A) $-u'' + au = f \, / \int_0^1 \cdot v(x)$

$$\int_0^1 (-u'' + au) v \, dx = \int_0^1 f v \, dx$$

$$-\int_0^1 u'' v \, dx + \int_0^1 a u v \, dx = \int_0^1 f v \, dx$$

$$-\int_0^1 u'' v \, dx = -u'v \Big]_0^1 + \int_0^1 u'v' \, dx =$$

$$-u'(1) v(1) + u'(0) v(0) + \int_0^1 u'v' \, dx$$

$$u'(1) = \frac{\partial u(1)}{\partial \vec{n}} = \alpha (g_b - u(1)) \quad v(0) = 0$$

$$-\alpha (g_b - u(1)) v(1) + \int_0^1 u'v' \, dx + \int_0^1 a u v \, dx = \int_0^1 f v \, dx$$

$$\alpha u(1) v(1) + \int_0^1 u'v' \, dx + \int_0^1 a u v \, dx = \int_0^1 \cancel{fvdx} + \alpha g_b v(1)$$

$$a(u,v) = \int_0^1 u'v' \, dx + \int_0^1 a u v \, dx + \alpha u(1) v(1)$$

$$\langle F, v \rangle = \int_0^1 \cancel{au}\, f v \, dx + \alpha g_b v(1)$$

$$u_h(x) = \sum_{j=0}^{n} u_j \varphi_j(x), \quad v_h = \sum_{i=0}^{n} v_i \varphi_i(x)$$

$$a(u_h, v_h) = \sum_j \sum_i \cancel{a(\varphi_j, \varphi_i)}\, u_j v_i \, a(\varphi_j \varphi_i)$$

$$= \sum_j \sum_i u_j v_i \left(\int_0^1 \varphi_j' \varphi_i' \, dx + \int_0^1 a \varphi_j \varphi_i \, dx + \alpha \varphi_j(1) \varphi_i(1) \right) =$$

$$\langle F, v_h \rangle = \sum_i v_i \langle F, \varphi_i \rangle = \sum_i v_i \left(\int_0^1 f \varphi_i \, dx + \alpha g_b \varphi_i(1) \right)$$

$\Rightarrow v^T K v = v^T f$

as in lecture

$Kv = f$

$K_{ij} = a(\varphi_j, \varphi_i) = \int_0^1 \varphi_j' \varphi_i' \, dx + \int_0^1 a \varphi_j \varphi_i \, dx + \alpha \varphi_j(1) \varphi_i(1)$

Case 1 $|i-j| \geq 2 \rightarrow \nexists z$ s.t $\varphi_j(z)\varphi_i(z) \neq 0$

$\varphi_j'(z)\varphi_i'(z) \neq 0$

Case 2 $i=j$

$\varphi_i(x) = \begin{cases} (x - x_{i-1})n & x \in [x_{i-1}, x) \\ (x_{i+1} - x)n & x \in [x_i, x_{i+1}) \\ 0 & \text{else} \end{cases}$

$\varphi_i'(x) = \begin{cases} n & x \in (x_{i-1}, x_i) \\ -n & x \in (x_i, x_{i+1}) \\ 0 & \text{else} \end{cases}$

2a) $i = j = 0$

$K_{00} = \int_0^1 (\varphi_0')^2 \, dx + \int_0^1 a (\varphi_0)^2 \, dx + 0, \quad \varphi_0(1) = 0$

$= \int_0^{x_1 = \frac{1}{n}} n^2 \, dx + \int_0^{\frac{1}{n}} a\left(\left(\frac{1}{n} - x\right)n\right)^2 dx$

$= n + a\int_0^{\frac{1}{n}} 1 - 2xn + x^2 n^2 = n + a\left(\frac{1}{n} - \frac{1}{n^2} n + \frac{1}{3n^3} n^2\right)$

$= n + \frac{a}{3n}$

2b) $i = j \quad i,j \in \{1 \ldots n-1\}$

$K_{ii} = \int_0^1 (\varphi_i')^2 dx + \int_0^1 a (\varphi_i)^2 dx + 0, \quad \varphi_i(1) = 0 \quad \forall i < n$

$= \int_{x_{i-1}}^{x_i} n^2 dx + \int_{x_i}^{x_{i+1}} n^2 dx + a\int_{x_{i-1}}^{x_i} \left(x - \frac{i-1}{n}\right)^2 n^2 dx + a\int_{x_i}^{x_{i+1}} \left(\frac{i+1}{n} - x\right)^2 n^2 dx$

substitute $= x - x_i$ leads to same Integral as in 2a)

$= 2n + \frac{2a}{3n}$

2c) $i = j = n$

$K_{nn} = \int_0^1 (\varphi_n')^2 dx + \int_0^1 a (\varphi_n)^2 dx + \alpha(\varphi_n(1))^2$

$= \int_{x_{n-1} = \frac{n-1}{n}}^1 n^2 \, dx + a\int_{\frac{n-1}{n}}^1 \left(\left(x - \frac{n-1}{n}\right)n\right)^2 dx + \alpha$

$v = x - \frac{n-1}{n}$

$$\int_{\frac{n-1}{n}}^{1} \left(\left(x - \frac{n-1}{n}\right)n\right)^2 dx = \int_0^{\frac{1}{n}} v^2 n^2 \, dv = \frac{1}{3n}$$

$$= n + \frac{a}{3n} + \alpha$$

Case 3 $|i-j| = 1$ $\quad j = i+1$ $\quad$ ($i = j+1$ is symmetric)

$$K_{i\,i+1} = \int_0^1 \varphi'_{i+1}\varphi'_i \, dx + a\int_0^1 \varphi_{i+1}\varphi_i \, dx + 0 \quad \left(\varphi_i(1) \overset{i<N}{=} 0\right)$$

$$= \int_{x_i}^{x_{i+1}} -n^2 \, dx + a \int_{x_i}^{x_{i+1}} (x_{i+1} - x)(x - x_i) n^2 \, dx$$

$v = x - x_i$

$$= -n + n^2 a \int_0^{\frac{1}{n}} \frac{v}{n} - v^2 \, dv$$

$$= -n + a\left(\frac{1}{2n^3} - \frac{1}{3n^3}\right)n^2 = -n + \frac{a}{6n^3}n^2 = -n + \frac{a}{6n}$$

$$\left(\begin{array}{cccc|c} d & s & 0 & 0 & f_1 \\ s & 2d & s & 0 & f_2 \\ 0 & s & 2d & s & f_3 \\ 0 & 0 & s & d+\alpha & f_4 + \alpha g \end{array}\right)$$

$$\left(\begin{array}{cccc|c} d & s & 0 & 0 & f_1 \\ s - \frac{2d^2}{s} & 0 & s & 0 & f_2 - \frac{f_1 d}{2} \\ -d & 0 & 2d & s & f_3 - f_1 \\ 0 & 0 & s & d+\alpha & \end{array}\right)$$

$$K = \begin{pmatrix} d & s & & \\ s & 2d & \ddots & 0 \\ & \ddots & \ddots & \\ 0 & & 2d & s \\ & & s & d+\alpha \end{pmatrix}$$

$d = n + \frac{a}{3n}$

$s = -n + \frac{a}{6n}$

Incorporating $v(0) = 0$

$$\rightarrow K = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ s & 2d & s & & \\ & s & 2d & \ddots & \\ & & \ddots & 2d & s \\ & & & s & d+\alpha \end{pmatrix}$$

for calculation use $f(x) = 1$ and $g_b = 1$

$$\cancel{f_i = \int_0^1 \varphi_i \, dx}$$

$f_0 = 0$ for b.c

$$f_i = \int_0^1 \varphi_i \, dx + \alpha \varphi_i(1)$$

for $i \in \{1 \ldots n-1\}$ ~~0 will be skipped anyways due to BC~~

$$f_i = \int_{x_{i-1}}^{x_i} n(x - x_{i-1}) \, dx + \int_{x_i}^{x_{i+1}} n(x_{i+1} - x) \, dx = \frac{1}{n}$$

$$f_n = \int_{\frac{n-1}{n}}^{n} n\left(x - \frac{n-1}{n}\right) dx + \alpha g_b = \frac{1}{2n} + \alpha g_b$$

see code