

A)

stationary heat equation (see Methode der finiten Elemente für Ingenieure)

$$(-\lambda u')' + \bar{q} u = f + \bar{q} u_A$$

$\bar{q}$ heat flux at the borders = 0 (case no convection and constant environment temp.)

$$(-\lambda u')' = f$$

In the strong setting u has to be in $C^2(0,1) \cap C[0,1]$ ← boundary cond

f has to be in $C(0,1)$ und $\lambda \in C^1(0,1)$

In the weak setting we can multiply the DE with φ , test func.

$$v \in H^1(0,1)$$

$$v(0)=v(1)=0$$

$$H^1(a,b) = \{u \in L_2(a,b) \mid \exists v \in L_2(a,b) : \int_{(a,b)} u w' dx = - \int_{(a,b)} w v dx\}$$

$$\forall w \in C_1(a,b) \\ w(a)=w(b)=0$$

and integrate

$$\int_0^1 (-\lambda u')' v dx = \int_0^1 f v dx$$

partial integrate

$$-\left[\lambda u' v\right]_0^1 + \int_0^1 \lambda u' v' dx = \int_0^1 f v dx$$

$$\int_0^1 \lambda u' v' dx = \int_0^1 f v dx$$

$$u \in H^1(0,1) \quad u(0)=0, u(1)=1$$

since $\int_0^1 f v$ should exist $f \in L^2(0,1)$

since $u', v' \in L^2$ in order for $\int_0^1 \lambda u' v'$ to exist $\lambda \in L^\infty(a,b)$

$$f = c \quad \lambda = \text{constant}$$

$$-\lambda u'' = c \quad u(0)=0, u(1)=1$$

$$-\lambda (u'')' = 0 \quad u'(0)=0, u'(1)=1$$

$$-\lambda (u'')'' = c \quad u''(0)=0, u''(1)=0$$

$$-\lambda (u'')'' = 0$$

$$(u'')'' = 0$$

$$u'' = kx + d \xrightarrow{\text{b.c.}} u'' = x$$

$$-\lambda (u'')'' = c$$

$$(u'')'' = -\frac{c}{\lambda}$$

$$u'' = -\frac{c}{2\lambda} x^2 + bx + d \quad b, d \text{ integrable constants}$$

$$b, c \rightarrow d=0 \quad -\frac{c}{2\lambda} + b = 0 \Rightarrow b = \frac{c}{2\lambda}$$

$$u'' = -\frac{c}{2\lambda} x^2 + \frac{c}{2\lambda} x$$

$$u = -\frac{c}{2\lambda} x^3 + \left(\frac{c}{2\lambda} + 1\right)x \quad u(0)=0 \quad u(1)=1$$

$$-\lambda u'' = -\lambda \left(-\frac{c}{\lambda}\right) = c \quad \checkmark$$

[-1]

$$B \quad (-\lambda v')' + \bar{q}v = f + \bar{q}_1 v_A \quad ??$$

$$(-\lambda v')' = 0$$

$$-\lambda v'' = 0$$

$$v'' = 0 \quad \text{for } x \in (0, 0.5) \cup (0.5, 1) \quad v(0) = 0 \wedge v(1) = 1$$

Interface condition

$$\lim_{x \rightarrow 0.5^-} v(x) = \lim_{x \rightarrow 0.5^+} v(x)$$

temperature continues

$$\lim_{x \rightarrow 0.5^-} -\lambda v'(x) = \lim_{x \rightarrow 0.5^+} -\lambda v'(x)$$

heat flow continues

$$\lim_{x \rightarrow 0.5^-} -v'(x) = \lim_{x \rightarrow 0.5^+} -10v'(x)$$

$$v \in C^2(0, \frac{1}{2}) \cap C^2(\frac{1}{2}, 1) \cap C([0, 1]) \leftarrow \text{boundary / interface cond}$$

$$\lambda \in C^1(0, \frac{1}{2}) \cap C^1(\frac{1}{2}, 1)$$

$$f \in C(0, \frac{1}{2}) \cap C(\frac{1}{2}, 1)$$

$$v'' = 0 \quad \text{for } x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$$

$$v = \begin{cases} a_1 x + b_1 & x \in (0, 0.5) \\ a_2 x + b_2 & x \in (0.5, 1) \end{cases} \quad \begin{aligned} v(0) = 0 &\Rightarrow b_1 = 0 \\ v(1) = 1 &\Rightarrow a_2 + b_2 = 1 \\ &b_2 = 1 - a_2 \end{aligned}$$

$$\lim_{x \rightarrow 0.5^-} v(x) = a_1 \cdot 0.5 = a_2 \cdot 0.5 + 1 - a_2 = 1 - a_2 \cdot 0.5 = \lim_{x \rightarrow 0.5^+} v(x)$$

$$\lim_{x \rightarrow 0.5^-} -\lambda(x) v'(x) = -a_1 = -10a_2 = \lim_{x \rightarrow 0.5^+} -\lambda(x) v'(x)$$

$$\text{I: } \frac{1}{2} a_1 = -\frac{1}{2} a_2 + 1$$

$$\text{II: } -a_1 = -10a_2 \Rightarrow a_1 = 10a_2$$

$$\text{I: } 5a_2 = -\frac{1}{2} a_2 + 1$$

$$a_2 = \frac{2}{11} \Rightarrow a_1 = \frac{20}{11}$$

$$\hookrightarrow b_2 = \frac{9}{11}$$

$$v = \begin{cases} \frac{20}{11} x & x \in (0, \frac{1}{2}) \\ \frac{2}{11} x + \frac{9}{11} & x \in (\frac{1}{2}, 1) \end{cases} \quad v(0) = 0 \wedge v(1) = 1$$

$$\lim_{x \rightarrow 0.5^-} v(x) = \frac{20}{11} \cdot \frac{1}{2} = \frac{10}{11} = \frac{2}{11} \cdot \frac{1}{2} + \frac{9}{11} = \lim_{x \rightarrow 0.5^+} v(x)$$

$$\lim_{x \rightarrow 0.5^-} -\lambda(x) v'(x) = -\frac{20}{11} = -10 \cdot \frac{2}{11} = \lim_{x \rightarrow 0.5^+} -\lambda(x) v'(x)$$

c) ~~z~~ $z := u'$ $-z' + pz = 0$ ~~z(x) = 0~~

$z' = pz$ If $z=0$ Fixed point $\rightarrow u=c$ trivial sol ~~but doesn't fit bc~~

If $p=0 \rightarrow z'=0 \rightarrow z=c \rightarrow u=cx+d \xrightarrow{bc} u=x$

~~$z' = pz$~~

Else $z = Ke^{px}$ $K \in \mathbb{R}$

$u = \int z dx = \frac{K}{p} e^{px} + c$ ~~$u(0) = \frac{K}{p} + c = 0$~~

$u(0) = \frac{K}{p} + c$ $u(1) = \frac{K}{p} e^p + c$

$u(0) - u(1) = \frac{K}{p} (1 - e^p) = -1$

$\frac{K}{p} (e^p - 1) = 1$

$K = \frac{p}{e^p - 1}$

$u(0) = \frac{1}{e^p - 1} + c = 0$

$c = -\frac{1}{e^p - 1}$

$\rightarrow u(x) = \frac{1}{e^p - 1} e^{px} - \frac{1}{e^p - 1} = \frac{e^{px} - 1}{e^p - 1}$ ✓