Exercise 4: PDEs in 1D and FEM

Lisa Pizzo

(A)

Given problem: Consider the PDE

$$-u''(x) + a \cdot u(x) = f(x), \quad x \in (0,1) = \Omega,$$

with boundary conditions

$$u(0) = 0, \qquad \frac{\partial u(1)}{\partial \vec{n}} = \alpha\big(g_B - u(1)\big),$$

where $a, \alpha, g_B \in \mathbb{R}$ are constants.

- Write the variational formulation of that PDE, define $a(\cdot,\cdot)$ and $\langle F, \cdot \rangle$.
- Write down the FEM representation using an equidistant discretization of the computational domain Ω and linear shape function in each of the n elements.
- Compute the elements of the stiffness matrix.
- Solve the system of equations.

Weak Formulation
We define the trial and test space

$$V = \{v \in H^1(0,1) : v(0) = 0\}.$$

Multiply the PDE by a test function $v \in V$ and integrate:

$$\int_0^1 (-u'')v\,dx + \int_0^1 auv\,dx = \int_0^1 fv\,dx.$$

Integrating the first term by parts and inserting the boundary data yields:

$$\int_0^1 u'v'\,dx + \int_0^1 auv\,dx + \alpha u(1)v(1) = \int_0^1 fv\,dx + \alpha g_B v(1).$$

Define the **bilinear form and linear functional**

$$a(u,v) = \int_0^1 u'v'\,dx + \int_0^1 auv\,dx + \alpha u(1)v(1),$$

$$\langle F, v\rangle = \int_0^1 fv\,dx + \alpha g_B v(1).$$

FEM Discretization

Let the interval be discretized uniformly with nodes $x_i = ih$ with $h = 1/n$. Then using the standard linear basis functions $\{\varphi_1, \ldots, \varphi_n\}$, with which we then have $u_h(x) = \sum_{j=1}^{n} u_j \varphi_j(x)$, stiffness matrix

$$A_{ij} = a(\varphi_j, \varphi_i)$$

and the right hand side vector

$$F_i = \langle F, \varphi_i \rangle.$$

Element of the stiffness matrix

In order to properly compute the stiffness matrix we separate the two parts of the bilinear form $a(u, v)$. Hence, on one element $[x_{k-1}, x_k]$, we have

$$K^e_{\text{diff}} = \int_{x_{k-1}}^{x_k} \varphi_i' \varphi_j' dx = \frac{1}{h} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

✓

$$K^e_{\text{react}} = a \int_{x_{k-1}}^{x_k} \varphi_i \varphi_j dx = a \frac{h}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Thus local stiffness matrix:

$$A^e = K^e_{\text{diff}} + K^e_{\text{react}}.$$

The global stiffness matrix A is built by adding overlapping contributions from all elements.

The local element level load vector corresponding to a constant source f on the element $[x_{k-1}, x_k]$ is F^e,

$$F^e = \int_{x_{k-1}}^{x_k} f \begin{pmatrix} \varphi_{k-1} \\ \varphi_k \end{pmatrix} dx = f \frac{h}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

✓

We can now modify the last entry to account for the Robin BC

$$A_{nn} = A_{nn} + \alpha$$

and

$$F_n = F_n + \alpha g_B.$$

Analytical solution

We firstly solve the homogeneous equation $-u'' + au = 0$, and the solution is

$$u_h(x) = C_1 e^{\sqrt{a}x} + C_2 e^{-\sqrt{a}x}, \quad a > 0.$$

To this we then apply Dirichlet boundary conditions at $x = 0$ and hence

$$u_h(0) = C_1 + C_2 = 0 \Rightarrow u_h(x) = C_1(e^{\sqrt{a}x} - e^{-\sqrt{a}x}) = C_1 \sinh(\sqrt{a}x).$$

Then we solve the particular solution for constant $f(x) = f_0$, the solution will then be

$$u_p(x) = \frac{f_0}{a}(1 - e^{-\sqrt{a}x}).$$

Hence the general solution

$$\boxed{u(x) = C_1 \sinh(\sqrt{a}x) + \frac{f_0}{a}(1 - e^{-\sqrt{a}x}).}$$

✓

And now we can apply Robin boundary conditions at $x = 1$, $u'(1) = \alpha(g_B - u(1))$.

$$u'(x) = C_1 \sqrt{a} \cosh(\sqrt{a}x),$$

and hence, plugging in $x = 1$, we obtain

$$C_1\sqrt{a}\cosh(\sqrt{a}) = \alpha(g_B - \big(C_1\sinh(\sqrt{a}) + \frac{f_0}{a})\big).$$

This leads to

$$C_1 = \frac{\alpha(g_B - \frac{f_0}{a})}{\alpha\sinh(\sqrt{a}) + \sqrt{a}\cosh(\sqrt{a})}$$

Visualization in Matlab
I implemented both the numerical and analytical solutions in MATLAB, and the figure below compares their results.

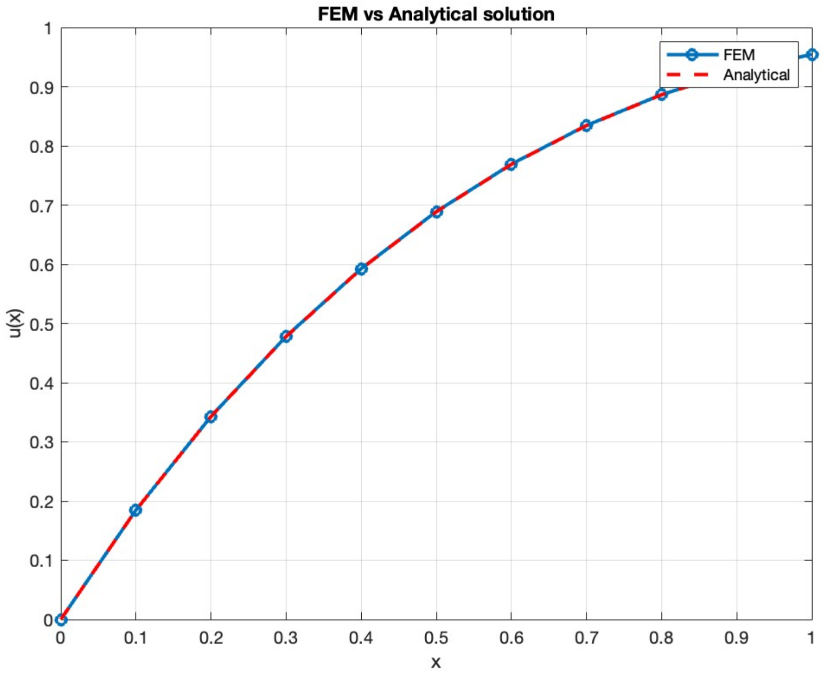

Figure 1: FEM vs Analytical solution

(B)

Given problem: Consider the PDE

$$-\big(\lambda(x)\cdot u'(x)\big)' = 0, \quad x \in (0,1) = \Omega, \quad u(0) = 0,\ u(1) = 1,$$

with

$$\lambda(x) = \begin{cases} 1, & x \in \left(0, \frac{1}{\sqrt{2}}\right), \\ 10, & x \in \left(\frac{1}{\sqrt{2}}, 1\right). \end{cases}$$

- Write the variational formulation and the FEM system.
- Solve the system of equations.

Variational Formulation

Test space with homogeneous Dirichlet conditions

$$V = H_0^1(0,1) = \{v \in H^1(0,1) \,:\, v(0) = v(1) = 0\}.$$

Multiply the PDE by $v \in V$ and integrate:

$$\int_0^1 -(\lambda u')' v \, dx = 0,$$

then integrate by parts

$$\int_0^1 \lambda(x) u'(x) v'(x) \, dx = 0 \qquad \forall v \in V.$$

Since $u \notin V$, we handle the non homogeneous boundary conditions by writing

$$u = u_D + w,$$

where $w \in V$ and u_D satisfies $u_D(0) = 0$ and $u_D(1) = 1$. A convenient choice is $u_D(x) = x$. Inserting $u = u_D + w$ into the weak form gives:

$$\int_0^1 \lambda(u_D' + w')v' \, dx = 0.$$

Thus the variational formulation for w is:

$$\text{Find } w \in V: \quad a(w,v) = F(v) \quad \forall v \in V,$$

with

$$a(w,v) = \int_0^1 \lambda \, w' v' \, dx, \qquad F(v) = -\int_0^1 \lambda \, u_D' \, v' \, dx.$$

Since $u_D'(x) = 1$, the right-hand side simplifies to

$$F(v) = -\int_0^1 \lambda \, v' \, dx.$$

FEM Discretization

Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a uniform grid with $h = 1/n$. The standard hat basis $\{\varphi_0, \ldots, \varphi_n\}$ satisfies $\varphi_i(x_j) = \delta_{ij}$.

Since $w(0) = w(1) = 0$, we approximate w in

$$V_h = \operatorname{span}\{\varphi_1, \ldots, \varphi_{n-1}\}.$$

We write

$$w_h(x) = \sum_{j=1}^{n-1} w_j \varphi_j(x).$$

The discrete system is

$$Aw = F,$$

with entries

$$A_{ij} = \int_0^1 \lambda\, \varphi_j' \varphi_i'\, dx, \qquad F_i = -\int_0^1 \lambda\, \varphi_i'\, dx.$$

Local Element Matrices

Consider an element $[x_{k-1}, x_k]$ of length h. The derivatives are constant:

$$\varphi_{k-1}' = -\frac{1}{h}, \qquad \varphi_k' = \frac{1}{h}.$$

Case A: The element is entirely inside a region where λ is constant.

For $\lambda(x) = \lambda$ (either 1 or 10), the local stiffness matrix is

$$A = \lambda \frac{1}{h} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The local load vector (using $u_D' = 1$) is:

$$F = -\lambda \begin{pmatrix} \int_{x_{k-1}}^{x_k} \varphi_{k-1}'\, dx \\ \int_{x_{k-1}}^{x_k} \varphi_k'\, dx \end{pmatrix} = \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix}.$$

Case B: The jump point $x_0 = \frac{1}{\sqrt{2}}$ lies *inside* the element.

Let

$$h_1 = x_0 - x_{k-1}, \qquad h_2 = x_k - x_0, \qquad h_1 + h_2 = h.$$

Split integrals:

$$A = \int_{x_{k-1}}^{x_0} 1 \cdot (\dots)\, dx + \int_{x_0}^{x_k} 10 \cdot (\dots)\, dx.$$

Compute:

$$A = \frac{1}{h^2} \begin{pmatrix} h_1 + 10h_2 & -(h_1 + 10h_2) \\ -(h_1 + 10h_2) & h_1 + 10h_2 \end{pmatrix}.$$

Local load contributions:

$$F = \begin{pmatrix} \dfrac{h_1 + 10h_2}{h} \\ -\dfrac{h_1 + 10h_2}{h} \end{pmatrix}.$$

Assembly and Final System

Assemble all local contributions into the global stiffness matrix A and load vector F. Solve the $(n-1) \times (n-1)$ linear system

$$Aw = F.$$

The final FEM approximation is

$$u_h(x_i) = u_D(x_i) + w_h(x_i) = x_i + w_i, \qquad i = 0, \dots, n.$$

Exact Solution (for verification)

From $-(\lambda u')' = 0$ we get $\lambda(x) u'(x) = J$ for some constant flux J. Thus

$$u'(x) = \frac{J}{\lambda(x)}.$$

Let $x_0 = \frac{1}{\sqrt{2}}$. For $0 \le x \le x_0$, $\lambda = 1$:

$$u(x) = Jx.$$

For $x_0 \le x \le 1$, $\lambda = 10$:

$$u(x) = J\Big(x_0 + \frac{x - x_0}{10}\Big).$$

Use $u(1) = 1$ to determine J:

$$1 = J\left(x_0 + \frac{1 - x_0}{10}\right) \quad \Rightarrow \quad J = \frac{1}{x_0 + \frac{1 - x_0}{10}}.$$

Thus the exact solution is

$$\boxed{u(x) = \begin{cases} Jx, & 0 \le x \le x_0, \\ J\left(x_0 + \dfrac{x - x_0}{10}\right), & x_0 \le x \le 1. \end{cases}}$$

✓

This provides a direct reference for validating the FEM implementation.

Visualization in Matlab

I implemented both the numerical and analytical solutions in MATLAB, and the figure below compares their results.

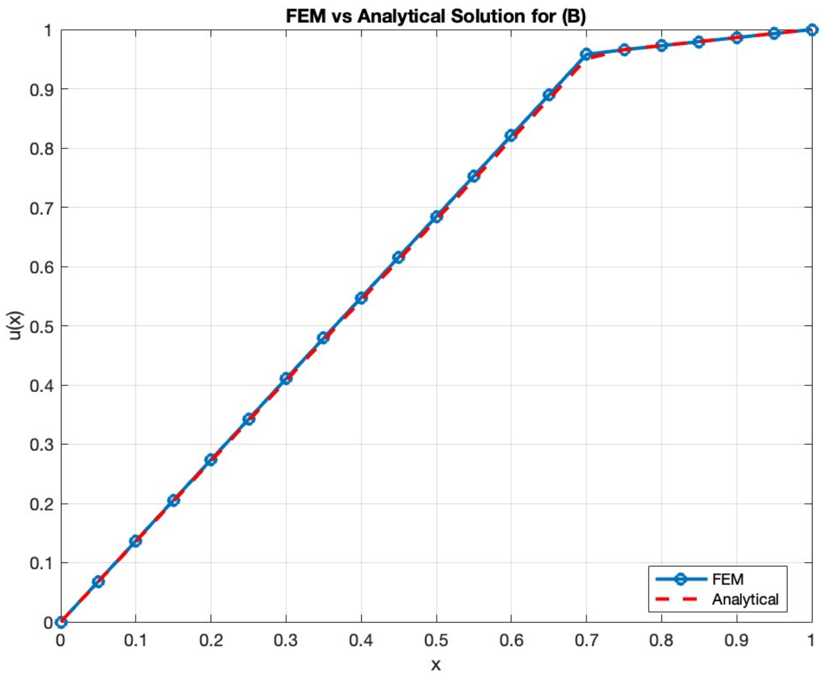

Figure 2: FEM vs Analytical solution

(C)

Given problem: Solve the Péclet problem

$$-u''(x) + p\,u'(x) = 0, \quad x \in (0,1), \quad u(0) = 0,\ u(1) = 1,$$

with FEM for a constant $p \in \mathbb{R}$.

- Write the variational formulation and the FEM system.
- Solve the system of equations with $p = 70$ and $n \in \{10, 20, 30, 40, 70\}$. Explain the behavior of the discrete solution u_h.

Variational formulation
Multiply the PDE by a test function $v \in H_0^1(0,1)$ and integrate:

$$\int_0^1 \big(-u''(x) + p\,u'(x)\big)v(x)\,dx = 0.$$

Integrate the second derivative term by parts:

$$\int_0^1 -u''v = \int_0^1 u'v'.$$

The boundary term vanishes because $v(0) = v(1) = 0$.
Thus the weak problem is:

$$\boxed{\text{Find } u \in H_0^1(0,1) \text{ such that } \int_0^1 u'(x)v'(x)\,dx + p\int_0^1 u'(x)v(x)\,dx = 0, \quad \forall v \in H_0^1(0,1).}$$

This gives the bilinear form:

$$a(u,v) = \int_0^1 u'v' + p\int_0^1 u'v.$$

FEM discretization
Let the mesh be uniform with nodes $x_i = ih$, $h = \frac{1}{n}$, and standard hat functions φ_i.
The discrete problem is:

$$\boxed{\text{Find } u_h \in V_h \subset H_0^1(0,1) \text{ such that } a(u_h, \varphi_j) = 0, \quad j = 1, \dots, n-1.}$$

We assemble the stiffness matrix:

$$A_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)\,dx \ + \ p\int_0^1 \varphi_i'(x)\varphi_j(x)\,dx.$$

For a uniform mesh and linear elements:

$$\int_{x_{i-1}}^{x_i} \varphi_i'\varphi_i' = \frac{1}{h}, \qquad \int_{x_{i-1}}^{x_i} \varphi_i'\varphi_{i-1}' = -\frac{1}{h},$$
$$\int_{x_{i-1}}^{x_i} \varphi_i'\varphi_i = \frac{1}{2}, \qquad \int_{x_{i-1}}^{x_i} \varphi_i'\varphi_{i-1} = -\frac{1}{2}.$$

Where the first two are used for the first term of the bilinear form a, and the second two for the second term of the bilinear form.
Thus each element contributes:

$$K = \frac{1}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad C = \frac{p}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Hence, the element matrix $A = K + C$ is

$$A = \begin{pmatrix} \frac{1}{h} - \frac{p}{2} & -\frac{1}{h} - \frac{p}{2} \\ -\frac{1}{h} + \frac{p}{2} & \frac{1}{h} + \frac{p}{2} \end{pmatrix}.$$

Now, we know that every interior node i receives contributions from the two elements touching it, meaning element $[x_{i-1}, x_i]$ and element $[x_i, x_{i+1}]$. Hence the global matrix is so assembled:
Global diagonal: each nodes gets K_{11} from the right element and K_{22} from the left element,

$$\boxed{A_{i,i} = \frac{1}{h} - \frac{p}{2} + \frac{1}{h} + \frac{p}{2} = \frac{2}{h}}$$

Global left off diagonal: contribution from element $[x_{i-1}, x_i]$

$$\boxed{A_{i,i-1} = -\frac{1}{h} - \frac{p}{2},}$$

Global right off diagonal: contribution from element $[x_i, x_{i+1}]$

$$\boxed{A_{i,i+1} = -\frac{1}{h} + \frac{p}{2},}$$

✓

The **load vector is zero**, and the **boundary condition** $u(1) = 1$ introduces:

$$b_{n-1} = \left(\frac{1}{h} - \frac{p}{2}\right) \cdot 1.$$

So the FEM linear system is:

$$A\mathbf{u}_h = \mathbf{b}.$$

Numerical results for $p = 70$
We solve the system for:

$$n \in \{10, 20, 30, 40, 70\}, \qquad h = \frac{1}{n}.$$

The exact solution is:

$$u(x) = \frac{e^{px} - 1}{e^p - 1}.$$

This function has a sharp boundary layer near $x = 1$ when p is large, such as $p = 70$.
Behavior of the discrete solution
For large p, the element Péclet number is:

$$\mathrm{Pe}_h = \frac{ph}{2}.$$

The Péclet number is the ratio between the two components of the bilinear form, often called diffusion and convection. This number gives us a clear way to understand what the FEM method produces.

- For small n (coarse mesh), h is large $\Rightarrow \mathrm{Pe}_h \gg 1$. The FEM solution shows: strong oscillations, loss of monotonicity, failure to capture the boundary layer. Hence it is very unstable. ✓

- As n increases: $\mathrm{Pe}_h = \frac{p}{2n}$ decreases.
  For $n = 70$:
  $$\mathrm{Pe}_h \approx \frac{70}{140} = 0.5,$$
  which is acceptable. The FEM solution becomes monotone and close to the exact exponential boundary layer.
- This demonstrates the classical issue: Standard FEM becomes unstable when convection dominates diffusion.

Hence, to capture the boundary layer without oscillations, one needs either:

$$h < \frac{2}{p} \quad \text{or stabilization.}$$

✓

Conclusion
The FEM system is well-defined, but standard linear FEM is unstable for large Péclet number. For $p = 70$, coarse meshes fail completely, while fine meshes (around $n \geq 70$) capture the exponential layer accurately.

Visualization in Matlab
I implemented both the numerical and analytical solutions in MATLAB, and the figure below compares their results.

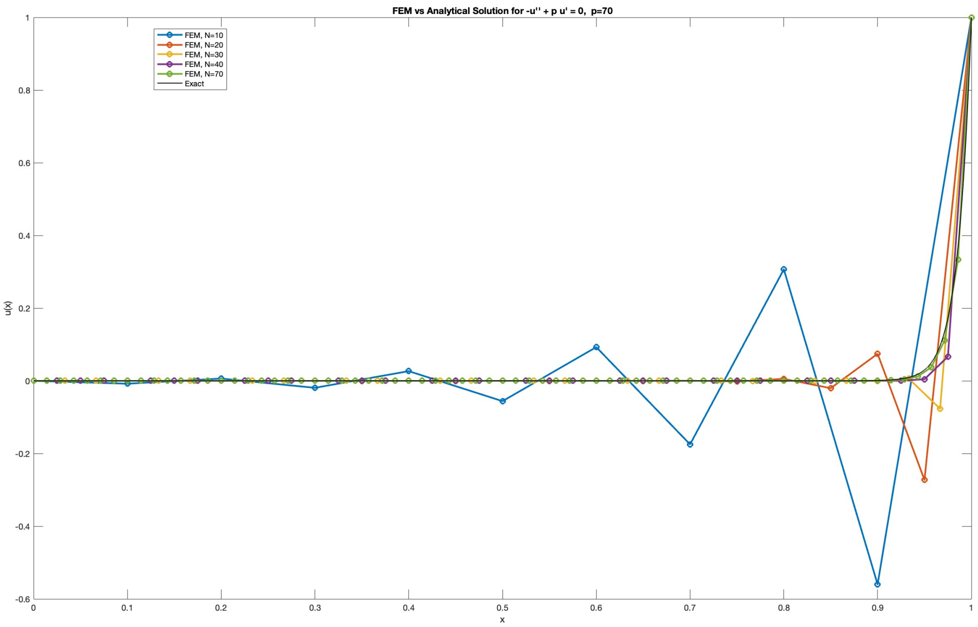

Figure 3: FEM vs Analytical solution

excellent!