```
%% FEM vs analytical solution for -u'' + a*u = f, mixed BC
clear; clc;
%% Parameters
n = 10;                     % number of elements
a = 2;                      % reaction term
alpha = 5;                  % Robin BC
gB = 1;                     % Robin BC value
f = 3;                      % constant RHS
h = 1/n;
x = linspace(0,1,n+1);

%% Assemble FEM matrices
A = zeros(n+1,n+1);
F = zeros(n+1,1);

% Linear shape functions, 1D FEM
K_diff = (1/h)*[1,-1;-1,1];
K_reac = a*(h/6)*[2,1;1,2];

for e=1:n
    nodes = [e, e+1];
    A(nodes,nodes) = A(nodes,nodes) + K_diff + K_reac;
    F(nodes) = F(nodes) + f*(h/2)*[1;1]; % f*v integral approx
end

% Dirichlet BC u(0)=0
A(1,:) = 0; A(:,1) = 0; A(1,1) = 1;
F(1) = 0;

% Robin BC at x=1
A(end,end) = A(end,end) + alpha;
F(end) = F(end) + alpha*gB;

%% Solve FEM
u_fem = A\F;

%% Analytical solution
syms C
C1 = (alpha*gB - (f/a)*alpha*(1 - exp(-sqrt(a))) - (f/sqrt(a))*exp(-sqrt(a))) ...
      / (alpha*sinh(sqrt(a)) + sqrt(a)*cosh(sqrt(a)));
u_exact = @(x) C1*sinh(sqrt(a)*x) + (f/a)*(1 - exp(-sqrt(a)*x));
u_an = u_exact(x);

%% Plot
plot(x,u_fem,'o-','LineWidth',2); hold on;
plot(x,u_an,'r--','LineWidth',2);
legend('FEM','Analytical');
```

```
xlabel('x'); ylabel('u(x)');
title('FEM vs Analytical solution');
grid on;
```

```
%% FEM vs analytical solution for -(lambda(x) u')' = 0 with Dirichlet BC
clear; clc;

%% Parameters
n = 20;                  % number of elements (increase for nicer plot)
h = 1/n;
x = linspace(0,1,n+1)'; % node coordinates

% Jump location
x0 = 1/sqrt(2);

% Piecewise lambda function
lambda_fun = @(x) (x < x0).*1 + (x >= x0).*10;

%% Assemble FEM matrix and RHS (lifting via u_D(x)=x)
A = zeros(n+1,n+1);
F = zeros(n+1,1);

% Derivatives of linear shape functions are constant on each element
phi_prime = [-1/h; 1/h];

for e = 1:n
    % element nodes
    nodes = [e, e+1];
    xL = x(nodes(1));
    xR = x(nodes(2));

    % CASE A: no jump inside the element
    if (xR <= x0) || (xL >= x0)
        lambda_e = lambda_fun((xL + xR)/2);  % constant lambda
        A_e = lambda_e * (1/h) * [1 -1; -1 1];
        % load due to u_D'(x)=1
        F_e = [ lambda_e; -lambda_e ];

    % CASE B: jump inside the element -> split integrals
    else
        h1 = x0 - xL;
        h2 = xR - x0;

        % Local stiffness contributions
        A_e = [ (1*h1 + 10*h2)/h^2 , -(1*h1 + 10*h2)/h^2 ;
               -(1*h1 + 10*h2)/h^2 ,  (1*h1 + 10*h2)/h^2 ];

        % load vector components
        F_e = [ (h1 + 10*h2)/h ;
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```
                    -(h1 + 10*h2)/h ];
    end

    % Assemble
    A(nodes,nodes) = A(nodes,nodes) + A_e;
    F(nodes) = F(nodes) + F_e;
end

%% Apply Dirichlet BC: u(0)=0, u(1)=1
% Dirichlet BC: w(0)=0, w(1)=0
A(1,:) = 0; A(:,1) = 0; A(1,1) = 1;
A(end,:) = 0; A(:,end) = 0; A(end,end) = 1;
F(1) = 0;
F(end) = 0;

%% Solve FEM
w = A \ F;
u_fem = x + w;

%% Analytical solution with piecewise Î»(x)
x0 = 1/sqrt(2);
J = 1 / (x0 + (1-x0)/10);

u_exact = @(x) (x <= x0) .* (J*x) + ...
               (x >  x0) .* ( J*x0 + (J/10)*(x - x0) );
u_an = u_exact(x);

%% Plot
plot(x,u_fem,'o-','LineWidth',2); hold on;
plot(x,u_an,'r--','LineWidth',2);
legend('FEM','Analytical','Location','Best');
xlabel('x'); ylabel('u(x)');
title('FEM vs Analytical Solution for (B)');
grid on;
```

```
clear; close all; clc;
p = 70;
Nvals = [10 20 30 40 70];

figure; hold on; box on;

for N = Nvals
    h = 1/N;
    x = linspace(0,1,N+1)';
    A = zeros(N-1);
    b = zeros(N-1,1);

    for i = 1:N-1
        A(i,i) = 2/h;

        if i > 1
            A(i,i-1) = -1/h - p/2;
        end

        if i < N-1
            A(i,i+1) = -1/h + p/2;
        end
    end
    b(end) = (1/h - p/2);
    uh_inner = A\b;
    uh = [0; uh_inner; 1];

    plot(x, uh, 'o-', 'LineWidth', 2, ...
         'DisplayName', ['FEM, N=', num2str(N)]);
end

x_exact = linspace(0,1,2000)';
u_exact = (exp(p*x_exact) - 1)/(exp(p) - 1);
plot(x_exact, u_exact, 'k', 'LineWidth', 1, 'DisplayName', 'Exact');

title("FEM vs Analytical Solution for -u'' + p u' = 0,  p=70");
xlabel('x'); ylabel('u(x)');
legend('Location','best');
```