

$$A) -u'' + \alpha u = f / \int_0^1 v(x)$$

$$+ \int_0^1 (-u'' + \alpha u) v dx = \int_0^1 f v dx$$

$$- \int_0^1 u'' v dx + \int_0^1 \alpha u v dx = \int_0^1 f v dx$$

$$- \int_0^1 u'' v dx = -u' v \Big|_0^1 + \int_0^1 u' v' dx =$$

$$- u'(1) v(1) + u'(0) v(0) + \int_0^1 u' v' dx$$

$$u'(1) = \frac{\partial u(1)}{\partial \vec{n}} = \alpha(g_b - u(1)) \quad v(0) = 0$$

$$- \alpha(g_b - u(1)) v(1) + \int_0^1 u' v' dx + \int_0^1 \alpha u v dx = \int_0^1 f v dx$$

$$\alpha u(1) v(1) + \int_0^1 u' v' dx + \int_0^1 \alpha u v dx = \int_0^1 f v dx + \alpha g_b v(1)$$

$$a(u, v) = \int_0^1 u' v' dx + \int_0^1 \alpha u v dx + \alpha u(1) v(1)$$

$$\langle F, v \rangle = \int_0^1 f v dx + \alpha g_b v(1)$$

$$u_h(x) = \sum_{j=0}^n u_j \varphi_j(x), \quad v_h = \sum_{i=0}^n v_i \varphi_i(x)$$

$$a(u_h, v_h) = \sum_j \sum_i u_j v_i a(\varphi_j, \varphi_i)$$

$$= \sum_j \sum_i u_j v_i \left(\int_0^1 \varphi_j' \varphi_i' dx + \int_0^1 \alpha \varphi_j \varphi_i dx + \alpha \varphi_j(1) \varphi_i(1) \right) =$$

$$\langle F, v_h \rangle = \sum_i v_i \langle F, \varphi_i \rangle = \sum_i v_i \left(\int_0^1 f \varphi_i dx + \alpha g_b \varphi_i(1) \right)$$

$$\Rightarrow v^T K v = v^T f$$

as in lecture

$$K v = f$$

$$K_{ij} = a(\varphi_j, \varphi_i) = \int_0^1 \varphi_j' \varphi_i' dx + \int_0^1 a \varphi_j \varphi_i dx + \alpha \varphi_j(1) \varphi_i(1)$$

Case 1 $|i-j| \geq 2 \rightarrow \nexists z \text{ s.t. } \varphi_j(z) \varphi_i(z) \neq 0$
 $\varphi_j'(z) \varphi_i'(z) = 0$

Case 2 $i=j$

2a) $i=j=00$

$$\varphi_0(x) = \begin{cases} (x-x_{i-1})n & x \in [x_{i-1}, x_i) \\ (x_i-x)n & x \in [x_i, x_{i+1}) \\ 0 & \text{else} \end{cases}$$

$$\varphi_0'(x) = \begin{cases} n & x \in (x_{i-1}, x_i) \\ -n & x \in (x_i, x_{i+1}) \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} K_{00} &= \int_0^1 (\varphi_0')^2 dx + \int_0^1 a(\varphi_0)^2 dx + 0, \varphi_0(1)=0 \\ &= \int_0^{\frac{1}{n}} n^2 dx + \int_0^{\frac{1}{n}} a \left(\left(\frac{1}{n} - x \right) n \right)^2 dx \\ &= n + \int_0^{\frac{1}{n}} 1 - 2xn + x^2 n^2 dx = n + a \left(\frac{1}{n} - \frac{1}{n^2} n + \frac{1}{3n^3} n^2 \right) \\ &= n + \frac{a}{3n} \end{aligned}$$

2b) $i=j, i, j \in \{1, \dots, n-1\}$

$$\begin{aligned} K_{ii} &= \int_0^1 (\varphi_i')^2 dx + \int_0^1 a(\varphi_i)^2 dx + 0, \varphi_i(1)=0 \quad \forall i < n \\ &= \int_{x_{i-1}}^{x_i} n^2 dx + \int_{x_i}^{x_{i+1}} n^2 dx + a \int_{x_{i-1}}^{x_i} \left(x - \frac{i-1}{n} \right)^2 n^2 dx + a \int_{x_i}^{x_{i+1}} \left(\frac{i+1}{n} - x \right)^2 n^2 dx \\ &= 2n + \frac{2a}{3n} \quad \checkmark \end{aligned}$$

substitute $u = x - x_i$ leads to same integral as in 2a)

2c) $i=j=n$

$$\begin{aligned} K_{nn} &= \int_0^1 (\varphi_n')^2 dx + \int_0^1 a(\varphi_n)^2 dx + \alpha (\varphi_n(1))^2 \\ &= \int_{x_{n-1}}^1 n^2 dx + a \int_{\frac{n-1}{n}}^1 \left(\left(x - \frac{n-1}{n} \right) n \right)^2 dx + \alpha \end{aligned}$$

$$u = x - \frac{n-1}{n}$$

$$\int_{\frac{n-1}{n}}^1 \left(x - \frac{n-1}{n}\right)^2 dx = \int_0^1 u^2 n^2 du = \frac{1}{3n}$$

$$= n + \frac{a}{3n} + \alpha$$

Case 3 $|i-j|=1$ $j=i+1$ ($i=j+1$ is symmetric)

$$K_{i,i+1} = \int_0^1 \psi'_{i+1} \psi'_i dx + a \int_0^1 \psi_{i+1} \psi_i dx + 0 \quad (\psi_i \notin 1) = 0$$

$$= \int_{x_i}^{x_{i+1}} -n^2 dx + a \int_{x_i}^{x_{i+1}} (x_{i+1} - x)(x - x_i) n^2 dx$$

$$= -n + n^2 \int_0^{\frac{1}{n}} \frac{u}{n} - u^2 du$$

$$= -n + a \left(\frac{1}{2n^3} - \frac{1}{3n^3} \right) n^2 = -n + \frac{a}{6n^3} n^2 = -n + \frac{a}{6n}$$

$$\begin{pmatrix} d & s & 0 & 0 \\ s & 2d & s & 0 \\ 0 & s & 2d & s \\ 0 & 0 & s & d+\alpha \end{pmatrix} \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 + \alpha g \end{matrix}$$

$$\begin{pmatrix} d & s & 0 & 0 \\ s - \frac{2d^2}{s} & 0 & s & 0 \\ -d & 0 & 2d & s \\ 0 & 0 & s & d+\alpha \end{pmatrix} \begin{matrix} f_1 \\ f_2 - \frac{f_1 d}{s} \\ f_3 - f_1 \\ f_4 + \alpha g \end{matrix}$$

$$K = \begin{pmatrix} d & s & & 0 \\ s & 2d & & 0 \\ & & \ddots & \\ 0 & & & 2d & s \\ & & & s & d+\alpha \end{pmatrix}$$

$$d = \frac{a}{3n} n + \frac{a}{3n}$$

$$s = -n + \frac{a}{6n}$$

Incorporating $u(0)=0$

$$\Rightarrow K = \begin{pmatrix} 1 & 0 & \dots & 0 \\ s & 2d & s & \\ & s & 2d & \\ & & & \ddots & 2d & s \\ & & & & s & d+\alpha \end{pmatrix}$$

for calculation use $f(x) = 1$ and $g_b = 1$

$$f_i = \int_0^1 \varphi_i dx$$

$$f_0 = 0 \quad \text{for b.c}$$

$$f_i = \int_0^1 \varphi_i dx + \alpha \varphi_i(1)$$

for $i \in \{1, \dots, n-1\}$

~~= 0 will be skipped anyway due to B.C~~

$$f_i = \int_{x_{i-1}}^{x_i} n(x - x_{i-1}) dx + \int_{x_i}^{x_{i+1}} n(x_{i+1} - x) dx = \frac{1}{n}$$

$$f_n = \int_{\frac{n-1}{n}}^1 n(x - \frac{n-1}{n}) dx + \alpha g_b = \frac{1}{2n} + \alpha g_b$$

see code