-\lambda u'' = c \quad u(0) = 0, u(1) = 1$

$-\lambda (u^H)'' = 0 \quad u^H(0) = 0, u^H(1) = 1$

$-\lambda (u^P)'' = c \quad u^P(0) = u^P(1) = 0$

$-\lambda (u^H)'' = 0$

$(u^H)'' = 0$

$u^H = kx + d \xrightarrow{b.c.} u^H = x$

$-\lambda (u^P)'' = c$

$(u^P)'' = \frac{c}{-\lambda}$

$u^P = -\frac{c}{2\lambda} x^2 + bx + d \qquad b, d$ integration constants

$\text{b.c.} \rightarrow d = 0 \;\wedge\; -\frac{c}{2\lambda} + b = 0 \Rightarrow b = \frac{c}{2\lambda}$

$u^P = -\frac{c}{2\lambda} x^2 + \frac{c}{2\lambda}x$

$u = -\frac{c}{2\lambda} x^2 + \left(\frac{c}{2\lambda} + 1\right) x \qquad u(0) = 0 \quad u(1) = 1$

$-\lambda u'' = -\lambda \left(-\frac{c}{\lambda}\right) = c$ ✓

B $(-\lambda u')' + \bar{q} u = f + \bar{q}_1 u_A$

$$(-\lambda u')' = 0$$

$$-\lambda u'' = 0$$

$$u'' = 0 \quad \text{for } x \in (0, 0.5) \cup (0.5, 1) \qquad u(0) = 0 \wedge u(1) = 1$$

Interface condition

$$\lim_{x \to 0.5-} u(x) = \lim_{x \to 0.5+} u(x) \qquad \text{temperature continues}$$

$$\lim_{x \to 0.5-} -\lambda_1(x)\, u'(x) = \lim_{x \to 0.5+} -\lambda_2\, u(x) \qquad \text{heat flow continues}$$

$$\lim_{x \to 0.5-} -u'(x) = \lim_{x \to 0.5+} -10 u'(x)$$

$u \in C^2(0, \frac{1}{2}) \cap C^2(\frac{1}{2}, 1) \cap C([0,1]) \leftarrow$ boundary / interface cond

$\lambda \in C^1(0, \frac{1}{2}) \cap C^1(\frac{1}{2}, 1)$

$f \in C((0, \frac{1}{2}) \cap C((\frac{1}{2}, 1)$

$$u'' = 0 \quad \text{for } x \in (0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1)$$

$$u = \begin{cases} a_1 x + b_1 & x \in (0, 0.5) \\ a_2 x + b_2 & x \in (0.5, 1) \end{cases} \qquad \begin{array}{l} u(0) = 0 \to b_1 = 0 \\ u(1) = 1 \to a_2 + b_2 = 1 \\ \qquad b_2 = 1 - a_2 \end{array}$$

$$\lim_{x \to 0.5-} u(x) = a_1 0.5 = a_2 \cdot 0.5 + 1 - a_2 = 1 - a_2 0.5 = \lim_{x \to 0.5+} u(x)$$

$$\lim_{x \to 0.5-} -\lambda(x) u'(x) = -a_1 = -10 a_2 = \lim_{x \to 0.5+} -\lambda(x) u'(x)$$

$$\text{I}: \frac{1}{2} a_1 = -\frac{1}{2} a_2 + 1$$

$$\text{II}: -a_1 = -10 a_2 \to a_1 = 10 a_2$$

$$\text{I}: 5 a_2 = -\frac{1}{2} a_2 + 1$$

$$a_2 = \frac{2}{11} \to a_1 = \frac{20}{11}$$

$$\hookrightarrow b_2 = \frac{9}{11}$$

$$u = \begin{cases} \frac{20}{11} x & x \in (0, \frac{1}{2}) \\ \frac{2}{11} x + \frac{9}{11} & x \in (\frac{1}{2}, 1) \end{cases} \qquad u(0) = 0 \wedge u(1) = 1$$

$$\lim_{x \to 0.5-} u(x) = \frac{20}{11} \cdot \frac{1}{2} = \frac{10}{11} = \frac{2}{11} \cdot \frac{1}{2} + \frac{9}{11} = \lim_{x \to 0.5+} u(x)$$

$$\lim_{x \to 0.5-} -\lambda(x) u'(x) = -\frac{20}{11} = -10 \cdot \frac{2}{11} = \lim_{x \to 0.5+} -\lambda(x) u'(x)$$

c) $z := u'$ $\quad -z' + pz = 0$

$z' = pz$ $\quad$ If $z = 0$ Fixpoint $\to u = c$ trivial sol
but doesn't fit bc.

If $p = 0 \to z' = 0$

$\to z = c \to u = cx + d \xrightarrow{bc} u = x$

Else $z = ke^{px}$ $\quad k \in \mathbb{R}$

$u = \int z \, dx = \frac{k}{p} e^{px} + c$

$u(0) = \frac{k}{p} + c \qquad u(1) = \frac{k}{p} e^{p} + c$

$u(0) - u(1) = \frac{k}{p}\left(1 - e^{p}\right) = -1$

$\frac{k}{p}\left(e^{p} - 1\right) = 1$

$k = \frac{p}{e^{p} - 1} \qquad u(0) = \frac{1}{e^{p} - 1} + c = 0$

$c = -\frac{1}{e^{p} - 1}$

$\to u(x) = \frac{1}{e^{p} - 1} e^{px} - \frac{1}{e^{p} - 1} = \frac{e^{px} - 1}{e^{p} - 1}$