Exercise 2 sheet: PDEs modelling and analysis

Lisa Pizzo

October 20, 2025

(A)

Given exercise
Model the stationary heat equation in the 1D domain $\Omega = (0,1)$ without convection and without advection. Assume Dirichlet boundary conditions $u(0) = 0$ and $u(1) = 1$ for the unknown solution u(x). Assume a constant heating $f(x) = c$ and a constant heat conductivity $\lambda(x)$ in the domain.

- Which function spaces are required for u, f and λ in that PDE?
- The solution can be expressed as $u(x) = u^H(x) + u^f(x)$ with $u^H(x)$ as homogeneous solution of the PDE with $f(x) = 0$ and $uf(x)$ as solution of the PDE with zero Dirichlet b.c. $u(0) = u(1) = 0$. Formulate both PDEs, determine $u^H(x)$ and $u^f(x)$ analytically and check correctness of $u(x)$ in the original PDE.

Solution:
1) Model
The stationary heat equation in one dimension is

$$-\big(\lambda(x)\, u'(x)\big)' = f(x), \qquad x \in (0,1),$$

with $u(0) = 0$, $u(1) = 1$. With constant λ and constant $f = c$, this becomes

$$-\lambda\, u''(x) = c, \qquad x \in (0,1), \qquad u(0) = 0,\ u(1) = 1.$$

Function spaces:

- $\lambda \in L^\infty(0,1)$ with $\lambda(x) \geq \lambda_{\min} > 0$,
- $f \in L^2(0,1)$,
- weak solution $u \in H^1(0,1)$ and the strong solution is $u \in C^2([0,1])$.

2) Decomposition $u = u_H + u_f$

Homogeneous problem:

$$-\lambda u_H''(x) = 0, \quad u_H(0) = 0,\ u_H(1) = 1.$$

Integrate twice:

$$u_H(x) = Ax + B.$$

Boundary conditions yield $B = 0$, $A = 1$. Hence

$$u_H(x) = x.$$

Particular problem:

$$-\lambda u_f''(x) = c, \quad u_f(0) = u_f(1) = 0.$$

Then $u_f''(x) = -\frac{c}{\lambda}$. Integrating twice:

$$u_f(x) = -\frac{c}{2\lambda}x^2 + Ax + B.$$

Boundary conditions give $B = 0$ and $A = \frac{c}{2\lambda}$. Thus

$$u_f(x) = \frac{c}{2\lambda}x(1-x).$$

Total solution:

$$u(x) = x + \frac{c}{2\lambda}x(1-x).$$

Verification.

$$u'(x) = 1 + \frac{c}{2\lambda}(1-2x), \quad u''(x) = -\frac{c}{\lambda}.$$

Then $-\lambda u''(x) = c$ and the boundary conditions hold.

(B)

Given exercise
Model the stationary heat equation in the 1D domain $\Omega = (0,1)$ without convection and without advection. Assume Dirichlet boundary conditions $u(0) = 0$ and $u(1) = 1$ for the unknown solution u(x). There are no internal source $f(x) = 0$ and we have heterogeneous material $\lambda(x) = \begin{cases} 1 & x \in (0, 0.5) \\ 10 & x \in (0.5, 1) \end{cases}$, i.e., the heat conductivity changes.

- Use the strong formulation of the PDE with interface conditions.
- Which function spaces are required for u, f and λ in that PDE?
- Determine the solution $u(x)$ analytically.

Solution: 1) Strong Formulation
In the two intervals we can consider λ to be constant and hence we end up with $-\lambda(x)u''(x) = 0$ and hence $u''(x) = 0$. We need to take care of the interface $x = \frac{1}{2}$, there we want

$$u(1/2_-) = u(1/2_+), \quad \lambda u'(1/2_-) = \lambda u'(1/2_+).$$

Function spaces:

- $\lambda(x)$ is a piecewise constant, hence $\lambda \in L^\infty(0,1)$ with $\lambda(x) \geq \lambda_{\min} > 0$.
- $f \equiv 0 \in L^2(0,1)$.
- Weak solution is $u \in H^1(0,1)$, $u(0) = 0$, $u(1) = 1$ and strong solution is $u \in C^0([0,1]) \cap C^2((0,0.5) \cup (0.5,1))$ with a continuous flux $\lambda u'$ across the interface.

2) Piecewise Solution

Let

$$u_1(x) = a_1 x + b_1, \quad x \in [0, 0.5], \qquad u_2(x) = a_2 x + b_2, \quad x \in [0.5, 1].$$

Boundary and interface conditions:

$$\begin{cases} u_1(0) = 0, \\ u_2(1) = 1, \\ u_1(0.5) = u_2(0.5), \\ a_1 = 10a_2. \end{cases}$$

$u_1(0) = 0 \rightarrow b_1 = 0$ hence $\frac{1}{2}a_1 = \frac{1}{2}a_2 + b_2 \rightarrow b_2 = \frac{1}{2}(a_1 - a_2)$ and together with $a_1 = 10a_2$ we have $b_2 = \frac{9}{2}a_2$.

$u_2(1) = 1$: $a_2 + b_2 = 1 \rightarrow \frac{11}{2}a_2 = 1 \rightarrow a_2 = \frac{2}{11} \rightarrow a_1 = \frac{20}{11}$. Hence we have $b_2 = \frac{9}{11}$.

Hence

$$u(x) = \begin{cases} \frac{20}{11}x, & 0 \le x \le 0.5, \\ \frac{2}{11}x + \frac{9}{11}, & 0.5 \le x \le 1. \end{cases}$$

Verification:
Continuity at $x = \frac{1}{2}$: both sides give $\frac{10}{11}$.
Derivative (flux): $1 \cdot \frac{20}{11} = 10 \cdot \frac{2}{11} = \frac{20}{11}$.
Boundary conditions: $u(0) = 0$, $u(1) = 1$.

(C)

Given exercise Solve the Peclet problem

$$-u''(x) + pu'(x) = 0, \quad x \in (0, 1),$$

with $u(0) = 0$, $u(1) = 1$, analytically with a constant $p \in \mathbb{R}$.

- Hint: Substitute $z(x) = u'(x)$, solve the ODE and incorporate the b.c..
- Visualize (Matlab, python,...) your solution for an increasing $p > 0$ as well as for a decreasing $p < 0$.

Solution: 1) Analytical Solution Let $z(x) = u'(x)$. Then

$$-z'(x) + pz(x) = 0 \to z'(x) = pz(x) \to z(x) = Ce^{px}.$$

Integrate, with $p \neq 0$

$$u(x) = \frac{C}{p}e^{px} + D.$$

$u(0) = 0 \to \frac{C}{p} + D = 0 \to D = -\frac{C}{p}$. $u(1) = 1 \to \frac{C}{p}(e^p - 1) = 1 \to \frac{C}{p} = \frac{1}{e^p-1}$. Hence $u(x) = \frac{e^{px}-1}{e^p-1}$.

If $p = 0$: The equation is $-u'' = 0 \to u(x) = Ax + B$. $u(0) = 0$, $u(1) = 1 \to u(x) = x$. Limit as $p \to 0$ gives x as expected. Hence

$$u(x) = \begin{cases} \dfrac{e^{px} - 1}{e^p - 1} & p \neq 0, \\ x & p = 0. \end{cases}$$

3) Visualization in Matlab

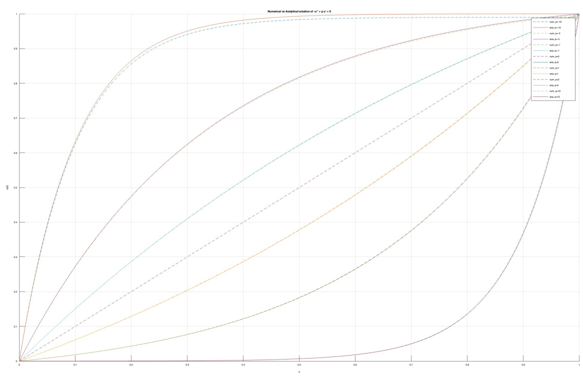

Figure 1: Solution of $-u'' + p\,u' = 0$ with $u(0) = 0$, $u(1) = 1$ for various p values.